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ON STOPPING RULES AND THE EXPECTED SUPREMUM OF S_n/a_n AND $|S_n|/a_n$ [1]

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Let $\{X_n\}$ be a sequence of i.i.d. mean zero random variables. Let $S_n = X_1 + \ldots + X_n$. This paper is devoted to determining the conditions whereby $E\sup_{n\geq 1} S_n/a_n < \infty$ and $E\sup_{n\geq 1}|S_n|/a_n < \infty$ for quite general sequences of increasing constants $\{a_n\}$. For the sequences $\{a_n\}$ considered, we find it sufficient to examine whether or not $\lim_{n\to\infty} E(\sum_{k=1}^n X_k/a_k)^+ < \infty$. The existence of optimal extended-valued stopping rules with finite expected reward for sequences $\{S_n/a_n\}$ or $\{|S_n|/a_n\}$ is a by-product of our results. This generalizes results of D. L. Burkholder, Burgess Davis, R. F. Gundy, B. J. McCabe and L. A. Shepp, who treat the case $a_n = n$.

Given i.i.d. mean zero random variables $X_1, X_2, \cdots$ and a sequence of positive constants $a_1, a_2, \cdots$, one may be interested in identifying the conditions under which an extended-valued stopping rule τ exists such that

$$E\frac{S_\tau^+}{a_\tau} = \sup_{t\in T_\infty} E\frac{S_t^+}{a_t} < \infty \tag{1}$$

or

$$E\frac{|S_\tau|}{a_\tau} = \sup_{t\in T_\infty} E\frac{|S_t|}{a_t} < \infty \tag{2}$$

where T_∞ is the class of randomized extended-valued stopping rules based on $\{X_1, \cdots, X_n\}$ and $S_n = X_1 + \cdots + X_n$.

Works by D. L. Burkholder [1], B. Davis [4], R. F. Gundy [5], B. J. McCabe and L. A. Shepp [7], provide solutions to such problems for $a_n = n$. Namely, (1) is satisfied iff $EX_1^+ \log^+ X_1 < \infty$ and so (2) is satisfied iff $E|X_1|\log|X_1| < \infty$.

Due to Theorem 4 of D. Siegmund [8] or Theorem 1 of M. Klass [6], for any sequence of returns $V_1, V_2, \cdots, V_\infty$ based on random variables $X_1, X_2, \cdots$ (i.e., $V_n(\omega) = V_n(X_1(\omega), \cdots, X_n(\omega))$ for $\omega \in \Omega$), there exists $\tau \in T_\infty$ such that

$$EV_\tau = \sup_{t\in T_\infty} EV_t < \infty \qquad \text{whenever} \tag{3}$$

$$E\sup_{1\leq n\leq\infty} V_n < \infty \qquad \text{and} \tag{4}$$

$$\limsup_{n\to\infty} V_n \leq V_\infty\,. \tag{5}$$

For the problems treated, (4) is also necessary to ensure the existence of $\tau \in T_\infty$ satisfying (3); and (4) implies that $\lim_{n\to\infty} V_n = 0$ a.s. Hence we define $V_\infty = 0$

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so that (4) $\Rightarrow$ (5). (Note: $V_n = S_n^+/a_n$ or $V_n = |S_n|/a_n$.) Throughout the entire paper we will regard V_∞ as zero, regardless of $\{V_n\}$ or $\limsup_{n\to\infty} V_n$.

The problem of bounding $E\max_{1\le k\le n}(|S_k|)/a_k$ above can be reduced by means of the proof of Kronecker's lemma to that of upper bounding $E\max_{1\le k\le n}|\sum_{j=1}^k X_j/a_j|$. Thanks to the fact that for independent mean zero random variables $Y_1, Y_2, \cdots, Y_n$,

$$E\max_{1\le k\le n}(\textstyle\sum_{j=1}^k Y_j)^+ \leqq 8E(\sum_{j=1}^n Y_j)^+ ,$$

the complexity of the problem at hand is considerably reduced, sufficiently so as to enable the derivation of a solution in a quite general setting.

Our primary results are twofold: Let $a(\cdot)$ be a nonnegative continuous function such that for some $\alpha > \frac{1}{2}$, $a(y)/y^\alpha \nearrow \infty$.

