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THE ROSENBLATT MIXING CONDITION AND BERNOULLI SHIFTS[1]

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If T is an automorphism on a Lebesgue space and P a finite generator for T, then T is a Bernoulli shift if

$$\sup\{|\mu(A \cap B) - \mu(A)\mu(B)| : A \in \bigvee_{-\infty}^{-1} T^j P,\ B \in \bigvee_k^{\infty} T^j P\}$$

is $o(|P|^{-a_k})$ where $a_k/k \to \infty$ as $k \to \infty$.

1. Introduction. Rosenblatt [7] introduced a concept of weak dependence for stochastic processes. A stationary stochastic process always gives rise to a measure preserving transformation on the path space of the process and, as such, the weak dependence condition of Rosenblatt's can be phrased in terms of measure preserving transformations on a probability space and partitions of this space as is done by Smorodinsky in [9]. In this form the condition can be compared with the ε-independence of Ornstein [3], [4], [1], [2]. It is well known that a stationary (strong sense) process which satisfies the Rosenblatt condition is regular, hence the induced measure preserving transformation on path space is a K-automorphism. Thus a measure preserving transformation with a generating partition satisfying a Rosenblatt condition is a K-automorphism. In this paper we show that if the Rosenblatt gage of dependence converges to zero rapidly enough then the transformation is (i.e. isomorphic to) a Bernoulli shift. The main tool used is the proof of Proposition 9 of [4] which actually proves something stronger than the statement of this proposition.

The notation, terminology, and definitions which we use are found in Ornstein [4] and we will give these as needed. In particular, if $P_j = \{p_j^1, \cdots, p_j^k\}$ is a collection of ordered partitions the symbol $\bigvee_{j=1}^n P_j$ denotes the ordered partition $\{p_1^{i_1} \cap p_2^{i_2} \cap \cdots \cap p_n^{i_n} : 1 \leqq i_j \leqq k\}$ where the order is the lexicographic order. We shall denote the entropy of an automorphism T given a partition P by $h(T, P)$ rather than by $E(P, T)$ which is the notation of Ornstein. We shall have occasion to distinguish between a partition and the σ-algebra generated by the partition. If P is a partition then $P^\wedge$ will denote the σ-algebra generated by the atoms (members) of P. If T is an automorphism of Ω and P a partition then $\bigvee_n^\infty T^i P$ will denote the σ-algebra generated by the partitions $\bigvee_n^l T^i P$ for all $l \geqq n$ and (T, P) will denote the dynamical system $(\Omega, \bigvee_{-\infty}^{\infty} T^i P, T)$, i.e., T restricted to the σ-algebra generated by $\bigvee_{-n}^{\infty} T^i P$ for $n \geqq 1$.

All spaces are Lebesgue spaces and all transformations are automorphisms.

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2. The main result.

2.1. DEFINITION. Sub σ-algebra $\mathscr{A}$ and $\mathscr{B}$ of a probability space $(\Omega, \mathscr{C}, \mu)$ are symmetrically ε-independent if

$$\sup\{|\mu(a \cap b) - \mu(a)\mu(b)| : a \in \mathscr{A}, b \in \mathscr{B}\} < \varepsilon .$$

Partitions P and Q of a space $(\Omega, \mathscr{C}, \mu)$ are symmetrically ε-independent if

$$\sum_{p \in P} \sum_{q \in Q} |\mu(p \cap q) - \mu(p)\mu(q)| < \varepsilon .$$

2.2. LEMMA. *Let P and Q be finite measurable partitions of a probability space.*

(a) *If P and Q are symmetrically ε-independent then $P^\frown$ and $Q^\frown$ are also.*

(b) *If $P^\frown$ and $Q^\frown$ are symmetrically ε-independent then P and Q are symmetrically $2k\varepsilon$-independent where $k = \min\{|P|, |Q|\}$ and $|P|$ denotes the number of atoms in P.*

