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STRONG LAWS OF LARGE NUMBERS FOR WEAKLY ORTHOGONAL SEQUENCES OF BANACH SPACE-VALUED RANDOM VARIABLES

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This paper studies Strong Laws of Large Numbers for Banach space-valued random variables which are subject to the Banach space analog of orthogonality called weak orthogonality.

1. Summary. The Strong Law of Large Numbers for Banach space-valued random variables, in its generic form states:

THEOREM. *Let $\mathscr{X}$ be a specified Banach space. Let $\{X_k\}$, $k = 1, 2, \cdots$, be a sequence of $\mathscr{X}$-valued random variables subject to a certain set of conditions. Then we have that $n^{-1} \sum_{k=1}^{n} X_k$ converges to a constant function in the norm topology of $\mathscr{X}$ almost surely.*

Strong Laws for independent $\mathscr{X}$-valued random variables have been studied in Beck [1]. In this paper we relax the requirement of independence and consider instead a form of orthogonality. For real-valued random variables, the following result is well known.

THEOREM (Radamacher [7] and Mensov [5]). *Let $\{X_k\}$ be a sequence of mutually orthogonal scalar-valued (real or complex) random variables with $\sigma_k^2 = \operatorname{Var} X_k < \infty$, $\forall k = 1, 2, \cdots$. If $\sum_{k=1}^{\infty} k^{-2}\sigma_k^2 \log^2 k < \infty$, then the sequence of random variables satisfies the Strong Law of Large Numbers.*

The authors have studied a Banach space analog of orthogonality called weak orthogonality (cf. Beck and Warren [2]). Here we prove a Strong Law for strictly stationary sequences of weakly orthogonal $\mathscr{X}$-valued random variables. In the case of independent sequences of $\mathscr{X}$-valued random variables, identical distribution is sufficient for the Strong Law. This is not so for weakly orthogonal sequences, as we show by means of an example. This same example shows that a sequence of identically distributed, uniformly bounded $\mathscr{X}$-valued random variables satisfying the Strong Law in the weak linear topology of $\mathscr{X}$ does not necessarily satisfy the Strong Law in the norm topology of $\mathscr{X}$. This is interesting because similar sequences of $\mathscr{X}$-valued (for $\mathscr{X}$ separable) random variables which satisfy the Weak Law of Large Numbers (convergence in probability) in the weak linear topology also must satisfy the Weak Law in the norm topology (cf. Taylor [8]). The notion of orthogonality is more natural in a Banach algebra

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where a multiplicative structure is available. One might expect to find, in that context, a counterpart of the Radamacher and Mensov Strong Law. However, the same example shows, in general, that such a conjecture is false. Finally, we give two theorems which state conditions when identical distribution and weak orthogonality are sufficient for the Strong Law.

2. Preliminaries. $\mathscr{X}$ denotes a Banach space with norm $||\cdot||_{\mathscr{X}}$, $\mathscr{X}^*$ denotes the dual of $\mathscr{X}$. $(\Omega, \beta, \Pr)$ denotes a probability space. A mapping $X: \Omega \to \mathscr{X}$ is *strongly measurable* if for every Borel set $B \subset \mathscr{X}$, $X^{-1}(B)$ is measurable. Strongly measurable functions from Ω into $\mathscr{X}$ are called *$\mathscr{X}$-valued random variables* (rv's). $L_p(\Omega, \beta, \Pr, \mathscr{X})$, for $1 \leqq p < \infty$, hereafter written simply $L_p(\Omega, \mathscr{X})$, denotes the space of all $\mathscr{X}$-valued rv's on Ω for which the norm

$$|||X||| = (\textstyle\int_\Omega ||X(\omega)||_{\mathscr{X}}^p \Pr(d\omega))^{1/p} < \infty .$$