Then

$$E|X_1| \textstyle\int_1^{a^{-1}|X_1|\vee 1}(1/a(y))\,dy < \infty \qquad \text{iff (2) holds.} \tag{6}$$

If, in addition, $\liminf_{y\to\infty}(a(y)/y) > 0$, then

$$E(X_1^+ \textstyle\int_1^{a^{-1}X_1^+\vee 1}(1/a(y))\,dy) < \infty \Leftrightarrow (1)\,. \tag{7}$$

As a result, (2) holds with $a_n = n^{1/\alpha}$ for some $1 < \alpha < 2$ iff $E|X_1|^\alpha < \infty$. Assuming $a_n = (n\log n)\vee 1$, (1) is equivalent to the requirement that $EX_1^+\log^+(\log X_1) < \infty$. For $a_n = (n\log\log n)\vee 1$, (1) is equivalent to the condition $EX_1^+\log^+(\log\log X_1) < \infty$, and so forth.

Other results relate to such quantities as $E\sup_{n\ge 1}(X_n/a_n)$ and the a.s. convergence of $\sum_{k=1}^\infty (X_k/a_k)$.

THEOREM 1. *Let $Y_1, Y_2, \cdots$ be independent mean zero random variables. Let $T_n = Y_1 + \cdots + Y_n$. Then*

$$ET_n^+ \leqq E\max_{1\le k\le n} T_k^+ \leqq 8ET_n^+\,.$$

PROOF. The left side is obvious. We first prove what is essentially Ottaviani's inequality (Chung [3] page 114). Fix $u \geqq 0$ and $n \geqq 1$. Let

$$\begin{aligned}\tau &= \text{1st} \quad k \leqq n : T_k \geqq 2u \qquad \text{if such } k \text{ exists}\\ &= \infty \qquad\qquad\qquad\qquad\quad \text{otherwise.}\end{aligned}$$

Since

$$\{\tau = k, T_n - T_k \geqq -u\} \subseteqq \{\tau = k, T_n \geqq u\}, \qquad \text{therefore}$$

$$\textstyle\sum_{k=1}^n P\{\tau = k, T_n - T_k \geqq -u\} \leqq \sum_{k=1}^n P(\tau = k, T_n \geqq u) \leqq P(T_n \geqq u)\,.$$

Also

$$1_{\{\tau=k\}} \quad\text{and}\quad 1_{\{T_n - T_k \ge -u\}} \quad \text{are independent.}$$

Hence

$$P(T_n \geqq u) \geqq \textstyle\sum_{k=1}^n P(\tau = k)P(T_n - T_k \geqq -u)$$

and therefore

$$P(T_n \geqq u) \geqq \min_{1\le k\le n} P(T_n - T_k \geqq -u)P(\max_{1\le k\le n} T_k \geqq 2u)\,. \tag{8}$$

$$\begin{aligned}
\min_{1\leqq k\leqq n} P(T_n - T_k \geqq -u) &\\
&= 1 - \max_{1\leqq k\leqq n} P(T_n - T_k < -u)\\
&= 1 - \max_{1\leqq k\leqq n} P(T_k - T_n > u)\\
&\geqq 1 - \max_{1\leqq k\leqq n} \frac{E(T_k - T_n)^+}{u}\\
&= 1 - \max_{1\leqq k\leqq n} \frac{E(T_n - T_k)^+}{u} = 1 - \frac{E(T_n - T_1)^+}{u}\\
&\geqq 1 - \frac{ET_n^+}{u}.
\end{aligned}$$

Thus $u \geqq 2ET_n^+$ implies

$$P(\max_{1\leqq k\leqq n} T_k \geqq 2u) \leqq \frac{P(T_n \geqq u)}{1 - ET_n^+/u} \leqq 2P(T_n \geqq u).$$

Estimating $E\max_{1\leqq k\leqq n} T_k^+$, we have

$$\begin{aligned}
E\max_{1\leqq k\leqq n} T_k^+ &= 2\int_0^\infty P(\max_{1\leqq k\leqq n} T_k^+ \geqq 2u)\,du\\
&\leqq 4ET_n^+ + 2\int_{2ET_n^+}^\infty P(\max_{1\leqq k\leqq n} T_k^+ \geqq 2u)\,du\\
&\leqq 4ET_n^+ + 4\int_{2ET_n^+}^\infty P(T_n \geqq u)\,du\\
&\leqq 8ET_n^+.
\end{aligned}$$

REMARK. Since $ET_n^- = ET_n^+$, any condition on $\{Y_n^+\}$ which is necessary to ensure that $\lim_{n\to\infty} E\max_{1\leqq k\leqq n} T_k^+ < \infty$ is a condition on $\{|Y_n|\}$.