Note that 2.2(b) is best possible. For if P and Q are partitions of the unit square given by

$$p^1 = \{(x, y) : 0 \leqq x \leqq 1, 0 \leqq y < \tfrac{1}{2}\}$$
$$p^2 = \{(x, y) : 0 \leqq x \leqq 1, \tfrac{1}{2} \leqq y \leqq 1\}$$

and

$$q^1 = \{(x, y) : \tfrac{1}{2} \leqq x \leqq 1, 0 \leqq y < x - \tfrac{1}{2}\}$$
$$q^2 = \{(x, y) : 0 \leqq x < \tfrac{1}{2}, 0 \leqq y \leqq x\}\{(x, y) : \tfrac{1}{2} \leqq x \leqq 1, x - \tfrac{1}{2} \leqq y \leqq x\}$$
$$q^3 = \{(x, y) : 0 \leqq x \leqq 1, x < y \leqq 1\}$$

then $P^\frown$ and $Q^\frown$ are symmetrically $\frac{1}{8}$-independent while P and Q are symmetrically $\frac{1}{2}$-independent.

In the following definition we need some more of Ornstein's notation. If $P = \{p^1, \cdots, p^k\}$ and $Q = \{q^1, \cdots, q^k\}$ are ordered partitions then $d(P, Q)$ is defined by

$$d(P, Q) = \textstyle\sum_1^k |\mu(p^i) - \mu(q^i)| .$$

If q is a measurable set with positive measure, P/q denotes the ordered partition $\{p^1 \cap q, \cdots, p^k \cap q\}$ of the Lebesgue space $(q, q \cap \mathscr{C}, \mu_q)$ where $\mu_q(c) = \mu(c \cap q)/\mu(q)$. Thus

$$d(P/q, Q) = \textstyle\sum_1^k |\mu(p^i \cap q)/\mu(q) - \mu(q^i)| .$$

2.3. DEFINITION. If P and Q are finite partitions then P is ε-independent of Q if there is a collection Q_1 of atoms of Q such that the sum of the measures of the atoms $q \in Q_1$ is greater than $1 - \varepsilon$ and $d(P/q, P) < \varepsilon$.

2.4. LEMMA. (a) *If P and Q are symmetrically ε^2-independent then P is ε-independent of Q.*

(b) *If P is ε-independent of Q then P and Q are symmetrically 3ε-independent.*

2.5. DEFINITION. Let T be an invertible measure preserving transformation on a Lebesgue space and P a finite ordered partition of the space. For each integer $k > 0$, define

$$\alpha(k) = \sup\{|\mu(A \cap B) - \mu(A)\mu(B)| : A \in \bigvee_{-\infty}^{-1} T^iP,\ B \in \bigvee_{k}^{\infty} T^iP\}.$$

The partition P satisfies the Rosenblatt condition for T, or (T, P) has the Rosenblatt property if $\alpha(k) \to 0$ as $k \to \infty$.

2.6. PROPOSITION. *If (T, P) has the Rosenblatt property then for every $\varepsilon > 0$ and integer N there is an integer $K = K(N, \varepsilon)$ such that*

$$\bigvee_{-N}^{-1} T^iP \quad \textit{is } \varepsilon\textit{-independent of} \quad \bigvee_{K}^{K+l} T^iP$$

for all $l > 0$. Hence (T, P) is a K-automorphism.

PROOF. Let ε and N be given. Since $\alpha(K) \to 0$, there is $K = K(N, \varepsilon)$ such that $\alpha(K) < \varepsilon^2/2|P|^N$ and hence

$$\bigvee_{-\infty}^{-1} T^iP \quad \text{and} \quad \bigvee_{K}^{\infty} T^iP \quad \text{are symmetrically} \quad \varepsilon^2/2|P|^N\text{-independent.}$$

Thus $(\bigvee_{-N}^{-1} T^iP)^\frown$ and $(\bigvee_{K}^{K+l} T^iP)^\frown$ are symmetrically $\varepsilon^2/2|P|^N$-independent for each l. Since $|\bigvee_{-N}^{-1} T^iP| \leqq |P|^N$, Lemma 2.2(b) implies that $\bigvee_{-N}^{-1} T^iP$ and $\bigvee_{K}^{K+l} T^iP$ are symmetrically ε^2-independent for each l and Lemma 2.4(a) implies that $\bigvee_{-N}^{-1} T^iP$ is ε-independent of $\bigvee_{K}^{K+l} T^iP$ for each l.