A finite collection of $\mathscr{X}$-valued rv's $X_1, \cdots, X_m$ is *independent* if, for every collection of m Borel sets $B_1, \cdots, B_m \subset \mathscr{X}$, we have

$$\Pr\{\omega: X_1(\omega) \in B_1, \cdots, X_m(\omega) \in B_m\} = \textstyle\prod_{i=1}^m \Pr\{\omega: X_i(\omega) \in B_i\} .$$

An infinite collection of $\mathscr{X}$-valued rv's is independent if every finite subcollection is independent. A collection of $\mathscr{X}$-valued rv's $X_1, X_2, \cdots$ is *identically distributed* if, for every Borel set $B \subset \mathscr{X}$, and for all positive integers i and j, $\Pr\{\omega: X_i(\omega) \in B\} = \Pr\{\omega: X_j(\omega) \in B\}$. A sequence of $\mathscr{X}$-valued rv's is *strictly stationary* if, for every nonnegative integer h and every positive integer n and each collection of Borel sets $B_i \subset \mathscr{X}$, $i = 1, 2, \cdots, n$, we have that $\Pr\{\omega: X_{h+1}(\omega) \in B_1, \cdots, X_{h+n}(\omega) \in B_n\}$ is independent of h. Note that the rv's of a strictly stationary sequence are identically distributed.

An $\mathscr{X}$-valued rv X is *strongly* (Bochner) *integrable* if $\int_\Omega ||X(\omega)||_{\mathscr{X}} \Pr(d\omega) < \infty$. This is not the usual definition of strong integrability, but is employed here to avoid a longer definition. If X is strongly integrable, then there exists an element $y \in \mathscr{X}$ such that, for every $x^* \in \mathscr{X}^*$, we have $x^*(y) = \int_\Omega x^*(X(\omega)) \Pr(d\omega)$. y is called the *integral*[1] of X and is written $y = \int_\Omega X(\omega) \Pr(d\omega)$. This integral, if it exists, is also called the *expectation* of X and is written $E(X)$. If $X \in L_2(\Omega, \mathscr{X})$, then we define a Variance:

$$\sigma^2 = \operatorname{Var} X = \textstyle\int_\Omega ||X(\omega) - E(X)||^2 \Pr(d\omega) .$$

A sequence of $\mathscr{X}$-valued rv's is *weakly orthogonal* if, for all $x^* \in \mathscr{X}^*$, we have

$$E(x^*X_i \cdot x^*X_j) = \textstyle\int x^*X_i(\omega) \cdot x^*X_j(\omega) \Pr(d\omega) = 0 .$$

Throughout this paper, unless otherwise noted, we shall consider only $\mathscr{X}$-valued rv's which have their expectation equal to the zero element of $\mathscr{X}$. This is not a serious restriction since all rv's which have an expectation are readily "centered"

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[1] In this definition, y is defined as the weak or Pettis integral. We rely on a theorem which states "every strongly integrable function is also weakly integrable and the integrals have the same value." Cf. E. Hille and R. S. Phillips [4] page 80.

at zero by subtraction of a constant. In particular, $X - E(X)$ always has expectation zero. A consequence of this assumption is that independent random variables, say X_1 and X_2, are weakly orthogonal. This follows since, for all $x^* \in \mathscr{X}^*$,

$$E(x^*X_1 \cdot x^*X_2) = E(x^*X_1) \cdot E(x^*X_2) = x^*E(X_1) \cdot x^*E(X_2) .$$

Finally, a sequence of $\mathscr{X}$-valued rv's $X_1, X_2, \cdots$, satisfies the *Strong Law of Large Numbers* (SLLN) iff

$$\lim_{n\to\infty} ||n^{-1} \textstyle\sum_{k=1}^{n} X_k(\omega)||_{\mathscr{X}} \to 0 \quad \text{a.s.}$$

3. Strong laws in Banach spaces. Our first theorem was originally proved by substantially more complex techniques. A result of E. Mourier, which we present next, considerably simplifies our original proof. Hereinafter, for a sequence of rv's $X_1, X_2, \cdots$, we use the notation $Y_n = n^{-1}(X_1 + \cdots + X_n)$.

