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BOUNDS ON DISTRIBUTION FUNCTIONS IN TERMS OF EXPECTATIONS OF ORDER-STATISTICS

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Suppose $x_1, \cdots, x_n$ are the order-statistics of a random sample from a distribution F. We assume that the expectations $\xi_{i:n} = E(x_i)$ are known, and derive sharp bounds on $F(x)$ for all x. These results are obtained by transforming the problem into a classical one involving ordinary power moments.

1. Introduction and summary. Kadane [5] has considered the problem of deciding when a triangular array of numbers $\xi_{i:n}$ can be represented as the expectations of order-statistics in random samples from some distribution on the positive half-line, so that a distribution function (df) F exists such that $F(0-) = 0$, and

$$\xi_{i:n} = B(i, n+1-i)^{-1} \int xF(x)^{i-1}(1-F(x))^{n-i}\, dF(x) \qquad 1 \leqq i \leqq n \tag{1}$$

where B is the complete beta function (see e.g. [3]). Here we treat the analogous problems when F has unrestricted or interval support. We also consider the question of finding sharp bounds for $F(x)$ when the expectations $\xi_{i:n}$ are known for $1 \leqq i \leqq n$, and sharp bounds for $\xi_{i:n+1}$ $(1 \leqq i \leqq n+1)$ under the same conditions.

Our basic device is to consider, instead of F the inverse function $G(p)$ defined by (3) below, treating this as a df. The (ordinary) moments of this new df are related very simply to the expectations of order-statistics in samples from F, so the problems considered here reduce to standard ones in the theory of moments and Chebyshev inequalities. Thus, necessary and sufficient consistency conditions on the expectations $\xi_{i:n}$ are given in Theorem 1 below for the finite-range case and in Theorem 2 for cases where F has semi-infinite or infinite support. In Section 3 we derive sharp bounds for $F(x)$ when the expectations of the order-statistics in samples of size n are known. Some explicit results appear in Section 4.

The author's original motivation for this work was his desire to understand to what degree a df can be determined from a compact summary of information from several random samples. Suppose one is able to collect (repeatedly) random samples of size n, but can only afford to record a total of n items of information. Then one could choose $n-1$ cell-boundaries $\{\theta_i : i = 1, \cdots, n-1\}$ and record the cumulative frequencies in the n cells; this would enable us to estimate only the $n-1$ values $\{F(\theta_i) : i = 1, \cdots, n-1\}$. A weakness of this method is that the cells have to be selected in advance; a poor choice will result in lost efficiency. Also, F remains completely unknown (except for monotonicity) between the θ_i's.

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An alternative method would be to accumulate estimates of the first n moments of the distribution, and to use these to provide an estimate of F. Here one can use the Chebyshev inequalities (see [7], [8]) to bound $F(x)$ for each x. More generally, one could estimate other linear functionals (generalized moments) of F.

Another possibility would be to record the cumulative averages of the order-statistics. This would provide estimates of $\xi_{i:n}$, and the question arises as to how effective these would be in determining the df. Here we assume that $\xi_{i:n}$, $i = 1, \cdots, n$ can be estimated to as high an accuracy as desired, and proceed to derive sharp bounds on $F(x)$ at each x. It turns out that in comparison to the Chebyshev bounds derived from power-moments, these are comparatively weak in the tails but stronger in the center, so that this method promises to be of value if the shape of the center of the distribution is of most interest.

A detailed comparison of these various techniques for estimating and bounding F, taking account of sampling variability, would be interesting but has not yet been accomplished.

Kadane [5] gives a direct application of his results; he is able to show that a conjectured structure for the expectations of the order-statistics is satisfied only by a degenerate distribution.

2. The basic transformation. Suppose $F(x)$ is any df, (monotone non-decreasing, right-continuous) with $F(\alpha-) = 0$, $F(\beta) = 1$. We assume throughout that $\int |x| \, dF(x) < \infty$ so that $|\xi_{i:n}| < \infty$ for all i, n.