LEMMA 1. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with common distribution function F. Let $a(y)$ be a nonnegative strictly increasing continuous function defined for $y \geqq 0$. Write $a_j = a(j)$. Then*

$$\begin{aligned}
-\frac{1}{a_1} EX_1^+ &+ \int_{a_1}^\infty \left(x \int_1^{a^{-1}(x)} (1/a(y))\,dy\right) dF(x)\\
&\leqq \sum_{n=1}^\infty \int_{a_n}^\infty \frac{x\,dF(x)}{a_n} \leqq \frac{1}{a_1} EX_1^+ + \int_{a_1}^\infty \left(x \int_1^{a^{-1}(x)} (1/a(y))\,dy\right) dF(x).
\end{aligned}$$

PROOF. Let $A_j = \{a_j < x \leqq a_{j+1}\}$.

$$\begin{aligned}
\sum_{n=1}^\infty \int_{a_n}^\infty \frac{x\,dF(x)}{a_n} &= \sum_{n=1}^\infty \sum_{j=n}^\infty \int_{A_j} \frac{x\,dF(x)}{a_n}\\
&= \sum_{j=1}^\infty \int_0^\infty \left(\sum_{n=1}^j \frac{1}{a_n}\right) x\, 1_{A_j}\, dF(x).
\end{aligned}$$

The right-hand inequality is easy because

$$\sum_{n=1}^j \frac{1}{a_n} 1_{A_j} \leqq \frac{1}{a_1} + \int_1^{a^{-1}(x)} (1/a(y))\,dy\, 1_{A_j}.$$

As for the left-hand side,

$$\begin{aligned}\left(\sum_{n=1}^{j}\frac{1}{a_n}\right)1_{A_j} &\geqq \left(\int_1^j \frac{1}{a(y)}\,dy\right)1_{A_j}\\ &= \left(\int_1^{a^{-1}(x)}(1/a(y))\,dy\right)1_{A_j} - \left(\int_j^{a^{-1}(x)}(1/a(y))\,dy\right)1_{A_j}\\ &\geqq \left(\int_1^{a^{-1}(x)}(1/a(y))\,dy - \frac{1}{a_j}\right)1_{A_j}\\ &\geqq \left(\int_1^{a^{-1}(x)}(1/a(y))\,dy - \frac{1}{a_1}\right)1_{A_j}.\end{aligned}$$

COROLLARY 1. *With $\{X_n\}$ and $a(y)$ as above,*

$$\begin{aligned}&-\frac{E|X_1|}{a_1} + \int_{\{|x|>a_1\}}\left(|x|\int_1^{a^{-1}(|x|)}(1/a(y))\,dy\right)dF(x)\\ &\leqq \sum_{n=1}^{\infty}\int_{\{|x|>a_n\}}\frac{|x|\,dF(x)}{a_n} \leqq \frac{E|X_1|}{a_1} + \int_{\{|x|>a_1\}}\left(|x|\int_1^{a^{-1}(|x|)}(1/a(y))\,dy\right)dF(x).\end{aligned}$$

LEMMA 2. *Let $\{X_n\}$ be a sequence of* i.i.d. *mean zero random variables. Let $\{a_n\}$ be a sequence of positive numbers such that for some $\varepsilon > 0$, $a_n/n^\varepsilon \nearrow$. Then for each $M > 0$, $P(X_n \geqq a_n$ i.o.$) = P(X_n \geqq Ma_n$ i.o.$)$.*

PROOF. Suppose $N^{1/\varepsilon}$ is a positive integer. If $P(X_n \geqq a_n \text{ i.o.}) = 1$, then

$$\begin{aligned}\infty &= \sum_{n=N^{1/\varepsilon}}^{\infty} P(X_n \geqq a_n)\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{N^{1/\varepsilon}-1} P(X_1 \geqq a(nN^{1/\varepsilon}+k))\\ &\leqq N^{1/\varepsilon}\sum_{n=1}^{\infty} P(X_1 \geqq a(nN^{1/\varepsilon}))\\ &\leqq N^{1/\varepsilon}\sum_{n=1}^{\infty} P(X_1 \geqq Na_n),\end{aligned}$$

so that $P(X_n \geqq Na_n \text{ i.o.}) = 1$.

For any $M \geqq 1$ $\exists N \geqq M$ such that $N^{1/\varepsilon}$ is a positive integer. Now

$$\begin{aligned}P(X_n \geqq a_n \text{ i.o.}) &\geqq P(X_n \geqq Ma_n \text{ i.o.})\\ &\geqq P(X_n \geqq Na_n \text{ i.o.}).\end{aligned}$$

The extreme terms are equal regardless of whether $P(X_n \geqq a_n \text{ i.o.})$ is zero or one. Hence

$$P(X_n \geqq a_n \text{ i.o.}) = P(X_n \geqq Ma_n \text{ i.o.}).$$

The case in which $0 < M < 1$ can be reduced to the previous situation by letting $b(y) = Ma(y)$ and observing that $b(y)/y^\varepsilon \nearrow$ so that $P(X_n \geqq b(n) \text{ i.o.}) = P(X_n \geqq M^{-1}b(n) \text{ i.o.})$.