THEOREM (E. Mourier, [6]). *If $\mathscr{X}$ is a separable Banach space, then every strictly stationary sequence of $\mathscr{X}$-valued* rv's $X_1, X_2, \cdots$, *with* $E(||X_1||) < \infty$, *has the property that there exists an $\mathscr{X}$-valued* rv Y *such that* $||Y_n - Y|| \to 0$ a.s.

LEMMA 1. *Let $\mathscr{X}$ be a Banach space and let $X_1, X_2, \cdots$, be a weakly orthogonal sequence of identically distributed $\mathscr{X}$-valued* rv's *with* $\operatorname{Var}(X_1) < \infty$. *Then, for each $x^* \in \mathscr{X}^*$, we have* $x^*(Y_n) \to 0$ a.s.

PROOF. $x^*(Y_n) = n^{-1}(x^*X_1 + \cdots + x^*X_n)$. Weak orthogonality implies that the sequence $x^*X_1, x^*X_2, \cdots$, is an orthogonal sequence of identically distributed scalar-valued rv's. Thus $\operatorname{Var}(x^*X_1) = \operatorname{Var}(x^*X_k)$, $\forall k = 2, 3, \cdots$. Furthermore, it is easy to see that $\operatorname{Var}(x^*X_1) \leqq 1 + ||x^*|| \operatorname{Var}(X_1) < \infty$. Now the lemma is an immediate consequence of Radamacher and Mensov's Strong Law.

LEMMA 2. *Let $\mathscr{X}$ be a separable Banach space and suppose* $||Y_n - Y|| \to 0$ a.s., *where Y_n and Y are $\mathscr{X}$-valued* rv's. *If* $x^*(Y_n) \to 0$ a.s., *for each $x^* \in \mathscr{X}^*$, then* $Y = 0$ a.s.

PROOF. Suppose, to the contrary, that there is a set of positive measure on which $Y \neq 0$. By hypothesis, $||Y_n - Y|| \to 0$ except possibly on a set B of measure zero. Let $x_1, x_2, \cdots$, be a countable dense subset of $\mathscr{X}$ and let $o_1, o_2, \cdots$, be a collection of spheres in $\mathscr{X}$ such that $o_k = \{x \in \mathscr{X}: ||x - x_k|| \leqq ||x_k||/4\}$. The spheres form a countable cover for $\mathscr{X} - \{0\}$. There must exist at least one k, say $k = p$, such that $\Pr\{\omega: Y(\omega) \in o_p\} > 0$. Otherwise $Y(\omega) = 0$ for almost every ω. Let $E_p = \{\omega: Y(\omega) \in o_p\} - B$. As a consequence of the Hahn–Banach theorem we can choose $x_p^* \in \mathscr{X}^*$ so that $||x_p^*|| = 1$ and $x_p^*(x_p) = ||x_p||$. For an arbitrary $\omega_0 \in E_p$ we can find n_0 such that $||Y_n(\omega_0) - Y(\omega_0)|| < ||x_p||/4$ for $n \geqq n_0$. This implies that $||Y_n(\omega_0) - x_p|| < ||x_p||/4$ for $n \geqq n_0$. Therefore, for $n \geqq n_0$,

$$\begin{aligned} ||x_p^*Y_n(\omega_0)|| &\geqq |x_p^*(x_p)| - |x_p^*(Y_n(\omega_0) - x_p| \\ &> ||x_p||/2 . \end{aligned}$$

Since ω_0 was chosen arbitrarily from E_p, it follows that $x_p^*(Y_n)$ does not converge to zero almost surely, in contradiction to the hypothesis.