Expanding the factor $(1 - F)^{n-i}$ in (1) by the binomial theorem, we can write $\xi_{i:n}$ as a linear combination of $\xi_{m:m}$ $(1 \leqq m \leqq n)$; this gives

$$\xi_{i:n} = B(i, n+1-i)^{-1} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{1}{i+j} \xi_{i+j:i+j} \,. \tag{2}$$

Conversely by introducing a unit factor written as $(F + 1 - F)^{n-m}$ and expanding we can write $\xi_{m:m}$ in terms of $\xi_{i:n}$ $(1 \leqq i \leqq n,\ n \leqq m)$. Many other identities can be obtained similarly; Kadane [5] gives another pair explicitly, and see also [9].

Thus instead of considering the whole triangular array $\xi_{i:n}$ $(1 \leqq i \leqq n)$, it is sufficient (and convenient) to consider only the sequence $\xi_{k:k}$ $(1 \leqq k \leqq n)$. Henceforth we shall write ξ_k for $\xi_{k:k}$. Chan [2] and Konheim [6] show that the sequence $\xi_1, \xi_2, \cdots$ determines F; from (2) this is clearly equivalent to the observation that the triangular array $\{\xi_{i:n} : 1 \leqq i \leqq n < \infty\}$ determines F. (In particular, $n^{-1} \sum_i (\frac{1}{2} + \frac{1}{2} \operatorname{sgn}(x - \xi_{i:n}))$ converges to F as $n \to \infty$; see [4].)

We now assume that both α and β are finite. We can define a function G inverse to F by

$$G(p) = \sup \{x : F(x) \leqq p\} \qquad 0 \leqq p < 1 \,. \tag{3}$$

Then G is monotone non-decreasing and right continuous, and so it defines a finite nonnegative measure on [0, 1] with total mass

$$\mu_0 = \textstyle\int_0^1 dG(p) = \beta - \alpha .$$

From (1), integrating by parts, for $k \geqq 1$,

$$\xi_k = \textstyle\int_\alpha^\beta x\, dF^k(x) = \beta - \int_0^1 p^k\, dG(p) = \beta - \mu_k \tag{4}$$

where μ_k is the kth moment of the measure G. Clearly it is convenient to define $\xi_0 = \alpha$, so that (4) holds for $k = 0$ also.

We have now transformed the problem into a standard form, namely that of the Hausdorff moment problem (for which see [8], Theorem 3.1 and Corollary 1.1, or [1], Section 6.4, page 73) and have the following result.

THEOREM 1. *Necessary and sufficient* (N *and* S) *conditions that the array* $\xi_{i:n}$ $(1 \leqq i \leqq n)$ *be representable as the expectations of order-statistics of random samples from some distribution on the finite interval* $[\alpha, \beta]$ *are that they satisfy the consistency conditions* (2) *and that the sequence*

$$\mu_k = \beta - \xi_k \qquad 0 \leqq k \leqq n \tag{5}$$

(*where* $\xi_0 - \alpha$, $\xi_k = \xi_{k:k}$) *are the moments of a nonnegative measure on* [0, 1]. *This moment-problem is always determinate, so that if a distribution F exists corresponding to the expectations* ξ_k, $k = 1, 2, \cdots$ *then F is unique.*

These N and S conditions are presented explicitly in the references cited above; to write them in a convenient form we introduce the following definition.

DEFINITION. A sequence $r_1, s_1, r_2, \cdots, s_n$ is *m-zero* if $r_k, s_k > 0$ for $k = 1, 2, \cdots, m - 1$, while $r_k = s_k = 0$, $k = m + 1, \cdots, n$. Now we define certain determinants. With $\mu_1, \mu_2, \cdots$ defined by (5), let

$$\begin{aligned}
A_{2k} &= |\mu_{i+j}|_{i,j=0,1,\cdots,k} && k = 1, 2, \cdots \\
A_{2k+1} &= |\mu_{i+j+1}|_{i,j=0,1,\cdots,k} && k = 0, 1, \cdots \\
B_{2k} &= |\mu_{i+j-1} - \mu_{i+j}|_{i,j=1,\cdots,k} && k = 1, 2, \cdots \\
B_{2k+1} &= |\mu_{i+j} - \mu_{i+j+1}|_{i,j=0,1,\cdots,k} && k = 0, 1, \cdots .
\end{aligned} \tag{6}$$

Notice that A_k and B_k are not functions of $\xi_{k+1}, \cdots, \xi_n$.