THEOREM 2. *Let $\{X_n\}$ be a sequence of* i.i.d. *mean zero random variables with common distribution function F. Let $a(y)$ be a nonnegative continuous function defined for $y \geqq 0$. Suppose*

$$\text{(9)} \qquad \textit{for some} \quad \varepsilon > 0, \quad \frac{a(y)}{y^\varepsilon} \nearrow \infty$$

$$\text{(10)} \qquad \int_{a_1}^{\infty} x \int_1^{a^{-1}(x)} (1/a(y))\,dy\,dF(x) = \infty.$$

Then $\lim_{n\to\infty} E(\sum_{k=1}^n (X_k)/a_k)^+ = \infty$, $E\sup_{n\geq 1}(S_n^+)/a_n = \infty$, *and* $E\sup_{n\geq 1}(X_n^+)/a_n = \infty$. (*Note that* $S_n = X_1 + \cdots + X_n$.)

PROOF. Assume that $P(X_n \geq a_n \text{ i.o.}) = 1$. Fix M large. For each $n \geq 1$, let

$$\begin{aligned} t_n &= \text{last} \quad k \leq n: X_k \geq Ma_k \quad && \text{if such } k \text{ exists} \\ &= \infty && \text{otherwise.} \end{aligned}$$

Observe that $P(t_n \neq \infty) = P(\bigcup_{k=1}^n \{X_k \geq Ma_k\}) \to 1$ as $n \to \infty$ by Lemma 2. From Theorem 1,

$$\begin{aligned} E\left(\sum_{k=1}^n \frac{X_k}{a_k}\right)^+ &\geq \tfrac{1}{8} E \max_{1\leq j\leq n}\left(\sum_{k=1}^j \frac{X_k}{a_k}\right)^+ \\ &\geq \tfrac{1}{8} \sum_{l=1}^n E \max_{1\leq j\leq n}\left(\sum_{k=1}^j \frac{X_k}{a_k}\right)^+ 1_{\{t_n=l\}} \\ &\geq \tfrac{1}{8} \sum_{l=1}^n E\left(\sum_{k=1}^l \frac{X_k}{a_k}\right) 1_{\{t_n=l\}} \end{aligned}$$

$\{t_n = l\} \subseteq B(X_l, X_{l+1}, \cdots, X_n)$. Hence for $k < l$, $(X_k)/a_k$ and $1_{\{t_n=l\}}$ are independent. Therefore

$$E\left(\sum_{k=1}^n \frac{X_k}{a_k}\right)^+ \geq \tfrac{1}{8} \sum_{l=1}^n \frac{EX_l 1_{\{t_n=l\}}}{a_l}.$$

Clearly,

$$E \sup_{k\leq n} \frac{X_k^+}{a_k} \geq \sum_{l=1}^n \frac{EX_l 1_{\{t_n=l\}}}{a_l}.$$

Similarly,

$$E \max_{1\leq k\leq n} \frac{S_k^+}{a_k} \geq \sum_{l=1}^n E\frac{S_l 1_{\{t_n=l\}}}{a_l} = \sum_{l=1}^n \frac{EX_l 1_{\{t_n=l\}}}{a_l}.$$

The three quantities of interest are each bounded below by

$$\tfrac{1}{8} \sum_{l=1}^n \frac{EX_l 1_{\{t_n=l\}}}{a_l} \geq \tfrac{1}{8} \sum_{l=1}^n EM1_{\{t_n=l\}} = \frac{MP(t_n \neq \infty)}{8} \to \frac{M}{8} \quad \text{as } n \to \infty,$$

which completes the proof when $P(X_n \geq a_n \text{ i.o.}) = 1$. Now assume $P(X_n \geq a_n \text{ i.o.}) = 0$. For each $n \geq 1$, let

$$\begin{aligned} t_n &= \text{last} \quad k \leq n: X_k \geq a_k \quad && \text{if such } k \text{ exists} \\ &= \infty && \text{otherwise.} \end{aligned}$$

$$\begin{aligned} P(t_n = \infty) &\geq P(\textstyle\bigcap_{k=1}^\infty \{X_k < a_k\}) \\ &= \textstyle\prod_{k=1}^\infty P(X_k < a_k) = \prod_{k=1}^\infty (1 - P(X_1 \geq a_k)) > 0 \end{aligned}$$

since $\sum_{k=1}^\infty P(X_1 \geq a_k) < \infty$ and $P(X_1 \geq a_1) < 1$. Let $c = \frac{1}{8}\prod_{k=1}^\infty P(X_1 < a_k)$. Again we obtain

