*The Annals of Probability*
1974, Vol. 2, No. 5, 954-957

# A NOTE ON STATIONARY GAUSSIAN SEQUENCES

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Let $\{\xi_n\}$ be a stationary Gaussian sequence with $E(\xi_0) = 0$; $E(\xi_0 \xi_n) = r_n$. If $n^\alpha r_n \to 0$ for some $\alpha > 1$, then Strassen's functional law of iterated logarithm applies to $\{\xi_n\}$.

**1. Main result.** Let $\{\xi_n : -\infty < n < \infty\}$ be a stationary Gaussian sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. Let $E\xi_0 = 0$ and $S_n = \sum_{j=1}^n \xi_j$. For $n > 3$, $\omega \in \Omega$, let $g_n(\cdot, \omega)$ be the function on $[0, 1]$ defined by

$$g_n(j/n, \omega) = (2n \log \log n)^{-\frac{1}{2}} S_j \,, \qquad j = 0, 1, \cdots, n \tag{1}$$

and $g_n(\cdot, \omega)$ is linear on the subintervals $[(j-1)/n, j/n]$, $j = 1, 2, \cdots, n$. For a nonnegative number $\sigma$ let $K_\sigma$ denote the set of absolutely continuous functions on $[0, 1]$ which vanish at zero and whose derivatives have $L_2$-norms less than or equal to $\sigma$. We say that $\{\xi_n\}$ satisfies Strassen's law of iterated logarithm if there exists a $\sigma \geqq 0$, and a set $\Omega_0$ with $P(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$ the sequence $\{g_n(\cdot, \omega)\}$ is precompact in the metric space $C[0, 1]$ and has $K_\sigma$ as the set of its limit points.

For $n \geqq 1$, let $r_n = (E\xi_0^2)^{-1} E(\xi_0 \xi_n)$. The object of this note is to give a simple sufficient condition in terms of the correlation sequence $\{r_n\}$, for $\{\xi_n\}$ to obey Strassen's law.

The case of strong-mixing, stationary Gaussian sequences was considered in Deo (1973). For a positive integer $n$, let $\alpha_n$ denote the sup $|P(AB) - P(A)P(B)|$ where the supremum is taken over all sets $A$, $B$ such that $A$ is in the $\sigma$-field generated by $\{\xi_j : j \leqq 0\}$ and $B$ in the $\sigma$-field generated by $\{\xi_j : j \geqq n\}$. It is shown in [2] that if

$$\sum \alpha_n{}^p < \infty \qquad \text{for some } \ 0 < p < 2 \,, \tag{2}$$

then Strassen's law applies to $\{\xi_n\}$. Furthermore, the condition (2) is satisfied whenever $\{\xi_n\}$ has a strictly positive spectral density which satisfies a Hölder condition of order greater than $\frac{1}{2}$.

In this note it is shown that another sufficient condition for Strassen's law to apply to $\{\xi_n\}$ is

$$\lim_{n\to\infty} n^\alpha r_n = 0 \qquad \text{for some } \ \alpha > 1 \,. \tag{3}$$

Also we give examples to show that conditions (2) and (3) overlap but neither of them implies the other. It is also shown that (3) cannot be weakened to $\alpha > 0$ (at least not with the norming used in the definition of $g_n$'s and with

Received September 26, 1973; revised January 15, 1974.
*AMS* 1970 *subject classifications.* Primary 60F15, 60G15; Secondary 60G10.
*Key words and phrases.* Gaussian sequences, Strassen's law of iterated logarithm.

Strassen's set $K_\sigma$ as the limit set). The main proposition here is derived as a consequence of a very general recent theorem of I. Berkes (1973).

Under (3), the series $\sum_{n=1}^{\infty} r_n$ converges absolutely. Let

$$\sigma^2 = E(\xi_0^2)\{1 + 2\sum_{n=1}^{\infty} r_n\}\,. \tag{4}$$

PROPOSITION. *Let* (3) *hold. Then the sequence* $\{\xi_n\}$ *obeys Strassen's law with* $\sigma$ *defined in* (4).