Then the N and S condition for the numbers $\xi_1, \xi_2, \cdots$ to be the expected values of largest order-statistics in samples from a distribution on $[\alpha, \beta]$ is that either $A_1, B_1, A_2, \cdots, B_n$ are all positive or else for some m, $1 \leqq m \leqq n$, this sequence is m-zero with either $A_m = 0$, $B_m > 0$ or with $A_m > 0$, $B_m = 0$.

The significance of the integer m appearing here is as follows. Iff (if and only if) $0 < B_1, A_1, B_2, \cdots, A_{m-1}, B_m$, $0 = A_m = B_{m+1} = A_{m+1} = \cdots = A_n$ then there is a *unique* distribution having the given expected order-statistics, and it has support at $1 + [\frac{1}{2}m]$ points, one of which is $x = \beta$, and another of which is $x = \alpha$ iff m is even. This distribution is determined by $\alpha, \beta, \xi_1, \cdots, \xi_m$, so that all higher μ's and ξ's are functions of these. Similarly, iff $0 < A_1, B_1, \cdots, B_{m-1}, A_m$, $0 = B_m = A_{m+1} = B_{m+1} = \cdots = B_n$, then there is a unique distribution with the given expectations, with support at $[\frac{1}{2}(m+1)]$ points, none of which is $x=\beta$, and one of which is $x=\alpha$ iff m is odd. Again, this distribution is determined by $\alpha, \beta, \xi_1, \cdots, \xi_m$.

If $A_n B_n > 0$ there are many distributions having the given expectations.

All the above facts follow easily from the material in [1] and [8] cited above, after re-interpretation using (3) and (4).

Now we can deduce N and S conditions on the ξ's for the cases in which F has infinite or semi-infinite support.

We now define

$$\begin{aligned}
A_1' &= 1\,, \qquad B_k' = B_k && k=1,2,\cdots\\
A_{2k}' &= |-\xi_{i+j-2} + 2\xi_{i+j-1} - \xi_{i+j}|_{i,j=1,2,\cdots,k} && k=1,2,\cdots\\
A_{2k+1}' &= |-\xi_{i+j-1} + 2\xi_{i+j} - \xi_{i+j+1}|_{i,j=1,\cdots,k} && k=1,2,\cdots\\
A_1'' &= B_1'' = A_2'' = 1 &&\\
A_{2k+1}'' &= A_{2k+1}' && k=1,2,\cdots\\
B_{2k}'' &= B_{2k} && k=1,2,\cdots\\
B_{2k+1}'' &= |\xi_{i+j+1} - \xi_{i+j}|_{i,j=1,\cdots,k} && k=1,2,\cdots\\
A_{2k}'' &= |-\xi_{i+j-2} + 2\xi_{i+j-1} - \xi_{i+j}|_{i,j=2,3,\cdots,k} && k=2,3,\cdots.
\end{aligned}$$

(In the above definitions, the determinants B_k' and B_{2k}'' are to be taken as the functions of $\xi_1, \xi_2, \cdots$ that are obtained by using (5) to substitute ξ's for μ's in the definitions (6). In each case the result is independent of β.)

THEOREM 2. (i) *If* $-\infty < \alpha$, $\beta = \infty$ *the* N *and* S *condition on the sequence* $\xi_1, \xi_2, \cdots, \xi_n$ *is that either* $A_1', B_1', A_2', \cdots, B_n'$ *are all positive, or else for some* m, $1 \leqq m \leqq n$ *this sequence is m-zero with* $B_m' = 0$, $A_m' > 0$.

(ii) *If* $\alpha = -\infty$, $\beta = +\infty$, *the* N *and* S *condition is that either* $A_1'', B_1'', \cdots, B_n''$ *are all positive, or else for some* m, $1 \leqq m \leqq n$ *this sequence is m-zero where* m *is even*, $B_m'' = 0$, $A_m'' > 0$.

The proof is straightforward.

