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A STABLE LOCAL LIMIT THEOREM

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Conditions are given which imply that the partial sums of a sequence of independent integer-valued random variables, suitably normalized, converge in distribution to a stable law of exponent α, $0 < \alpha < 2$, and imply as well that a strong version of the corresponding local limit theorem holds.

1. Introduction. If $\{X_k\}_{k=1}^{\infty}$ is a sequence of independent integer-valued random variables whose partial sums $S_n = \sum_{i=1}^{n} X_i$, after suitable normalization, converge in distribution to a stable limit law with exponent α, $0 < \alpha < 2$, i.e.,

$$P\{S_n/B_n - A_n < x\} \rightarrow G_\alpha(x) , \tag{1}$$

$\{A_n\}$ and $\{B_n\}$ being sequences of constants, $G_\alpha(x)$ being the distribution function of a stable law, then $\{X_n\}$ are said to satisfy a stable local limit theorem if, in addition,

$$\lim_{n\to\infty} B_n P\{S_n = x\} - g_\alpha(x/B_n - A_n) = 0 \tag{2}$$

uniformly for all integer x, where $g_\alpha(x) = (d/dx)G_\alpha(x)$. $P\{\cdot\}$ is, of course, the product measure defined by the distribution functions $\{F_n(x)\}$ of the sequence. If such a theorem holds for all sequences $\{X_{k+m}\}_{k=1}^{\infty}$, then $\{X_k\}_{k=1}^{\infty}$ is said to satisfy a strong stable local limit theorem. Rozanov [5] has shown that if $B_n \rightarrow \infty$, a necessary condition for a strong local limit theorem is that

$$\prod_{k=1}^{\infty} [\max_{0\leqq x<h} P\{X_k \equiv x \,(\mathrm{mod}\, h)\}] = 0 , \qquad \text{for all } h \geqq 2 . \tag{A}$$

We note that if $\{X_k\}$ satisfies (A), so also does $\{X_k'\}$, the symmetrization of $\{X_k\}$, i.e., $X_k' = X_k - Y_k$, where Y_k is independent of X_k and has the same distribution.

Stable local limit theorems have been proved by Gnedenko [1] in the identically distributed case, and by Mitalauskas [3], [4] in the non-identically distributed case. In this paper similar results are obtained under weaker hypotheses by modifying the methods of [2].

2. Local limit theorem. We employ the notation

$$P_k(x) = P(X_k = x) , \qquad P_k'(x) = P(X_k' = x) ,$$
$$\varphi_k(t) = \sum e^{itx} P_k(x) , \qquad \psi_k(t) = \prod_{k=1}^{n} \varphi_k(t) .$$

Note $P_k'(x) = \sum P_k(y)P_k(x+y)$.

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THEOREM. *If* $\{X_n\}$ *satisfies* (A) *and for some* $\{B_n\}$, $B_n \to \infty$, $0 < \alpha < 2$,

(C₁) $$\sup_{k \leqq n} P\{|X_k| > \varepsilon B_n\} \to 0\,, \qquad \textit{for all } \varepsilon > 0\,,$$

(C₂) $$\textstyle\sum_{k=1}^{n} F_k(x) = (c_1 B_n^{\alpha} + e_n(x))|x|^{-\alpha}\,, \qquad \textit{for } x < 0, \textit{ and}$$
$$\textstyle\sum_{k=1}^{n} (1 - F_k(x)) = (c_2 B_n^{\alpha} + e(x))|x|^{-\alpha}\,, \qquad \textit{for } x > 0\,,$$

where $c_1, c_2 \geqq 0$, $c_1 + c_2 > 0$, *and* $e_n(B_n x)B_n^{-\alpha} \to 0$ *for any* x,

(C₃) $$\exists\{L_n\} \quad \textit{satisfying} \quad n(L_n/B_n)^{\eta} \to 0\,,$$
$$\eta = 1 \ \textit{for} \ \alpha < 1, \eta = 2 \ \textit{for} \ \alpha \geqq 1\,,$$

and such that $\max_{L_n < |x|} |e_n(x)|B_n^{-\alpha}$ *is uniformly bounded, and, if* $\alpha < 1$, *approaches zero as* $n \to \infty$,

(C₄) $\exists\{M_n\}$ *and* L *such that, setting* $Q_n = \sum_{k=1}^{n} P\{0 < |X_k'| \leqq L\}$,

(a₁) $$\max_{M_n < x} |e_n(x) + e_n(-x)|B_n^{-\alpha} < (1 - \delta)(c_1 + c_2)\,, \qquad \delta > 0\,,$$