$$\lim_{n\to\infty} \min\left(E\left(\sum_{k=1}^n \frac{X_k}{a_k}\right)^+, E\max_{1\le k\le n} \frac{S_k^+}{a_k}, E\max_{1\le k\le n} \frac{X_k^+}{a_k}\right)$$
$$\geqq \tfrac{1}{8} \lim_{n\to\infty} \sum_{k=1}^n \frac{EX_k 1_{\{t_n=k\}}}{a_k}$$
$$= \tfrac{1}{8} \lim_{n\to\infty} \sum_{k=1}^n \frac{(\int_{a_k}^\infty x\,dF(x))}{a_k} \left(\prod_{j=k+1}^n P(X_j < a_j)\right)$$
$$\geqq c \sum_{k=1}^\infty \frac{\int_{a_k}^\infty x\,dF(x)}{a_k}$$

which by Lemma 1 is at least

$$c\left(-\frac{EX_1^+}{a_1} + \int_{a_1}^\infty \left(x \int_1^{a^{-1}(x)} (1 - a(y))\,dy\right) dF(x)\right)$$

and is consequently infinite.

LEMMA 3. *Let $\{X_n\}$ be a sequence of* i.i.d. *random variables with common distribution function F. Let $a(y)$ be a strictly increasing continuous function defined for $y \geqq 0$. Assume $a(0) = 0$ and $a(\infty) = \infty$. Set $Y_n = X_n 1_{\{0 \le X_n \le a_n\}}$. Then*

$$-P(X_1 > 0) + \int_0^\infty \left(x^2 \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x)$$
$$\leqq \sum_{n=1}^\infty \frac{EY_n^2}{a_n^2} \leqq P(X_1 > 0) + \int_0^\infty \left(x^2 \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x)\,.$$

PROOF. Let $A_j = \{a_{j-1} < x \leqq a_j\}$ for $j \geqq 1$.

$$\sum_{n=1}^\infty \frac{EY_n^2}{a_n^2} = \sum_{n=1}^\infty \sum_{j=1}^n \int_0^\infty \frac{x^2 1_{A_j}\,dF(x)}{a_n^2}$$
$$= \sum_{j=1}^\infty \int_0^\infty x^2 \left(\sum_{n=j}^\infty \frac{1}{a_n^2}\right) 1_{A_j}\,dF(x)\,.$$

We deduce the inequality on the right by noting that

$$x^2\left(\sum_{n=j}^\infty \frac{1}{a_n^2}\right) 1_{A_j} \leqq x^2\left(\frac{1}{a_j^2} + \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) 1_{A_j}$$
$$\leqq \left(1 + x^2 \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) 1_{A_j}\,.$$

To produce the left side:

$$x^2\left(\sum_{n=j}^\infty \frac{1}{a_n^2}\right) 1_{A_j} \geqq x^2\left(\int_j^\infty \frac{1}{a^2(y)}\,dy\right) 1_{A_j}$$
$$= x^2\left(\int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy - \int_{a^{-1}(x)}^j \frac{1}{a^2(y)}\,dy\right) 1_{A_j}$$
$$\geqq x^2\left(\int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy - \frac{j - a^{-1}(x)}{x^2}\right) 1_{A_j}$$
$$\geqq x^2\left(\int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy - 1\right) 1_{A_j}\,.$$

LEMMA 4.

$$\int_0^\infty \left(x^2 \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x) \leqq \int_{a_1}^\infty \left(x^2 \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x) + P(X_1 > 0) + a_1^2 \int_1^\infty \frac{1}{a^2(y)}\,dy\,.$$

PROOF. This follows from the fact that

$$\begin{aligned}\int_0^{a_1} \left(x^2 \int_{a^{-1}(x)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x) &= \int_0^{a_1} x^2 \int_{a^{-1}(x)}^1 \frac{1}{a^2(y)}\,dy + \int_0^{a_1} \left(x^2 \int_1^\infty \frac{1}{a^2(y)}\,dy\right) dF(x)\\ &\leqq \int_0^{a_1} (1 - a^{-1}(x))\,dF(x) + a_1^2 \left(\int_1^\infty \frac{1}{a^2(y)}\,dy\right)\\ &\leqq P(X_1 > 0) + a_1^2 \int_1^\infty \frac{1}{a^2(y)}\,dy\,.\end{aligned}$$

COROLLARY 2. Let $\{X_n\}$ be a sequence of i.i.d. random variables with common distribution function F. Let $a(y)$ be a strictly increasing continuous function defined for $y \geqq 0$. Assume $a(0) = 0$ and $a(\infty) = \infty$.

