The Annals of Probability
1974, Vol. 2, No. 1, 143–146

GROUPS OF TRANSFORMATIONS WITHOUT FINITE INVARIANT MEASURES HAVE STRONG GENERATORS OF SIZE 2[1]

BY AMY J. KUNTZ

The Ohio State University, Columbus

A size 2 generator of a measure space $(X, \mathscr{F}, p)$ under a set S of transformation of X is a partition $\{A, A^c\}$ of X such that $\mathscr{F}$ is the smallest σ-algebra containing $\{s^{-1}A : s \in S\}$ up to sets of p-measure zero. Let S be a semigroup of invertible nonsingular measurable transformations on a separable measure space $(X, \mathscr{F}, p)$ with $p(X) = 1$. Suppose that S does not preserve any finite invariant measure absolutely continuous with respect to p. Then $\mathscr{F}$ has a size 2 generator $\{A, A^c\}$ and the orbit of A under S is dense in $\mathscr{F}$.

1. Introduction. U. Krengel (1970) (see also Jones and Krengel) has shown that if T is a nonsingular invertible transformation on a finite separable measure space $(X, \mathscr{F}, p)$ such that T does not preserve any finite measure absolutely continuous with respect to p then there exist strong generators of size 2, in fact, sets with dense orbits in $\mathscr{F}$. In this paper I will extend Krengel's result to the following for groups of invertible transformations: Let G be a group of nonsingular transformations on a finite separable measure space $(X, \mathscr{F}, p)$. Assume that S, a subsemigroup of G, does not preserve any finite measure absolutely continuous with respect to p. Then $\mathscr{F}$ has a generator of size 2 whose orbit under S is dense in $\mathscr{F}$. Note that if T is as above: a nonsingular invertible transformation without a finite invariant measure then $\mathscr{F}$ has a generator of size 2 under $S = \{T^i : i \geqq 1\}$, i.e. a strong generator.

2. Definitions. By a generator under S of size 2 for a σ-algebra $\mathscr{F}$ I mean a partition of X: $\{A, A^c\}$ such that the smallest σ-algebra containing $\{s^{-1}A : s \in S\}$ is $\mathscr{F}$. A weakly wandering set under S is a set W for which there exists a sequence $(s_j)_{j=1}^{\infty}$ in S such that the sets $s_j^{-1}W$ are pairwise disjoint. $A \triangle B$ (the symmetric difference) is $(A \cap B^c) \cup (A^c \cap B)$. In this paper S will always be a semigroup of invertible nonsingular transformations on $(X, \mathscr{F}, p)$, a finite measure space with $p(X) = 1$.

3. Weakly wandering sets. Y. N. Dowker (1955) showed that for an invertible nonsingular transformation T a necessary and sufficient condition that there exist a finite T-invariant measure m equivalent to p is that for every measurable

Received March 27, 1973; revised May 10, 1973.

[1] The results in this paper are part of the author's doctoral dissertation for the Department of Mathematics, Ohio State University, Columbus. It was partially supported by National Science Foundation Grant GP 14594.

AMS 1970 *subject classifications.* Primary 28A65; Secondary 20M20.

Key words and phrases. Size-2 generator, weakly wandering sets, no finite invariant measure.

set A such that $p(A) > 0$ we have $\liminf p(T^n A) > 0$ as $n \to \infty$. Hajian and Kakutaini (1964) proved that the existence of a finite invariant measure is equivalent to the nonexistence of weakly wandering sets. Hajian and Itô (1969) extended these previous results to groups G showing that the following are equivalent:

(i) $\inf\{p(gA) : g \in G\} = 0$,
(ii) G has no finite invariant measure equivalent to p,
(iii) there exists a weakly wandering set under G.