(a₂) $$M_n^2 \log n/Q_n \to 0\,,$$

and either

(a₃′) $$\inf_{k \leqq n} P\{|X_k| < M_n\} \geqq U > 0\,, \qquad \textit{for all } n\,,$$

or

(a₃″) $$\sup_{k \leqq n} \sup_{M_n < t} [t^{-1} \textstyle\int_{|x|<t} x\, dF_k(x)] \to 0 \qquad \textit{as } n \to \infty\,,$$

then $\{X_k\}$ *satisfies a strong local limit theorem of the form* (2), *i.e.*,

$$\lim_{n \to \infty} B_n P\{S_n = x\} - g_\alpha(x/B_n - A_n) = 0$$

uniformly in integer x, *where* $\{A_n\}$ *is defined by*

$$A_n = B_n^{-1} \textstyle\sum_{k=1}^{n} EX_k \qquad \textit{if} \quad \alpha > 1$$
$$A_n = B_n^{-1} \textstyle\sum_{k=1}^{n} \int_{|x| < \tau B_n} x\, dF_k(x) \qquad \textit{if} \quad \alpha = 1$$
$$A_n = 0\,, \qquad \textit{if} \quad \alpha < 1$$

and where $g_\alpha(x)$ *is determined by*

$$\begin{aligned}\psi(t) = \textstyle\int e^{itx} g_\alpha(x)\, dx = \exp\{&i\gamma(\tau)t + c_1 \textstyle\int_{-\infty}^{-\tau} (e^{itx} - 1)|x|^{-\alpha-1}\, dx \\ &+ c_1 \textstyle\int_{-\tau}^{0} (e^{itx} - 1 - itx)|x|^{-\alpha-1}\, dx \\ &+ c_2 \textstyle\int_0^{\tau} (e^{itx} - 1 - itx)x^{-\alpha-1}\, dx + c_2 \textstyle\int_{\tau}^{\infty} (e^{itx} - 1)x^{-\alpha-1}\, dx\}\,,\end{aligned}$$

in which $\gamma(\tau) = 0$ *when* $\alpha = 1$, *and* $\gamma(\tau) = (c_2 - c_1)\alpha\tau^{1-\alpha}/1 - \alpha$ *otherwise.*

REMARKS. No requirement is made that B_n be of strict order of magnitude $n^{1/\alpha}$ as in [3], [4]. Of course, hypothesis (a₁) implies the bound $B_n \leqq O(n^{1/\alpha} M_n)$. The most unsatisfactory hypothesis is the alternative (a₃′) or (a₃″) which places a uniform restriction on the distributions of the individual X_k. It seems quite difficult to remove.

A more standard form for the characteristic function $\psi(t)$ is

$$\psi(t) = \exp\{i\gamma' t - c|t|^{\alpha}[1 + i\beta t/|t| \tan \pi\alpha/2]\} \qquad \text{if } \alpha \neq 1, \text{ or}$$
$$\psi(t) = \exp\{i\gamma' t - c|t|^{\alpha}[1 + 2it\beta/\pi|t| \cdot \log |t|]\} \qquad \text{if } \alpha = 1\,,$$

where $\beta = (c_1 - c_2)/(c_1 + c_2)$ and $c = -(c_1 + c_2)I(\alpha)$, $I(\alpha)$ being a constant depending on α. We use the nonstandard form to facilitate the application of Gnedenko's theorem.

If $1 < \alpha < 2$, setting $L_n = M_n = n^\gamma$, the following simplified result is obtained from the theorem:

COROLLARY. *If* $\{X_k\}$ *satisfies* (A), (C_1), *and* (C_2) *holds with* $1 < \alpha < 2$,

(D_1) $$B_n \cdot n^{-\beta} \geqq A > 0\,, \qquad \textit{where } \beta > \tfrac{1}{2}\,,$$

(D_2) $$\inf_k \min_{|x|<L} P\{X_k \neq x,\ |X_k| < L\} > 0\,, \qquad \textit{and}$$

(D_3) $$\max_{n^\gamma < x} |e_n(x) + e_n(-x)| B_n^{-\alpha} \leqq (1 - \delta)(c_1 + c_2)\,, \qquad \textit{where } \gamma < \min(\tfrac{1}{2}, \beta - \tfrac{1}{2}),\ \delta > 0\,,$$

then $\{X_k\}$ *satisfies a strong local limit theorem of the form* (2).

An analogous result can be given in the case $0 < \alpha < 1$.