Let $Y_n = X_n 1_{\{|X_n| \leqq a_n\}}$. Then

$$\begin{aligned}-1 + \int_{-\infty}^\infty \left(x^2 \int_{a^{-1}(|x|)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x) &\leqq \sum_{n=1}^\infty \frac{EY_n^2}{a_n^2} \leqq 1 + \int_{-\infty}^\infty \left(x^2 \int_{a^{-1}(|x|)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x)\\ &\leqq 2 + a_1^2 \int_1^\infty \frac{1}{a^2(y)}\,dy + \int_{\{a_1 \leqq |x|\}} \left(x^2 \int_{a^{-1}(|x|)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x)\,.\end{aligned}$$

THEOREM 3. *Let* $\{X_n\}$ *be a sequence of* i.i.d. *non-degenerate mean zero random variables with common distribution function* F. *Let* $a(\cdot)$ *be a nonnegative continuous function defined for* $y \geqq 0$ *such that for some* $\varepsilon > 0$, $a(y)/y^\varepsilon \nearrow \infty$. *Then the following are equivalent*:

(11) $$\lim_{n\to\infty} E\left|\sum_{k=1}^n \frac{X_k}{a_k}\right| < \infty$$

(12) $$\lim_{n\to\infty} E \max_{k \leqq n}\left|\sum_{j=1}^k \frac{X_j}{a_j}\right| < \infty$$

(13) (i) $\int_{\{|x| \geqq a_1\}} (|x| \int_1^{a^{-1}(|x|)} (1/a(y))\,dy)\,dF(x) < \infty$ *and*

(ii) $\int_{-\infty}^\infty \left(x^2 \int_{a^{-1}(|x|)}^\infty \frac{1}{a^2(y)}\,dy\right) dF(x) < \infty\,.$

REMARK. (ii) may be replaced by the two conditions

$$\int_1^\infty \frac{1}{a^2(y)}\,dy < \infty$$

and

$$\int_{\{|x|\geqq a_1\}} \left(x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy\right) dF(x) < \infty .$$

PROOF. According to Theorem 1, (11) $\Rightarrow$ (12). We next establish that (12) $\Rightarrow$ (13).

By the contrapositive of Theorem 2 applied to $E(\sum_{k=1}^{n} (X_k)/a_k)^+$ and $E(\sum_{k=1}^{n} (X_k)/a_k)^-$, (12) $\Rightarrow$ (13) (i). When (12) holds we may invoke the Martingale Convergence Theorem to conclude that $\sum_{k=1}^{\infty} (X_k)/a_k$ converges a.s. Let $Y_n = X_n 1_{\{|X_n|\leqq a_n\}}$ and $Z_n = X_n - Y_n$. By the Three Series Theorem (Chung [3] page 112),

$$\sum_{n=1}^{\infty} \frac{EY_n^{\,2} - (EY_n)^2}{a_n^{\,2}} < \infty .$$

Since

$$\frac{|EY_n|}{a_n} \leqq 1 , \qquad \left(\frac{EY_n}{a_n}\right)^2 \leqq \frac{|EY_n|}{a_n} = \frac{|EZ_n|}{a_n}$$

and so

$$\sum_{n=1}^{\infty} \left(\frac{EY_n}{a_n}\right)^2 \leqq \sum_{n=1}^{\infty} \frac{E|Z_n|}{a_n}$$

which, by Corollary 1 is

$$\leqq \frac{E|X_1|}{a_1} + \int_{\{|x|>a_1\}} (|x| \int_1^{a^{-1}(|x|)} (1/a(y))\, dy)\, dF(x) < \infty$$

by (13) (i) and the fact that X_1 has finite mean. Thus

$$\infty > \sum_{n=1}^{\infty} \frac{EY_n^{\,2}}{a_n^{\,2}} \geqq -1 + \int_{-\infty}^{\infty} \left(x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy\right) dF(x)$$

(see Corollary 2). Hence (12) also implies (13) (ii).