2.7. LEMMA. *Let a_k be a sequence of positive integers and (T, P) a dynamical system, where P is a finite partition. If $\alpha(k) = o(|P|^{-a_k})$ then for every $\varepsilon > 0$, there is a $K = K(\varepsilon)$ such that if $k \geqq K$ then*

$$\bigvee_{k}^{k+a_k} T^iP \quad \textit{is } \varepsilon\textit{-independent of} \quad \bigvee_{-l}^{-1} T^iP$$

for each $l > 0$.

PROOF. Let $\varepsilon > 0$ be given. There is $K = K(\varepsilon)$ such that for all $k \geqq K$, $\alpha(k) < \varepsilon^2/2|P|^{a_k}$ so that for any $l > 0$, $k \geqq K$,

$$(\bigvee_{k}^{k+a_k} T^iP)^\frown \quad \text{and} \quad (\bigvee_{-l}^{-1} T^iP)^\frown \quad \text{are symmetrically} \quad \varepsilon^2/2|P|^{a_k}\text{-independent.}$$

Since $|\bigvee_{k}^{k+a_k} T^iP| < |P|^{a_k}$, Lemma 2.2(b) implies that $\bigvee_{k}^{k+a_k} T^iP$ and $\bigvee_{-l}^{-1} T^iP$ are symmetrically ε^2-independent and Lemma 2.4(a) implies that $\bigvee_{k}^{k+a_k} T^iP$ is ε-independent of $\bigvee_{-l}^{-1} T^iP$ for each $l > 0$.

For the next lemma we shall need some more terminology of Ornstein's. If $P = \{p^1, \cdots, p^k\}$ is a finite ordered partition, $d(P)$ denotes the distribution of P and is defined to be k-tuple $(\mu(p^1), \cdots, \mu(p^k))$, so that $d(P, Q) = \|d(P) - d(Q)\|_1$, where $\|\ \|_1$ is the l^1-norm. The metric D is defined by $D(P, Q) = \sum_1^k \mu(p^i \triangle q^i)$ where $\triangle$ denotes symmetric difference. If $\{P_i\}_{i=1}^n$ is a collection of ordered partitions, $P_i = \{p_i^1, \cdots, p_i^k\}$ and $\{Q_i\}_1^n$ is another such collection, $\bar{d}$ is defined by

$$\bar{d}(\{P_i\}_1^n, \{Q_i\}_1^n) = \sup\left\{\frac{1}{n}\sum_1^n D(\bar{P}_i, \bar{Q}_i)\right\}$$

where the supremum is taken over all collections $\{\bar{P}_i\}_1^n$ and $\{\bar{Q}_i\}_1^n$ such that $d(\bigvee_1^n \bar{P}_i) = d(\bigvee_1^n P_i)$ and $d(\bigvee_1^n \bar{Q}_i) = d(\bigvee_1^n Q_i)$.

A collection $\{P_i\}_1^n$ of finite partitions is ε-independent of a partition Q if there is a collection Q_1 of atoms of Q such that the sum of the measures of $q \in Q_1$ is greater than $1 - \varepsilon$ and $\bar{d}(\{P_i/q\}_1^n, \{P_i\}_1^n) < \varepsilon$.

2.8. LEMMA. *Let a_k be a sequence of positive integers such that $a_k/k \to \infty$ as $k \to \infty$ and (T, P) a dynamical system where P is a finite partition. If $\alpha(k) = o(|P|^{-a_k})$ then for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ such that*

$$\{T^iP\}_0^N \quad \textit{is } \varepsilon\textit{-independent of} \quad \bigvee_{-l}^{-1} T^iP$$

for each $l > 0$.