PROOF. Assume first $\sigma > 0$; with this assumption there is no loss of generality in taking $\sigma = 1$. Let $\delta_n = \sup_{j \geqq n} |r_j|$. Then (3) is equivalent to

$$\lim_{n\to\infty} n^\alpha \delta_n = 0 \qquad \text{for some } \alpha > 1\,. \tag{5}$$

Clearly we can assume $1 < \alpha < 2$. We first show that there exists a constant $C$ (depending only on the sequence $\{\xi_n\}$) such that for all positive integers $m$, $n$, $p$ we have

$$|E\{(mn)^{-\frac{1}{2}} \sum_{i=1}^{m} \xi_i \sum_{j=m+p}^{m+p+n-1} \xi_j\}| \leqq C[\min(m, n)]^{1-\alpha} \tag{6}$$

and

$$|E\{n^{-\frac{1}{2}} \sum_{i=1}^{n} \xi_i\}^2 - 1| \leqq Cn^{1-\alpha}\,. \tag{7}$$

To prove (6) assume for the sake of definiteness $m \leqq n$. The left side of (6) is $\leqq$

$$\begin{aligned}(mn)^{-\frac{1}{2}}E(\xi_0^2) \sum_{i=1}^{m} \sum_{j=m_1+p}^{m_1+p+n-1} |r_{j-i}| &\leqq E(\xi_0^2)m^{-1}\{\sum_{j=1}^{m} j\delta_j + m\sum_{j=m+1}^{\infty} \delta_j\}\\ &\leqq E(\xi_0^2)m^{-1}\{C_1 \sum_{j=1}^{m} j^{1-\alpha} + C_2\, mm^{1-\alpha}\}\\ &\leqq E(\xi_0^2)m^{-1}\{C_3 m^{2-\alpha} + C_2 m^{2-\alpha}\}\\ &\leqq C_4 m^{1-\alpha}\,.\end{aligned}$$

Here, $C_1$, $C_2$, $C_3$, $C_4$, are positive constants which do not depend on $m$, $n$, $p$. Similarly, the left side of (7) is equal to

$$\begin{aligned}&|E(\xi_0^2) + 2E(\xi_0^2)\sum_{j=1}^{n-1}(1 - j/n)r_j - 1|\\ &\qquad = |E(\xi_0^2) + 2E(\xi_0^2)\sum_{n=1}^{j-1}(1 - j/n)r_j - \sigma^2|\\ &\qquad \leqq 2E(\xi_0^2)\left\{\frac{1}{n}\sum_{j=1}^{n-1} j\delta_j + \sum_{j=n}^{\infty}\delta_j\right\} \leqq C_5 n^{1-\alpha}\,.\end{aligned}$$

Thus (6) and (7) hold with $C = \max(C_4, C_5)$.

Now to prove the proposition let us apply Theorem 2 of Berkes (1973); and toward this end verify conditions $A$ and $B_3$ in that paper. Verification of condition $A$ is straightforward in view of the easily-checked fact that for all choices of positive integer $n$ and numbers $a_1, a_2, \cdots, a_n$ the random variable $(\sum a_i^2)^{-\frac{1}{2}} \sum_{i=1}^{n} a_i \xi_i$ has normal distribution with mean zero and variance at most equal to $\{E(\xi_0^2) + DE(\xi_0^2)\sum_{k=1}^{\infty} k^{-\alpha}\}$ where $D = \sup_k k^\alpha \delta_k$.

It remains to verify the condition $B_3$ of [1]. In the notation there the left side of (2.6) is equal to

$$|\exp(-\tfrac{1}{2}\sum_{i=1}^{r} \lambda_i^2 \rho(i, i) - \tfrac{1}{2}\sum_{i\neq j} \lambda_i \lambda_j \rho(i, j)) - \exp(-\tfrac{1}{2}\sum_1^r \lambda_i^2)| \tag{8}$$

where we have written, for $1 \leqq i \leqq r$ and $1 \leqq j \leqq r$;

$$\rho(i,j) = E\{n[(t_i' - t_i)(t_j' - t_j)]^{-\frac{1}{2}} \sum_{k=[nt_i]+1}^{[nt_i']} \xi_k \sum_{l=[nt_j]+1}^{[nt_j']} \xi_l\} .$$

By (6) and (7),

$$|\rho(i,j)| \leqq C(nt)^{1-\alpha} \qquad 1 \leqq i \neq j \leqq r , \tag{9}$$

and

$$|\rho(i,i) - 1| \leqq C(nt)^{1-\alpha} \qquad 1 \leqq i \leqq r \tag{10}$$

where $t = \min_{1 \leqq i \leqq r} (t_i' - t_i)$.