We next assert that (13) $\Rightarrow$ (11).

$$\begin{aligned} E\left|\sum_{k=1}^{n} \frac{X_k}{a_k}\right| &= E\left|\sum_{k=1}^{n} \frac{Y_k + Z_k}{a_k}\right| \leqq E\left|\sum_{k=1}^{n} \frac{Y_k}{a_k}\right| + E\sum_{k=1}^{n} \frac{|Z_k|}{a_k} \\ &\leqq \left(E\left(\sum_{k=1}^{n} \frac{Y_k}{a_k}\right)^2\right)^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{E|Z_k|}{a_k} \\ &= \left(\mathrm{Var}\left(\sum_{k=1}^{n} \frac{Y_k}{a_k}\right) + \left(E\sum_{k=1}^{n} \frac{Y_k}{a_k}\right)^2\right)^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{E|Z_k|}{a_k} \\ &\leqq \left(\sum_{k=1}^{\infty} \frac{EY_k^{\,2}}{a_k^{\,2}} + \left(\sum_{k=1}^{\infty} \frac{E|Z_k|}{a_k}\right)^2\right)^{\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{E|Z_k|}{a_k} . \end{aligned}$$

As has been previously shown, (13) (i) implies

$$\sum_{k=1}^{\infty} \frac{E|Z_k|}{a_k} < \infty \qquad \text{(Corollary 1)}$$

and (13) (ii) implies

$$\sum_{k=1}^{\infty} \frac{EY_k^{\,2}}{a_k^{\,2}} < \infty \qquad \text{(Corollary 2)} .$$

Consequently

$$\lim_{n\to\infty} E\left|\sum_{k=1}^{n} \frac{X_k}{a_k}\right| < \infty .$$

It is of interest and importance to know when (ii) implies (i) and vice versa. Lemmas 5 and 6 deal with this subject.

LEMMA 5. *Let $a(y)$ be a nonnegative strictly increasing continuous function defined for $y \geqq 0$ such that for some $\frac{1}{2} < \alpha < 1$, $a(y)/y^\alpha \searrow$. Then*

$$|x| \int_1^{a^{-1}(|x|)} (1/a(y))\, dy \leqq \frac{a^{-1}(|x|)}{1-\alpha} \leqq \left(\frac{2\alpha - 1}{1-\alpha}\right) x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy .$$

PROOF.

$$\begin{aligned}|x| \int_1^{a^{-1}(|x|)} (1/a(y))\, dy &= |x| \int_1^{a^{-1}(|x|)} (y^\alpha/a(y))(1/y^\alpha)\, dy \\ &\leqq |x| \frac{[a^{-1}(|x|)]^\alpha}{|x|} \int_1^{a^{-1}(|x|)} (1/y^\alpha)\, dy \\ &\leqq [a^{-1}(|x|)]^\alpha \frac{[a^{-1}(|x|)]^{1-\alpha}}{1-\alpha} = \frac{[a^{-1}(|x|)]}{1-\alpha}\end{aligned}$$

$$\begin{aligned}x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy &= x^2 \int_{a^{-1}(|x|)}^{\infty} \left(\frac{y^{2\alpha}}{a^2(y)}\right)\left(\frac{1}{y^{2\alpha}}\right) dy \\ &\geqq x^2 \frac{[a^{-1}(|x|)]^{2\alpha}}{x^2} \int_{a^{-1}(|x|)}^{\infty} \frac{1}{y^{2\alpha}}\, dy \\ &= [a^{-1}(|x|)]^{2\alpha} \frac{[a^{-1}(|x|)]^{1-2\alpha}}{2\alpha - 1} = \frac{a^{-1}(|x|)}{2\alpha-1} ,\end{aligned}$$

which proves the lemma.

THEOREM 4. *Let $\{X_n\}$ and a be as in Theorem 3. Assume further that for some $\frac{1}{2} < \alpha < 1$, $a(y)/y^\alpha \searrow$. Then* (11), (12), *and* (13)(ii) *are also each equivalent to*

$$\sum_{k=1}^{\infty} \frac{X_k}{a_k} \text{ converges a.s.} \tag{14}$$

PROOF. From the proof of Theorem 3, (12) $\Rightarrow$ (14) and (14) $\Rightarrow$

$$\sum_{n=1}^{\infty} \frac{EY_n^2 - (EY_n)^2}{a_n^2} < \infty$$

where $Y_n = X_n 1_{\{|X_n| \leqq a_n\}}$. The proof of Theorem 3 also gives

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{EY_n}{a_n}\right)^2 &\leqq \frac{E|X_1|}{a_1} + \int_{\{|x|>a_1\}} |x| \int_1^{a^{-1}(|x|)} (1/a(y))\, dy\, dF(x) \\ &\leqq \frac{E|X_1|}{a_1} + \int_{\{|x|>a_1\}} \frac{a^{-1}(|x|)}{1-\alpha}\, dF(x) \qquad \text{(by Lemma 5)} \\ &< \infty \Leftrightarrow \sum_{n=1}^{\infty} P(a^{-1}(|X_n|) > n) < \infty \\ &\Leftrightarrow \sum_{n=1}^{\infty} P(|X_n| \geqq a_n) < \infty \Leftrightarrow \sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty .\end{aligned}$$

Referring once again to the Three Series Theorem, $\sum_{k=1}^{\infty}(X_k)/a_k$ converges a.s. $\Rightarrow \sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty$. Hence $\infty > \sum_{n=1}^{\infty}(EY_n^2)/a_n^2$ and so (13)(ii) holds (see Corollary 2). By virtue of Lemma 5, (13)(ii) $\Rightarrow$ (13)(i). Therefore (14) $\Rightarrow$ (13) $\Rightarrow$ (12). Consequently, (11), (12), (13)(ii) and (14) are equivalent.