The above results can be used to obtain sharp bounds for the expectations of order-statistics in samples of all sizes; if $\xi_{1:m}, \cdots, \xi_{m:m}$ are given such that a distribution exists having these expectations, then for any $n > m$ the expectations $\xi_{1:n}, \cdots, \xi_{n:n}$ must satisfy the conditions expounded above (after Theorem 1 and in Theorem 2).

3. Bounds for the df **when the expectations are known.** We turn now to the problem of giving sharp bounds for $F(x)$ when $\xi_1, \cdots, \xi_n$ are known. We start from the classical Chebyshev inequalities for $G(p) - \alpha$ (given $p, \alpha, \beta, \mu_0, \mu_1, \cdots, \mu_n$), and have only to translate these into bounds for p given x (and $\alpha, \beta, \xi_1, \cdots, \xi_n$) using (3) and (5). The algebraic details in the case α, β both finite are messy, and we shall merely describe these results informally. (Details of the classical results can be found in [8], Lemma 3.1, page 79.) Letting $\beta \to \infty$ and/or $\alpha \to -\infty$ does lead to some simplifications.

For α, β finite, the Chebyshev bounds for $G(p_0) - \alpha$ given $p_0, \mu_0, \mu_1, \cdots, \mu_n$ are obtained by fitting a distribution G_{p_0} to the given moments, having exactly $\frac{1}{2}n + 1$ points of increase in [0, 1], one of which is p_0; here the end-points (0 and 1) count as halves. Thus for $n = 2$ we either have a distribution with support $\{p_0, p_0'\}$ for some p_0' ($\neq 0, p_0, 1$) or else a distribution with support $\{0, p_0, 1\}$. This construction is always possible if $A_m B_m > 0$ for $m = 1, \cdots, n$ (if not, $G(p_0)$ is unique!) except when p_0 takes one of n special values $\pi_1, \cdots, \pi_n$. These are the zeros of two polynomials A_n^*, B_n^* of degree $[\frac{1}{2}(n+1)]$, $[\frac{1}{2}n]$ respectively which can be constructed from the determinants A_{n+1}, B_{n+1} by replacing their bottom rows by the row $(1, \pi, \pi^2, \cdots)$. The zeros of A_n^* interlace with those of B_n^*. When $p = \pi_m$ for some m, it is possible to construct a distribution with support at exactly $\frac{1}{2}(n+1)$ points (end-points still counting half), one of which is π_m.

Once the appropriate "extremal" distribution G_{p_0} has been constructed, the Chebyshev bounds are

$$G_{p_0}(p_0 - 0) \leqq G(p_0) \leqq G_{p_0}(p_0 + 0) .$$

Using (3) and (5) we can translate these bounds into inequalities for $F(x)$ when $x, \alpha, \beta, \xi_1, \cdots, \xi_n$ are given. This gives

THEOREM 3. *Given $x_0, \alpha, \beta, \xi_1, \cdots, \xi_n$ with $A_1, \cdots, B_n$ all positive, sharp bounds for $F(x_0)$ can be obtained by constructing a distribution $F_{x_0}(x)$ on $[\alpha, \beta]$ having the correct order-statistic expectations, and having exactly $\frac{1}{2}n + 1$ points of increase in $[\alpha, \beta]$ (end-points counting half), one of which is x_0; this construction is always possible unless x_0 takes one of n special values $\zeta_1, \zeta_2, \cdots, \zeta_n$ in (α, β), when a distribution with $\frac{1}{2}(n+1)$ points of increase can be constructed. The bounds are then*

$$F_{x_0}(x_0 - 0) \leqq F(x_0) \leqq F_{x_0}(x_0) . \tag{7}$$

Now we can let $-\alpha$ and/or $\beta \to \infty$. It is clear that provided the determinants $A_1', B_1', \cdots$ etc. are all positive, sharp bounds for $F(x_0)$ will be obtained if we take β large, proceeding as in Theorem 3, and then let $\beta \to +\infty$, and so on. On examination of the various cases that arise we have