THEOREM 3. *If $\mathscr{X}$ is a separable Banach space, then every sequence of $\mathscr{X}$-valued* rv's $X_1, X_2, \cdots$, *with* Var $(X_1) < \infty$ *which is weakly orthogonal and strictly stationary satisfies the Strong Law of Large Numbers.*

PROOF. Since the random variables of a strictly stationary sequence are necessarily identically distributed, it follows from Lemma 1 that, for each $x^* \in \mathscr{X}^*$, $x^*(Y_n) \to 0$ a.s. Mourier's theorem assures us that there exists an $\mathscr{X}$-valued rv Y such that $||Y_n - Y|| \to 0$ a.s. Lemma 2 now implies that $Y = 0$ a.s. which is what we wanted to prove.

It is clear that identical distribution of the rv's $X_1, X_2, \cdots$ is enough for the Strong Law in conjunction with some other adequately stringent condition. Mutual independence, when $\mathscr{X}$ is separable, is one such condition (cf. Beck [1]). In this connection, we give a pair of theorems which state conditions when identical distribution and weak orthogonality are sufficient for the SLLN.

THEOREM 4. *Let $\mathscr{X}^*$ be a separable Banach space, $\mathscr{S}$ a convex subset of $\mathscr{X}$ with compact closure, and let $X_1, X_2, \cdots$ be a weakly orthogonal sequence of identically distributed $\mathscr{X}$-valued* rv's *with* Var $(X_1) < \infty$. *If the range of $X_k \subseteqq \mathscr{S}$ for each k, then the sequence satisfies the* SLLN.

PROOF. Since $\mathscr{S}$ is convex, it is clear that the range of Y_n is contained in $\mathscr{S}$ for each n. Furthermore, Lemma 1 implies that, for each $x^* \in \mathscr{X}^*$, $x^*(Y_n) \to 0$ except possibly on a set $B(x^*)$ of measure zero. Let $x_1^*, x_2^*, \cdots$ be a countable dense subset of $\mathscr{X}^*$. Let $B = \bigcup_{k=1}^{\infty} B(x_k^*)$. Then B has measure zero and, if $\omega_0 \notin B$, we have $x_k^*(Y_n(\omega_0)) \to_n 0$ for each k.

Let $A(\omega_0) = \{Y_1(\omega_0), Y_2(\omega_0), \cdots\}$. For each ω_0, $A(\omega_0)$ has at least one limit point in the closure of $\mathscr{S}$ since $\bar{\mathscr{S}}$ is compact. Now suppose that $\omega_0 \notin B$ and $A(\omega_0)$ has at least two distinct limit points, say y_1 and y_2. Let $M = \max\{||y_1||, ||y_2||\}$. Then, since $\mathscr{X}^*$ separates points, there exists some $\hat{x}^* \in \mathscr{X}^*$ such that $|M(\hat{x}^*(y_1/M) - \hat{x}^*(y_2/M))| = |\hat{x}^*(y_1) - \hat{x}^*(y_2)| = \varepsilon > 0$. Since the x_k^* are dense in $\mathscr{X}^*$, we can find a k, say $k = \rho$, such that $||\hat{x}^* - x_\rho|| < \varepsilon/4M$. This implies that $|\hat{x}^*(y_i/M) - x_\rho^*(y_i/M)| < \varepsilon/4M$ for $i = 1, 2$. It follows that $|x_\rho^*(y_1) - x_\rho^*(y_2)| > \varepsilon/2$. This, however, contradicts the fact that $x_\rho^*(Y_n(\omega_0)) \to 0$. Indeed $A(\omega_0)$ has a unique limit point.

Let $\hat{Y}: \Omega \to \mathscr{X}$ be defined as the function which equals the unique limit point of $A(\omega)$ when $\omega \notin B$ and otherwise is arbitrary. Clearly $||Y_n - \hat{Y}|| \to 0$ a.s. Since $\mathscr{X}$ is separable, Lemma 2 implies that $\hat{Y} = 0$ a.s. which completes the proof of this theorem.