Thus, the expression in (8) is dominated by

$$|\exp(-\tfrac{1}{2} \textstyle\sum_1^r \lambda_i^2(\rho(i,i) - 1) - \tfrac{1}{2} \sum_{i \neq j} \lambda_i \lambda_j \rho(i,j)) - 1| . \tag{11}$$

For any real $u$, $|e^{-u} - 1| \leqq |u|e^{|u|}$. Using this and (9) and (10) we see that the expression (11) is at most $Le^L$ where $L$ stands for

$$\tfrac{1}{2}C(nt)^{1-\alpha} \textstyle\sum_1^r \lambda_i^2 + \tfrac{1}{2}C(nt)^{1-\alpha} \sum_{i \neq j} |\lambda_i||\lambda_j| .$$

Write $||\lambda||^2 = \sum_1^r \lambda_i^2$. Then $L$ itself is dominated by

$$C(nt)^{1-\alpha}||\lambda||^2(1 + r^2) .$$

Thus we have shown that the left side of (2.6) in [1] is at most

$$C(nt)^{1-\alpha}(1 + r^2)||\lambda||^2 \exp\{C(nt)^{1-\alpha}(1 + r^2)||\lambda||^2\} . \tag{12}$$

For $||\lambda||^2 \leqq C^{-1}(1 + r^2)^{-1}(nt)^{(\alpha-1)/2}$, and $nt > 1$, (12) is at most $(nt)^{(\alpha-1)/2}$. Thus the left side of (2.6) in [1] is at most $\max(2, (nt)^{(1-\alpha)/2})$ whenever $||\lambda||^2 \leqq C^{-1}(1 + r^2)^{-1}(nt)^{(\alpha-1)/2}$. This verifies the condition $B_3$ of [1] and the proof of the proposition is complete when $\sigma > 0$. The degenerate case $\sigma = 0$ for which we need to show that $g_n$ converges uniformly to zero for almost all $\omega$ can be handled along the lines of Lemma 9 in [2]. We omit the details.

## 2. Examples.

EXAMPLE 1. Let $\{\xi_n\}$ have spectral density

$$h(\lambda) = M + 2 \textstyle\sum_{n=1}^{\infty} 2^{-3n/4} \cos 2^n\lambda , \qquad -\pi < \lambda < \pi ;$$

where $M$ is chosen to make $h$ strictly positive. It is known that this Weirstrass function satisfies Hölder condition of order $\frac{3}{4}$; see e.g. page 47 of Zygmund (1968). In this case therefore the condition (2) is satisfied but not (3).

EXAMPLE 2. If the spectral density of $\{\xi_n\}$ vanishes on a subinterval of $(-\pi, \pi)$ and has two bounded derivatives then the condition (3) is satisfied. However, such a process cannot be strong-mixing because it is not even purely non-deterministic.

EXAMPLE 3. Let $\frac{1}{2} < \alpha < 1$, and $\{\eta_n: -\infty < n < \infty\}$ be an i.i.d. sequence of standard normal variables. Define

$$\xi_j = \textstyle\sum_{k=j+1}^{\infty} \eta_k/(k - j)^\alpha , \qquad -\infty - j < \infty .$$

Then $E(\xi_0 \xi_n) = O(n^{-\alpha})$. However, the variance of $S_n = \sum_{j=1}^{n} \xi_j$ is easily seen to be of the order of magnitude $n^{3-2\alpha}$ and thus Strassen's law clearly cannot apply with the norming used in the definition of $g_n$ functions. In this case, the proper norming would be to divide by $[2 \operatorname{Var}(S_n) \log \log n]^{\frac{1}{2}}$ rather than by $[2n \log \log n]^{\frac{1}{2}}$. However, even with this different norming, it is unlikely that Strassen's set $K_\sigma$ will appear as the limit set. This is because the finite-dimensional distributions of the sequence $\{(\operatorname{Var} S_n)^{-\frac{1}{2}} S_{[nt]} : 0 \leqq t \leqq 1\}$ converge, not to those of the Brownian motion, but to those of the Gaussian process $\{\zeta(t): 0 \leqq t \leqq 1\}$ with the covariance function

$$E(\zeta(s)\zeta(t)) = \text{const.}\,[s^{3-2\alpha} + t^{3-2\alpha} - |t - s|^{3-2\alpha}]\,, \qquad 0 \leqq s, t \leqq 1\,.$$

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