In Lemma 1 I shall use similar methods to show:

Lemma 1. *If S is a semigroup of nonsingular invertible transformations on a finite measure space X, $p(X) = 1$, then S has no finite invariant measure absolutely continuous with respect to p if and only if for all $e > 0$ there exists a weakly wandering set W such that $p(W) > 1 - e$.*

Proof. If there exist weakly wandering sets with measure arbitrarily close to 1 then any set of positive measure contains a weakly wandering set hence no set of positive measure can be the support of a finite invariant measure for S. Now suppose that S has no finite invariant measure absolutely continuous with respect to p. Following Hajian and Itô and Dowker I define $L_2(X)$ operators U_s such that $U_s(r(x)) = r(s^{-1}x)w_s^{\frac{1}{2}}$ where w_s is the Radon–Nikodym derivative of ps^{-1} with respect to p. U_s is a unitary operator and $U_s U_t = U_{st}$. Let $T = \{U_s 1 : s \in S,\ 1(x) = 1 \text{ for all } x \in X\}$ and let T^* be the closed convex hull of T in $L_2(X)$. $L_2(X)$ is a uniformly convex Banach space. So there exists a unique element t_0 in T^* such that $||t_0|| = \inf\{||t|| : t \in T^*\}$ (Wilansky (1964) page 110). Since we have $U_s T^* \subset T^*$ for all $s \in S$ we have that $U_s t_0 = t_0$. Let $m(E) = \int_E t_0^2\, dp$. Then

$$m(s^{-1}E) = \int_{s^{-1}E} t_0^2\, dp = \int_E t_0^2(s^{-1}x) w_s\, dp = \int_E (U_s t_0)^2\, dp = \int_E t_0^2\, dp = m(E)$$

so that m is a finite S-invariant measure absolutely continuous with respect to p. Since t_0 is a strong limit of convex combinations of $U_s 1$, $\int_E t_0\, dp \geqq \inf\{\int_E U_s 1\, dp : s \in S\}$. By the Cauchy–Schwartz inequality we have:

$$m(E) = \int_E t_0^2\, dp \geqq \left(\int_E t_0\, dp\right)^2 / p(E) \geqq \left(\inf_{s \in S} \int_E U_s 1\, dp\right)^2 / p(E)\,.$$

Since m is a finite S-invariant measure m must be identically zero. So $0 = m(X) \geqq \inf_{s \in S} \int_X U_s 1\, dp$ and there exists a sequence $U_{s_i} 1$ which converges to 0 pointwise a.e. Egorov's theorem implies that for all $e > 0$ we can find a set X' such that $p(X') > 1 - e$ and $U_{s_i} 1$ converges to 0 uniformly for x in X'. Then $p(s_i^{-1}X') = \int_{X'} (U_{s_i} 1)^2\, dp$ converges to 0 so $\inf\{p(s^{-1}X') : s \in S\} = 0$. Let $W = X' - (\bigcup_{i=1}^{\infty} s_i^{-1}X' \cup \bigcup_{i=2}^{\infty} \bigcup_{j<i} s_j s_i^{-1})$, s_i chosen so that $p(s_i^{-1} X') < d$ where d, $d < e/2^{i+1}$, is chosen so that if $p(A) < d$ then $p(s_j A) < e/i2^{i+1}$ for $j = 1, 2, \cdots, i - 1$. Then W is a weakly wandering set under s_i and $p(W) > p(X') - e > 1 - 2e$. (cf. Hajian and Kakutani (1964).)

LEMMA 2. *If for all* $e > 0$ *there exists a weakly wandering set* W *under* S *with* $p(W) > 1 - e$ *then for any decreasing sequence of positive numbers* e_k *there exist weakly wandering sets* W_k *and transformations* s_k *in* S *such that*:

(i) $p(W_k) > 1 - e_k$
(ii) $p(s_i^{-1}s_k W_k) < e_k/2^k$ *for* $i < k$
(iii) $p(s_k^{-1}(\bigcup_{i<k} s_i W_i)) < e_k$

PROOF. Suppose that W_i^{n-1}, s_i, $1 \leqq i \leqq n-1$, have been chosen so that:

(i) $p(W_i^{n-1}) > 1 - e_i(1 - 1/2^{n-i})$, $1 \leqq i \leqq n-1$
(ii) $p(s_j^{-1}s_i W_i^{n-1}) < e_i/2^i$, $j < i \leqq n-1$
(iii) $p(s_j^{-1}(\bigcup_{i<j} s_i W_i^{n-1})) < e_j$, $1 \leqq j \leqq n-1$.