PROOF OF THEOREM. (C_1) is simply the natural requirement that the random variables $\{X_k/B_n\}$ be infinitesimal. Verifying the conditions of Gnedenko's theorem ([1] page 124), we first establish that the relevant integral limit theorem (1) is satisfied. It is obvious that (C_2) implies that for $x < 0$,

(3) $$\textstyle\sum_{k=1}^n F_k(xB_n) \to c_1|x|^{-\alpha}$$

and a similar result for $x > 0$.

Next we must show

(4) $$\lim_{\varepsilon\to 0} \lim_{n\to\infty} B_n^{-2} \textstyle\sum_{k=1}^n \{\int_{|x|<B_n\varepsilon} x^2\, dF_k(x) - (\int_{|x|<B_n\varepsilon} x\, dF_k(x))^2\} = 0$$

for which it suffices that

(5) $$\limsup_{n\to\infty} B_n^{-2} \textstyle\sum_{k=1}^n \int_{|x|<B_n\varepsilon} x^2\, dF_k(x) \to 0 \qquad \text{as } \varepsilon \to 0\,.$$

For the sum of the integrals over the positive range, we have

(6) $$\begin{aligned} &B_n^{-2} \textstyle\sum_{k=1}^n \int_0^{L_n} x^2\, dF_k(x) + B_n^{-2} \sum_{k=1}^n \int_{L_n}^{\varepsilon B_n} x^2\, dF_k(x) \\ &\leqq nB_n^{-2}L_n^2 + B_n^{-2}[\textstyle\sum x^2(F_k(x) - 1)]_{L_n}^{\varepsilon B_n} \\ &\quad + B_n^{-2} \textstyle\int_{L_n}^{\varepsilon B_n} 2x^{1-\alpha}(c_2 B_n^\alpha + e_n(x))\, dx\,. \end{aligned}$$

The negative of the second term is equal to

$$\begin{aligned} &B_n^{-2}[-L_n^{2-\alpha}(c_2 B_n^\alpha + e_n(L_n)) + (\varepsilon B_n)^{2-\alpha}(c_2 B_n^\alpha + e_n(\varepsilon B_n))] \\ &= -(L_n/B_n)^{2-\alpha}(c_2 + e_n(L_n)B_n^{-\alpha}) + \varepsilon^{2-\alpha}(c_2 + e_n(\varepsilon B_n)B_n^{-\alpha}) \end{aligned}$$

which by (C_3) is $o(1) + \varepsilon^{2-\alpha}(c_2 + o(1))$.

The third term in (6) is bounded by

$$\begin{aligned} &\frac{2B_n^{-2}}{2-\alpha}(c_2 B_n^\alpha + \max_{L_n<x} |e_n(x)|)((\varepsilon B_n)^{2-\alpha} - L_n^{2-\alpha}) \\ &\leqq \frac{2}{2-\alpha}\varepsilon^{2-\alpha}(c_2 + \max_{L_n<x} |e_n(x)|B_n^{-\alpha}) = \frac{2}{2-\alpha}\varepsilon^{2-\alpha}(c_2 + O(1))\,. \end{aligned}$$

Treating the integrals over the negative range similarly, since ε is arbitrary, (5) follows.

We have now only to establish the value of the constants A_n in (1). If $\alpha > 1$, it suffices to take $A_n = ES_n/B_n$, since if $EX_n = 0$, for all n, then

$$B_n^{-1} \sum \int_{|x| \leqq \tau B_n} x \, dF_k(x) = B_n^{-1} \sum \int_{|x| > \tau B_n} x \, dF_k(x)$$

which, for a fixed value of τ approaches $((c_2 - c_1)\alpha/(\alpha - 1))\tau^{1-\alpha}$ by (C_2) and (C_3).

If $\alpha = 1$, we take, for an arbitrary τ, the constants $A_n(\tau)$ given by Gnedenko's theorem, i.e., $A_n(\tau) = B_n^{-1} \sum_{k=1}^{n} \int_{|x| < \tau B_n} x \, dF_k(x)$.

If $a < 1$, we have

$$\begin{aligned}
&B_n^{-1} \sum \int_0^{\tau B_n} x \, dF_k(x) \\
&\quad = \int_0^{L_n} + \int_{L_n}^{\tau B_n} \\
&\quad = B_n^{-1} \sum \int_0^{L_n} x \, dF_k(x) \\
&\qquad + B_n^{-1}[L_n(c_2 B_n^{\alpha} + e_n(L_n))L_n^{-\alpha} - \tau B_n(c_2 B_n^{\alpha} + e_n(\tau B_n))(\tau B_n)^{-\alpha}] \\
&\qquad + B_n^{-1+\alpha} c_2[(\tau B_n)^{1-\alpha} - L_n^{1-\alpha}]1/(1-\alpha) + B_n^{-1} \int_{L_n}^{\tau B_n} e_n(x) x^{-\alpha} \, dx
\end{aligned}$$

and by (C_3) this is equal to the sum of $c_2 \tau^{1-\alpha}\alpha/(1 - \alpha)$ and terms which approach zero as $n \to \infty$. A similar treatment of the integral over the negative range gives us the value of the constant $\gamma(\tau)$, concluding the proof of the integral limit theorem.