THEOREM 4. *Sharp upper and lower bounds for $F(x_0)$ when $\xi_{1:n}, \cdots, \xi_{n:n}$ are known are given by (7) where the distribution $F_{x_0}(x)$ has a finite number of points of increase, one of which is x_0, and is constructed as prescribed below.*

(i) $-\infty < \alpha$, $\beta = \infty$. There are $n + 2$ numbers $\alpha = \zeta_0 < \zeta_1 < \cdots < \zeta_n < \zeta_{n+1} = \infty$ such that if $\zeta_{2m} \leqq x_0 \leqq \zeta_{2m+1}$ then H_{x_0} has the correct expectations $\xi_{1:n}, \cdots, \xi_{n:n}$, while if $\zeta_{2m+1} < x_0 < \zeta_{2m}$ then F_{x_0} has only $\xi_{1:n}, \cdots, \xi_{n-1:n}$ correct. F_{x_0} has $[\frac{1}{2}n] + 1$ points of increase except when n is even and $x_0 = \zeta_{2m+1}$ for some m, when it has $[\frac{1}{2}n]$.

The numbers $\zeta_1, \cdots, \zeta_n$ can be found by constructing two distributions H_1, H_2, where H_1 has $[\frac{1}{2}(n+1)]$ points of increase, namely $\zeta_1, \zeta_3, \zeta_5, \cdots$ and has the correct expectations $\xi_{1:n}, \cdots, \xi_{k:n}$ where k is the largest odd integer $\leqq n$; and

H_2 has $[\frac{1}{2}n] + 1$ points of increase, namely, $\alpha, \zeta_2, \zeta_4, \cdots$ and has correct expectations $\xi_{1:n}, \cdots, \xi_{2n-1-k:n}$.

(ii) $-\infty = \alpha$, $\beta = +\infty$, *n odd. There are n numbers* $\zeta_1 < \cdots < \zeta_n$ such that if $x_0 \leqq \zeta_1$ or $\zeta_{2m} \leqq x_0 \leqq \zeta_{2m+1}$ then F_{x_0} has $\xi_{1:n}, \cdots, \xi_{n-1:n}$ correct, while if $\zeta_{2m+1} \leqq x_0 \leqq \zeta_{2m+2}$ or $\zeta_n \leqq x_0$ then F_{x_0} has $\xi_{2:n}, \cdots, \xi_{n:n}$ correct. F_{x_0} has $\frac{1}{2}(n+1)$ points of increase except when $x_0 = \zeta_{2m}$. The numbers $\zeta_1, \zeta_3, \cdots, \zeta_n$ are the points of increase of a distribution having $\xi_{1:n}, \cdots, \xi_{n:n}$ all correct; the numbers $\zeta_2, \cdots, \zeta_{n-1}$ are the points of increase of a distribution having $\xi_{2:n}, \cdots, \xi_{n-1:n}$ correct.

(iii) $-\infty = \alpha$, $\beta = +\infty$, n even. Again there are n numbers $\zeta_1 < \cdots < \zeta_n$. If $x_0 \leqq \zeta_1$, or $\zeta_{2m} < x_0 < \zeta_{2m+1}$, F_{x_0} has $\xi_{1:n}, \cdots, \xi_{n:n}$ all correct, and has $\frac{1}{2}n + 1$ points of increase; if $\zeta_{2m+1} \leqq x_0 \leqq \zeta_{2m+2}$, F_{x_0} has $\xi_{2:n}, \cdots, \xi_{n-1:n}$ correct and only $\frac{1}{2}n$ points of increase. The numbers $\zeta_1, \zeta_3, \cdots$ are the points of increase of a distribution having $\xi_{1:n}, \cdots, \xi_{n-1:n}$ correct; the numbers $\zeta_2, \zeta_4, \cdots$ are the points of increase of a distribution having $\xi_{2:n}, \cdots, \xi_{n:n}$ correct.

REMARKS. (i) Notice that in each case there are values of x_0 such that the bounds of $F(x_0)$ do not depend on $\xi_{n:n}$, but only on $\xi_{1:n}, \cdots, \xi_{n-1:n}$. When n is even and $\alpha = -\infty$, $\beta = +\infty$ there are values of x_0 such that neither $\xi_{1:n}$ or $\xi_{n:n}$ play any role in the inequalities.