THEOREM 5. *Let $\mathscr{X}$ be a finite dimensional Banach space. Let $X_1, X_2, \cdots$, be a weakly orthogonal sequence of identically distributed $\mathscr{X}$-valued* rv's *with* Var $(X_1) < \infty$. *Then the sequence satisfies the* SLLN.

PROOF. Let $b_1, \cdots, b_m$ be a Hamel basis for $\mathscr{X}$. Every $x \in \mathscr{X}$ can be uniquely written as $x = \sum_{i=1}^{m} \alpha_i b_i$, where the α_i are suitable scalars. Define linear maps $b_i^*: \mathscr{X} \to \Phi$, into the field of scalars, by $b_i^*(x) = \alpha_i$ if $x = \sum_{i=1}^{m} \alpha_i b_i$. These maps are continuous since every linear operator on a finite dimensional space is continuous. Thus $b_i^* \in \mathscr{X}^*$ for $i = 1, 2, \cdots, m$ and $x = \sum_{i=1}^{m} b_i^*(x) b_i$. Clearly if $x_1, x_2, \cdots$ is a sequence such that for each i, $b_i^*(x_k) \to 0$, then $x_k \to 0$ as $k \to \infty$.

Suppose $\mathscr{X}$ is m-dimensional. Then, by Lemma 1, for $i = 1, 2, \cdots, m$, $b_i^*(Y_n) \to 0$ as $n \to \infty$ except possibly on a set B_i of measure zero. Let $B = \bigcup_{i=1}^{m} B_i$, so that B has measure zero. Thus, for $\omega \notin B$, $b_i^*(Y_n(\omega)) \to 0$ for each $i = 1, 2, \cdots, m$. Hence $Y_n(\omega) \to 0$ as $n \to \infty$ except on a set of measure zero which is what we needed to prove.

It is reasonable to wonder whether Theorem 3 might be strengthened by relaxing the hypothesis of strict stationarity to, say, identical distribution of the rv's in our sequence. This is not the case as we show next.

EXAMPLE 6. Let $\mathscr{X} = c_0$, the subspace of l_∞ consisting of sequences which converge to zero. Then there exists a sequence of $\mathscr{X}$-valued rv's which are

(i) uniformly bounded in norm,
(ii) identically distributed,
(iii) weakly orthogonal, but which
(iv) do not satisfy the Strong Law of Large Numbers.

CONSTRUCTION.[2] We proceed by constructing first a probability space (Ω, P) and then describing a sequence of c_0-valued rv's $X_1, X_2, \cdots$, which satisfy the requirements of Example 6.

(Ω, P) is constructed as a product probability space with Ω of the form $\Omega_1 \times \Omega_2 \times \cdots$ where, for each k, (Ω_k, P_k) is a probability space. The probability measure P on Ω is the direct product measure of the factor spaces (Ω_k, P_k).

Before describing the details of the construction of the factor spaces, we shall choose two increasing sequences of positive integers $\{M(k)\}$ and $\{N(k)\}$, $k = 0, 1, 2, \cdots$. These sequences involve the essential parameters for the factor spaces and are chosen in order that the SLLN will fail. Let $M(0) = 0$ and define $Q(k) = \sum_{i=0}^{k} M(i)$. Let $\{\delta_k\}$ be a positive sequence converging monotonically to 0. Let $M(k)$ be the first integer larger than $Q(k-1)/\delta_k$. It follows that

$$\frac{M(k) - Q(k-1)}{M(k)} \geqq 1 - \delta_k .$$

[2] Editorial space limitations prohibit our expostulating the rather long and difficult proof of this example and we therefore restrict ourselves to a description of the rv's and probability space. The details will appear in a subsequent paper wherein the technique of this example is expanded to show that, in addition to (i), (ii) and (iii), there is a single set S_0 of probability zero, such that $x^*(Y_n(\omega)) \to 0$ as $n \to \infty$ for each $x^* \in \mathscr{X}^*$ and $\omega \notin S_0$. Furthermore, a small modification of the method used here will produce the same example in a Banach space which is not only separable, but which is also reflexive and, in fact, locally uniformly convex. Cf. Beck and Warren [3].