Therefore, for any interval $(-A, A)$ of values of t,

$$e^{-itA_n}\psi_n(t/B_n) \to \psi(t) \tag{7}$$

uniformly. By the Fourier inversion formula,

$$\begin{aligned}
&2\pi[B_n P\{S_n = x\} - g(x/B_n - A_n)] \\
&\quad = B_n \int_{-\pi}^{\pi} \varphi_n(t) e^{-itx} \, dt - \int_{-\infty}^{\infty} \psi(t) e^{-(x/B_n - A_n)it} \, dt \\
&\quad = [\int_{|t| \leqq A} \varphi_n(t/B_n) e^{-itx/B_n} \, dt - \int_{|t| \leqq A} \psi(t) e^{-itx/B_n + iA_n t} \, dt] \\
&\qquad - \int_{|t| > A} \psi(t) e^{-(x/B_n - A_n)it} \, dt + B_n \int_{A/B_n < |t| \leqq B/M_n} \varphi_n(t) e^{-itx} \, dt \\
&\qquad + B_n \int_{B/M_n < |t| \leqq C} \varphi_n(t) e^{-itx} \, dt + B_n \int_{C < |t| \leqq \pi} \varphi_n(t) e^{-itx} \, dt \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 .
\end{aligned}$$

By (7), for any fixed A, $|I_1|$ can be made arbitrarily small for n sufficiently large. Since $\psi(t)$ is absolutely integrable, $|I_2| < \varepsilon$ if A is sufficiently large.

To bound I_3 we use the fact that

$$|\varphi_k(t)| \leqq \exp \tfrac{1}{2}\{|\varphi_k(t)|^2 - 1\} = \exp \tfrac{1}{2}\{\textstyle\sum_x (\cos tx - 1) P_k'(x)\} , \tag{8}$$

and the bounds $1 - \cos u \geqq b_R > 0$ for $\pi/2R < |u| < (4R - 1)\pi/2R$ and $1 - \cos u \geqq u^2/6$ for $|u| < 2$.

If (a_3') holds, setting $A_R = (\pi/R, (2R - 1)\pi/R)$, then for $|t| < \pi/2RM_n$, $k \leqq n$,

$$\begin{aligned}
1 - |\varphi_k(t)|^2 &\geqq b_R \textstyle\sum_{|xt| \in A_{2R}} P_k'(x) \geqq b_R \sum_{|yt| < \pi/2R} P_k(y) \sum_{|xt| \in A_R} P_k(x) \\
&\geqq U b_R P\{|tX_k| \in A_R\}
\end{aligned}$$

so that by (a_1), if R is taken sufficiently large

$$(9)\qquad \sum_{k=1}^{n}(1-|\varphi_k(t)|^2) \geqq Ub_R(c_1+c_2)B_n{}^{\alpha}\pi^{-\alpha}\left(\delta R^{\alpha}|t|^{\alpha}-(2-\delta)\frac{R^{\alpha}|t|^{\alpha}}{(2R-1)^{\alpha}}\right)$$
$$\geqq c'B_n{}^{\alpha}|t|^{\alpha}$$

where $c'>0$ does not depend on n, or t.

If (a_3'') holds, let $G_k(t)=\sum_{|x|<1/|t|}P_k(x)$, $E_k(t)=\sum_{|x|<1/|t|}xP_k(x)$, $\sigma_k(t)=\sum_{|x|<1/|t|}x^2P_k(x)$. We prove (9) by using the inequalities

$$\sum_{|x|<2/|t|}x^2P_k'(x) \geqq G_k(t)\sigma_k(t)-(E_k(t))^2 \geqq 0$$

from which it follows that

$$(10)\qquad 1-|\varphi_k(t)|^2 \geqq \tfrac{1}{6}\sum_{|xt|<2}(xt)^2P_k'(x) - t^2/6[G_k(t)(\sigma_k(t)-\sigma_k(Rt)) - (E_k(t)-E_k(Rt))(E_k(t)+E_k(Rt))]\,.^{1}$$