(ii) In the semi-infinite case the inequalities are generated by distributions of the same shapes as arise in the case of the ordinary Chebyshev inequalities, when n power moments are given. (See for example [7] page 366.) This is not so in the doubly-infinite case.

(iii) Since the transformed moment-problem is always determinate (Theorem 1), the bounds converge to $F(x)$ as n increases.

4. Some explicit results. We present some of the inequalities derived above explicitly. For brevity we write $a_1 = \xi_{1:1}$, $b_1 = \xi_{1:2}$, $b_2 = \xi_{2:2}$, $c_1 = \xi_{1:3}$, $c_2 = \xi_{2:3}$, $c_3 = \xi_{3:3}$ etc. From the results of Section 2, when α, β are both finite, we have

$$\begin{gathered}
\alpha + (a_1 - \alpha)^2/(\beta - \alpha) \leqq b_1 \leqq a_1 \leqq b_2 \leqq \beta - (\beta - a_1)^2/(\beta - \alpha) \\
b_1 + \tfrac{1}{2}(b_2 - b_1)^2/(\beta - a_1) \leqq c_2 \leqq b_2 - \tfrac{1}{2}(b_2 - b_1)^2/(a_1 - \alpha) \\
b_2 + \tfrac{1}{4}(b_2 - b_1)^2/(a_1 - \alpha) \leqq c_3 \leqq \beta - (\beta - b_2)^2/(\beta - a_1)\,.
\end{gathered}$$

Letting $\alpha \to -\infty$, $\beta \to \infty$ we obtain results for the unrestricted case:

$$\begin{gathered}
b_1 \leqq a_1 \leqq b_2 \\
2b_1 - a_1 \leqq c_1 \leqq b_1 \leqq c_2 \leqq b_2 \leqq c_3 \leqq 2b_2 - a_1 \\
2c_1 - b_1 \leqq d_1 \leqq c_1 - \tfrac{1}{3}X \\
c_1 + X \leqq d_2 \leqq c_2 \leqq d_3 \leqq c_3 - Y \\
c_3 + \tfrac{1}{3}Y \leqq d_4 \leqq 2c_3 - b_2
\end{gathered}$$

where $X = (c_2 - c_1)^2/(c_3 - c_1)$, $Y = (c_3 - c_2)^2/(c_3 - c_1)$.

Applying the results of Section 3 for the unrestricted case with $n = 2$, the bounds for $F(x)$ are

$$\begin{array}{ll} 0 \leqq F(x) \leqq (b_2 - a_1)/(a_1 - x) & x \leqq b_1 \\ 0 \leqq F(x) \leqq 1 & b_1 \leqq x \leqq b_2 \\ (x - b_2)/(x - a_1) \leqq F(x) \leqq 1 & b_2 \leqq x\,. \end{array}$$

Obtaining the bounds generated by $\xi_{i:n}$ $(1 \leqq i \leqq n)$ for $n \geqq 3$ requires the solution of polynomial equations of degrees higher than unity; computationally a simpler approach is to construct the extremal distributions described in Theorems 3 and 4 by direct numerical methods.

The form of the bounds as $|x| \to \infty$ is of some interest; simple computations show that the upper bound for $F(x)$ as $x \to -\infty$ when ξ_j is given for $1 \leqq j \leqq 2m$ is asymptotically $c/|x|$ where

$$c = \begin{vmatrix} 1 & 1 & 1 & & & 1 \\ \xi_1 & \xi_2 & \xi_3 & \cdot & \cdot & \xi_{m+1} \\ \xi_2 & \xi_3 & \cdot & \cdot & \cdot & \xi_{m+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_m & \cdot & \cdot & \cdot & \cdot & \xi_{2m} \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 & & & 1 \\ \xi_3 & \xi_4 & \xi_5 & \cdot & \cdot & \xi_{m+2} \\ \xi_4 & \xi_5 & \cdot & \cdot & \cdot & \xi_{m+3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_m & \cdot & \cdot & \cdot & \cdot & \xi_{2m} \end{vmatrix}.$$

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