Choose $N(k)$ such that

$$(1 - 2^{-M(k)})^{N(k)} < \tfrac{1}{2}, \qquad \forall k = 1, 2, \cdots,$$

and define $R(k) = \sum_{i=1}^{k} N(i)$.

For each k, we shall construct the factor space (Ω_k, P_k) to have the property there exists a sequence of measurable subsets $\{\Omega_{k,j}\}_{j=1}^{\infty}$ such that

$$\Omega_k = \Omega_{k,1} \supset \Omega_{k,2} \supset \cdots \supset \Omega_{k,j} \supset \cdots,$$

$$P_k(\Omega_{k,j}) = \frac{1}{j}$$

and

$$\bigcap_{j=1}^{\infty} \Omega_{k,j} = \phi .$$

For simplicity, let

$$D_{k,j} = \{\Omega_{k,j}/\Omega_{k,j+1}\}$$

so that

$$\Omega_k = \bigcup_{j=1}^{\infty} D_{k,j} .$$

Let ω_k be a point of Ω_k. To describe these points it will thus be sufficient to characterize the points in each of the sets $D_{k,j}$, $j = 1, 2, \cdots$. If $\omega_k \in D_{k,j}$, then, in our construction, we let ω_k correspond to an $M(k)$-tuple of non-terminating row vectors having 0 for the first $R(j-1)$ coordinates, $+1$ or -1 in the next $N(j)$ coordinates and 0's thereafter. There are exactly $2^{M(k)N(j)}$ distinct points, ω_k, in $D_{k,j}$. The points in $D_{k,j}$ are taken to be equiprobable. Thus, we assign probabilities

$$P_k(\omega_k) = \left(\frac{1}{j} - \frac{1}{j+1}\right) 2^{-M(k)N(j)} \qquad \text{for} \quad \omega_k \in D_{k,j} .$$

According to the assignment of probabilities

$$P_k(D_{k,j}) = \frac{1}{j} - \frac{1}{j+1}$$

and

$$P_k(\Omega_{k,j}) = \sum_{n=j}^{\infty} P_k(D_{k,n}) = \frac{1}{j}$$

from which it follows that

$$P_k(\Omega_k) = P_k(\Omega_{k,1}) = 1$$

so that (Ω_k, P_k) is indeed a probability space.

We proceed now to define the random variables X_i, $i = 1, 2, \cdots$. For any positive integer i, there exist uniquely integers s and t such that

$$i = Q(s-1) + t \qquad \text{where} \quad 1 \leqq t \leqq M(s) .$$

Define $Y_{s,t}\colon \Omega_s \to c_0$, for $1 \leqq t \leqq M(s)$, by

$$Y_{s,t}(\omega_s) = t\text{th row of} \quad \omega_s$$

and define $X_i\colon \Omega \to c_0$ by

$$Y_i(\omega) = Y_{s,t}(\omega_s) \qquad \text{when} \quad i = Q(s-1) + t .$$

As defined, the random variable X_i picks out the tth row vector from points in Ω_s where s is determined by $i = Q(s-1) + t$. Clearly $X_i(\omega) \in c_0$ since each row vector becomes zero after finitely many entries.

Property (i) is immediate whereas verification of (ii) and (iii) depend on the equiprobability of the points in the sets $D_{k,j}$ and on a straightforward cardinality argument. The verification of (iv) is, however, quite intricate. The key idea is that certain collections of coordinate rv's are independent relative to the sets $D_{k,j}$. In particular, if we let $[X_i]_u$ denote the value of the uth coordinate of X_i, we can show that the elements of

$$C_{k,j} = \{[X_i]_u : R(j-1) < u \leqq R(j) \text{ and } Q(k-1) < i \leqq Q(k)\}$$

are independent relative to the conditional probability measure on Ω_k with respect to the set $D_{k,j}$. This fact enables us to show that

$$\left\| \frac{1}{M(k)} \sum_{k=Q(k-1)+1}^{Q(k)} X_i(\omega) \right\| \geqq 1 - \delta_k$$

with probability at least $1/2k$. The Borel Cantelli lemmas and some further estimates produce the final result. This completes our discussion of Example 6.