By (a_3''), for arbitrary $\varepsilon>0$, if n is sufficiently large, $k\leqq n$, $|t|<1/RM_n$,

$$(11)\qquad E_k(t)+E_k(Rt)\leqq \varepsilon t^{-1}\,,$$

and from (a_1), for $|t|<1/RM_n$,

$$(c_1+c_2)B_n{}^{\alpha}((2-\delta)(R|t|)^{\alpha}-\delta|t|^{\alpha}) \geqq \sum_{k=1}^{n}[G_k(|t|)-G_k(R|t|)]$$
$$\geqq (c_1+c_2)B_n{}^{\alpha}(\delta(R|t|)^{\alpha}-(2-\delta)|t|^{\alpha})$$
$$\geqq c_R'B_n{}^{\alpha}|t|^{\alpha}$$

and $\sum_{k=1}^{n}G_k(|t|)\geqq n-2(c_1+c_2)B_n{}^{\alpha}|t|^{\alpha}=n-c''B_n{}^{\alpha}|t|^{\alpha}$, c_R' being a constant dependent on R.

Let $I_n(t)=\{k/k\leqq n,\ G_k(t)<\frac{1}{2}\}$. Since the cardinality of this set is bounded by $2c''B_n{}^{\alpha}|t|^{\alpha}$, if R is chosen so that $c_R'>4c''$,

$$\sum_{k\notin I_n(t)}G_k(|t|)-G_k(R|t|)\geqq 2c''B_n{}^{\alpha}|t|^{\alpha}\,.$$

Therefore,

$$(12)\qquad \sum_{k=1}^{n}(1-|\varphi_k(t)|^2) \geqq t^2/6\{\tfrac{1}{2}(Rt)^{-2}\sum_{k\notin I_n(t)}[G_k(|t|)-G_k(R_k|t|)] - \varepsilon t^{-1}\cdot t^{-1}\sum_{k=1}^{n}[G_k(|t|)-G_k(R|t|)]\}$$
$$\geqq c'''B_n{}^{\alpha}|t|^{\alpha}$$

if ε is chosen sufficiently small in (11). It follows, by either (9) or (12) that, for an appropriate choice of B, and c,

$$|I_3|\leqq B_n\int_{A/B_n<|t|\leqq B/M_n}\exp\{-cB_n{}^{\alpha}|t|^{\alpha}\}\,dt < \int_{A<|t|}e^{-c|t|^{\alpha}}\,dt$$

which can be made small by choice of A.

If $|t|\leqq \pi/2L$, then

$$|\varphi_k(t)|^2\leqq 1-P\{0<X_k'\leqq L\}+P\{0<X_k'\leqq L\}(1-t^2/3)\,,$$

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[1] Subtract the inequality $G_k(R|t|)\sigma_k(R|t|)-(E_k(R|t|))^2\geqq 0$ from the previous inequality.

so that, setting $C = \pi/2L$,

$$|I_4| \leqq 2B_n \int_{B/M_n}^{C} \exp\{-Q_n t^2/6\}\, dt \leqq B_n/Q_n^{\frac{1}{2}} \int_{BQ_n^{\frac{1}{2}}/M_n}^{\infty} e^{-u^2/6}\, du$$
$$\leqq 6B_n M_n/BQ_n \exp\left\{-\frac{B^2 Q_n}{M_n^{\ 2}}\right\}$$

which by (a_2), for any fixed B, approaches zero as $n \to \infty$, since $B_n = O(n^{\frac{1}{2}} M_n)$.

$|I_5|$ is bounded by using the procedure of [2]: let $\{t_i\}$ be the set of points in $[C, \pi]$ of the form $2h/j$, h and j relatively prime and $2 \leqq j \leqq L$. Let $\{\Delta_i\}$ be the intervals covering $[C, \pi]$ of the form $\Delta_i = [\frac{1}{2}(t_{i-1} + t_i), \frac{1}{2}(t_i + t_{i+1})]$, $\{t_i\}$ indexed in increasing order, and $\Delta_1 = [C, \frac{1}{2}(t_i + t_2)]$, $\Delta_m = [\frac{1}{2}(t_{m-1} + t_m), \pi]$. For each Δ_i, write $u = t - t_i$ and

$$B_n \int_{\Delta_i} \psi_n(t)\, dt = \int_{|u| \leqq D/B_n} + \int_{D/B_n < |u| \leqq E/M_n} + \int_{E/M_n < |u|, u + t_i \in \Delta_i} .$$

The second and third integrals can be bounded in the same manner as I_3 and I_4. A bound on the first integral is established by use of condition (A). Details are included in [2].

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