PROOF. From Lemma 2.7 there is a $K_0 = K_0(\varepsilon)$ such that if $k \geqq K_0$,

$$\bigvee_k^{k+a_k} T^iP \quad \text{is } \varepsilon/2\text{-independent of} \quad \bigvee_{-l}^{-1} T^iP$$

for each $l > 0$, and from Lemma 2 of [4],

$$\{T^iP\}_k^{k+a_k} \quad \text{is } \varepsilon/2\text{-independent of} \quad \bigvee_{-l}^{-1} T^iP .$$

Since $a_k/k \to \infty$, there is $K_1 = K_1(\varepsilon)$ such that if $k \geqq K_1$, $k/(k + a_k) < \varepsilon/2$. Let $K = \max\{K_0, K_1\}$. Since $K \geqq K_0$,

$$\{T^iP\}_K^{K+a_K} \quad \text{is } \varepsilon/2\text{-independent of} \quad \bigvee_{-l}^{-1} T^iP$$

and there exist partitions P_j' and $\bar{P}_j$, $j = K, K+1, \cdots, K + a_K$ such that

$$d(\bigvee_K^{K+a_K} P_j') = d(\bigvee_K^{K+a_K} T^iP)$$
$$d(\bigvee_K^{K+a_K} \bar{P}_j) = d(\bigvee_K^{K+a_K} T^iP/q)$$

for all $q \in Q_1$, where Q_1 is a collection of atoms of $\bigvee_{-l}^{-1} T^iP$ whose union has measure greater than $1 - \varepsilon/2$ and

$$\frac{1}{a_K} \sum_K^{K+a_K} D(P_j', \bar{P}_j) < \varepsilon/2 .$$

Select partitions P_j', $\bar{P}_j$, $j = 0, 1, \cdots, K-1$, such that

$$d(\bigvee_0^{K+a_K} P_j') = d(\bigvee_0^{K+a_K} T^iP)$$
$$d(\bigvee_0^{K+a_K} \bar{P}_j) = d(\bigvee_0^{K+a_K} T^iP/q) , \qquad q \in Q_1 .$$

Then

$$\frac{1}{K + a_K} \sum_0^{K+a_K} D(P_j', \bar{P}_j) < \frac{1}{K + a_K} \sum_0^{K-1} D(P_j', \bar{P}_j) + \varepsilon/2$$
$$< \frac{K}{K + a_K} + \varepsilon/2$$

since $D(P_j', \bar{P}_j) < 1$. By the choice of K, $K/K + a_K < \varepsilon/2$ and taking $N = K + a_K$ we have

$$\{T^iP\}_0^N \quad \varepsilon\text{-independent of} \quad \bigvee_{-l}^{-1} T^iP .$$

For the conclusion of the next lemma we need the following definition of Ornstein from [4]. A partition P is *finitely determined* for T if given $\varepsilon > 0$ there is a $\eta = \eta(\varepsilon)$ and $n_1 = n_1(\varepsilon)$ such that if $\bar{T}$ is any mixing transformation and $\bar{P}$ a partition such that

(i) $d(\bigvee_0^{n_1} T^i P, \bigvee_0^{n_1} \bar{T}^i \bar{P}) < \eta$, and
(ii) $|h(T, P) - h(\bar{T}, \bar{P})| < \eta$

then $\bar{d}(\{T^i P\}_0^n, \{\bar{T}^i \bar{P}\}_0^n) < \varepsilon$ for all n.

2.9. LEMMA. *Let (T, P) be a dynamical system where P is a finite partition such that for every $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that*

$$\{T^i P\}_0^N \quad \text{is } \varepsilon\text{-independent of} \quad \bigvee_{-l}^{-1} T^i P$$

for all $l > 0$. Then P is finitely determined for T.

A careful analysis of the proof of Proposition 9 of [4] will show that this proof only requires the hypothesis of Lemma 2.9 rather than the stronger hypothesis of very weak Bernoulli. Thus the proof of Lemma 2.9 is given by Ornstein in [4].