4. Banach algebras. This paper would not be complete without some investigation of the consequences of investing a general Banach space with a multiplicative structure. Indeed it is the absence of multiplicative structure which motivates our interest in this notion of weak orthogonality. We limit our attention to commutative B^*-algebras (Banach algebras with an involution) for the convenience of having $B(S)$ and $C(S)$ as examples. These are, respectively, the algebras of all bounded functions and all continuous functions on a topological space S.

Let $\mathscr{X}$ be a Banach algebra. Two $\mathscr{X}$-valued functions f and g are *orthogonal* provided $\int_\Omega f(\omega) \cdot g(\omega)\mu(d\omega) = 0$ where the multiplication is that of the algebra $\mathscr{X}$.

THEOREM 7. *Let $\mathscr{X} = C(S)$ or $B(S)$. If f and g are weakly orthogonal $\mathscr{X}$-valued functions, then f and g are orthogonal.*

PROOF. The argument for $C(S)$ and $B(S)$ is the same so we will assume that f and g are $C(S)$-valued functions such that, for each $\mathscr{X}^*$ in the dual of $C(S)$,

$$\int_\Omega x^*(f(\omega))x^*(g(\omega)) = 0 .$$

Each valuation at a point $s \in S$ induces a multiplicative linear functional x_s^* defined by $x_s^*(\phi) = \phi(s)$ for all $\phi \in C(S)$. The set $\{x_s^* : s \in S\}$ is norm determining since, if $\phi \in C(S)$, then $\|\phi\| = \sup_{s \in S} |\phi(s)| = \sup_{s\in S} |x_s^*(\phi)|$. Since these functionals are multiplicative, we have that

$$0 = \int_\Omega x_s^*(f(\omega))x_s^*(g(\omega)) = \int_\Omega x_s^*(f(\omega) \cdot g(\omega)) = x_s^* \int_\Omega f(\omega) \cdot g(\omega)$$

for each $s \in S$. It follows that $\int_\Omega f(\omega) \cdot g(\omega) = 0$ which proves this theorem.

The strong analogy which obtains here with Hilbert space suggests that an analog of the Radamacher–Mensov SLLN or Theorem 3 may be true in the context of a commutative B^*-algebra.

CONJECTURE. Let $\mathscr{X}$ be a commutative B^*-algebra and let X_k, $k = 1, 2, \cdots$, be a sequence of $\mathscr{X}$-valued rv's with $\operatorname{Var}(X_k) < \infty$ for $k = 1, 2, \cdots$. Suppose that both of the following are satisfied:

(i) the rv's are either a weakly orthogonal or an orthogonal sequence,
(ii) either the rv's are identically distributed or they satisfy

$$\sum_{k=1}^{\infty} \frac{\operatorname{Var} X_k}{k^2} \log^2 k < \infty .$$

Then this sequence satisfies the SLLN.

In general, this conjecture is false. The c_0-valued rv's of Example 6 satisfy both conditions of (i) in view of Theorem 7 and also satisfy both conditions of (ii). The space c_0 can be imbedded in the B^*-algebra $C(Z)$, where Z is the space of integers with the discrete topology. Here, as shown in Example 6, the SLLN does not hold. Also, it is worth noting that the conjecture is false in the case $\mathscr{X} = C(S)$ where S is compact and Hausdorff. This follows from the Gelfand–Naĭmark theorem which tells us that the commutative B^*-algebra $C(Z)$ is isometrically isomorphic to $C(S)$ for some compact Hausdorff space S. In particular, S is the structure space (or maximal ideal space) of $C(Z)$.

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