Choose W_n^n so that $p(W_n^n) > 1 - d$ where $d < e_n/2$ and $p(A) < d$ implies $p(s_i^{-1}A) < e_n/2^n < e_i/2^{n-i+1}$ for $i < n$. Choose s_n so that $s_n W_n^n$ is disjoint from W_n^n and $p(s_n^{-1}W_n^n) < d$. Then we have for $W_i^n = W_i^{n-1} \cap s_i^{-1}W_n^n$:

(i) $p(W_i^n) > 1 - e_i(1 - 1/2^{n-i+1})$, $1 \leqq i \leqq n$ (since $p(s_i^{-1}W_n^n) > 1 - e_i/2^{n-i+1}$)
(ii) $p(s_j^{-1}s_i W_i^n) < e_i/2^i$, $j < i \leqq n$ (since $p(s_n W_n^n) < d$)
(iii) $p(s_n^{-1}(\bigcup_{i<n} s_i W_i^n)) < e_n$, (since $s_i W_i^n \subset W_n^n$).

Let $W_i = \bigcap_{n \geqq i} W_i^n$ then $p(W_i) > 1 - e_i$ and W_i satisfies 2 and 3 since $W_i \subset W_i^n$, $n \geqq i$.

4. Generators.

THEOREM. *If* S *is a semigroup of invertible nonsingular transformations on a finite separable measure space* $(X, \mathscr{F}, p)$, $p(X) = 1$, *where* S *does not preserve any finite invariant measure absolutely continuous with respect to* p *then* $\mathscr{F}$ *has a generator of size* 2 *in* $\mathscr{F}$ *under* S *and the orbit of that generator under* S *is dense in* $\mathscr{F}$.

PROOF. Without loss of generality assume that $(F_i)_{i=1}^\infty$ is a dense generating set for $\mathscr{F}$ (i.e. $(F_i)_{i=1}^\infty$ is dense in $\mathscr{F}$ and $\mathscr{F}$ is the smallest σ-algebra which contains $(F_i)_{i=1}^\infty$) and that every F_i appears infinitely often in the sequence $(F_i)_{i=1}^\infty$.

Let $A = \bigcup_{k=1}^\infty s_k(F_k \cap W_k)$ where W_k, s_k satisfy Lemma 2 for a sequence e_k which decreases to 0. Then $(s_k^{-1}A)_{k=1}^\infty$ is dense in $(F_i)_{i=1}^\infty$ since

$$p(s_i^{-1}A \triangle F_i) \leqq p(F_i - F_i \cap W_i) + p(\textstyle\bigcup_{k<i} s_i^{-1}s_k W_k) + \sum_{\{k:i>k\}} p(s_i^{-1}s_k W_k) < 3e_i .$$

So $\{A, A^c\}$ is a generator for $\mathscr{F}$ under S and the orbit of A under S is dense in $\mathscr{F}$. As in the case for a single transformation these generators are a dense G_δ in the symmetric difference topology on $\mathscr{F}$ (cf. Krengel (1970)).

Acknowledgment. I wish to thank my advisors, Professors U. Krengel and W. Krieger, for their guidance and encouragement in this research.

REFERENCES

[1] DOWKER, Y. N. (1955). On measurable transformations in finite measure spaces. *Ann. Math.* 2nd series, **62** 504–516.

[2] HAJIAN, A. B. and ITÔ, Y. (1969). Weakly wandering sets and invariant measures for a group of transformations. *J. Math. Mech.* **18** 1203–1216.

[3] HAJIAN, A. B. and KAKUTANI, S. (1964). Weakly wandering sets and invariant measures. *Trans. Amer. Math. Soc.* **110** 136–151.

[4] JONES, L. K. and KRENGEL, U. (1973). On transformations without finite invariant measure. To appear.

[5] KRENGEL, U. (1970). Transformations without finite invariant measure have finite strong generators. First Midwestern Conference, Ergodic Theory and Probability. Springer Lecture Notes 160, 133–157.

[6] WILANSKY, A. (1964). *Functional Analysis.* Blaisdell, New York.

DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210