2.10. THEOREM. *Let $\{a_k\}$ be a sequence of positive integers such that $a_k/k \to \infty$ as $k \to \infty$ and (T, P) a dynamical system where P is a finite partition. If $\alpha(k) = o(|P|^{-a_k})$ then (T, P) is isomorphic to a Bernoulli shift.*

PROOF. Lemmas 2.8 and 2.9 imply that P is finitely determined for T. Let S be a Bernoulli shift such that $h(T) = h(S)$. Since both T and S have generating partitions which are finitely determined, Ornstein's results, [4], give that they are isomorphic.

2.11. THEOREM. *There exists a Bernoulli automorphism (T, P) such that P does not satisfy a Rosenblatt condition.*

PROOF. The example given in Smorodinsky [9] of a two element partition P of a regular stationary Gaussian process is such that (T, P) is isomorphic to a Bernoulli shift but P does not satisfy the Rosenblatt condition.

Notice that the above example also shows that a partition can be finitely determined for T without satisfying a Rosenblatt condition, and that every partition of a K-automorphism does not have to satisfy a Rosenblatt condition.

2.12. THEOREM. *There exist an automorphism (T, P) such that T is Bernoulli, P satisfies the Rosenblatt condition, but given any sequence $\{a_k\}$ such that $a_k/k \to \infty$,*

$$|P|^{a_k} \alpha(k) \to \infty \qquad \text{as } k \to \infty .$$

PROOF. Let (T, P) be the two sided Markov shift on $\{0, 1\}$ with transition matrix

$$M = \begin{pmatrix} \frac{1}{2} + \beta & \frac{1}{2} - \beta \\ \beta & 1 - \beta \end{pmatrix}$$

where $0 < \beta < \frac{1}{2}$. The unique invariant initial distribution is given by $p_0 = 2\beta$, $p_1 = 1 - 2\beta$. The partition P is the zero time partition, i.e.,

$$P = (\{\omega_0 = 0\}, \{\omega_0 = 1\}) .$$

Since P is weak Bernoulli, (T, P) is isomorphic to a Bernoulli shift. Moreover

$\alpha(k) \leqq c\delta^k$ where c and δ are constants with $0 < \delta < 1$, so that P satisfies the Rosenblatt condition.

It can be shown, (see for example, Genedenko's *Theory of Probability*, page 148) that

$$M^n(0, 0) = p_0 + p_1 2^{-n},$$

and from this it follows that

$$\alpha(k) \geqq p_0 p_1 2^{-k}.$$

REFERENCES

[1] ORNSTEIN, D. (1970a). Bernoulli shifts with the same entropy are isomorphic. *Advances in Math.* **4** 337–352.

[2] ORNSTEIN, D. (1970b). Two Bernoulli shifts with infinite entropy are isomorphic. *Advances in Math.* **5** 339–348.

[3] ORNSTEIN, D. (1970c). Factors of Bernoulli shifts are Bernoulli shifts. *Advances in Math.* **5** 349–364.

[4] ORNSTEIN, D. (1970d). Imbedding Bernoulli shifts in flows. Contributions to Ergodic Theory and Probability. *Lecture Notes in Mathematics* **160** 178–218.

[5] FRIEDMAN, N. and ORNSTEIN, D. (1970). On isomorphisms of weak Bernoulli transformations. *Advances in Math.* **5** 365–394.

[6] FRIEDMAN, N. and ORNSTEIN, D. (1970). Entropy and the isomorphism problem. Mimeo. lecture notes.

[7] ROSENBLATT, M. (1961). Independence and dependence. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **4** 431–443.

[8] SMORODINSKY, M. (1971a). Ergodic theory, entropy. *Lecture Notes in Mathematics* **214**.

[9] SMORODINSKY, M. (1971b). A partition on a Bernoulli shift which is not weak Bernoulli. *Math. Systems Theory* **5** 201–203.

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