An example will illustrate that if $a(y)/y^\alpha$ is not decreasing for some $\alpha < 1$ then (14) may not imply (11), (12) or (13).

EXAMPLE 1. Let $\{X_n\}$ be a sequence of i.i.d. symmetric random variables with mean zero and common distribution function F. Assume $E|X_1| \log^+ |X_1| = \infty$. Let $a(y) = y$ and $Y_n = X_n 1_{\{|X_n| \leq n\}}$. $\sum_{n=1}^{\infty}(X_n)/n$ converges a.s. iff

(a) $\sum_n P(X_n \neq Y_n) < \infty$,
(b) $|\sum_n E(Y_n)/n| < \infty$, and
(c) $\sum_n (EY_n^2 - (EY_n)^2)/n^2 < \infty$ (Three Series Term)

$$E|X_1| < \infty \quad \text{so} \quad \sum_n P(|X_n| > n) < \infty .$$

Thus (a) holds. $EY_n = 0$ so (b) holds.

$$\sum_{n=1}^{\infty} \int_{\{|x| \leq n\}} \frac{x^2\, dF(x)}{n^2} \leqq 1 + \int_{-\infty}^{\infty} \left(x^2 \int_{|x|}^{\infty} \frac{1}{y^2}\, dy\right) dF(x) \qquad \text{(by Lemma 3)}$$
$$= 1 + \int_{-\infty}^{\infty} |x|\, dF(x) < \infty .$$

So (c) holds and hence $\sum_{n=1}^{\infty}(X_n)/n$ converges. However,

$$\int_1^{\infty} |x| \int_1^{|x|} \frac{1}{y}\, dy\, dF(x) = \int_1^{\infty} |x| \log |x|\, dF(x)$$
$$= E|X_1| \log^+ |X_1| = \infty$$

so that (13)(i) fails to hold.

Whenever $\sum_{k=1}^{\infty}(X_k)/a_k$ converges, Kronecker's lemma can be utilized to give $(S_n)/a_n \to 0$ a.s. What follows is another condition sufficient to guarantee that $S_n/a_n \to 0$ a.s. This will be useful when attempting to establish the existence of optimal extended-valued stopping rules.

THEOREM 5. *Let $\{X_n\}$ be a sequence of* i.i.d. *mean zero random variables. Let $a(y)$ be a nonnegative strictly increasing continuous function defined for $y \geqq 0$ such that $a(y)/y \searrow$. Let $S_n = X_1 + \cdots + X_n$. Assume*

$$\int_{-\infty}^{\infty} \left(x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy\right) dF(x) < \infty .$$

Then $S_n/a_n \to 0$ a.s.

PROOF. Let $Y_n = X_n 1_{\{|X_n| \leq a_n\}}$. We claim that $\sum_{n=1}^{\infty}(Y_n - EY_n)/a_n$ converges a.s.

$$\sum_{n=1}^{\infty} \frac{\sigma^2(Y_n)}{a_n^2} \leqq \sum_{n=1}^{\infty} \frac{EY_n^2}{a_n^2} < \infty \qquad \text{(Corollary 2).}$$

$((Y_n - EY_n)/a_n)$ is bounded and each term has mean zero. Applying the Three

Series Theorem, $\sum_{n=1}^{\infty}(Y_n - EY_n)/a_n$ converges a.s. By Kronecker's lemma,

$$\frac{1}{a_n}\sum_{k=1}^{n}(Y_k - EY_k) \to 0 \quad \text{a.s.}$$

Referring to the proof of Lemma 5,

$$x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy \geqq \frac{a^{-1}(|x|)}{2\alpha - 1}.$$

Therefore $Ea^{-1}(|X_1|) < \infty$ implies $P(|X_n| > a_n \text{ i.o.}) = 0 \Rightarrow P(X_n \neq Y_n \text{ i.o.}) = 0$ and therefore

$$\sum_{k=1}^{n} \frac{X_k - EY_k}{a_n} \to 0 \quad \text{a.s.}$$

$$\begin{aligned}\left|\sum_{k=1}^{n} \frac{EY_k}{a_n}\right| &\leqq \sum_{k=1}^{n} \int_{\{|x|>a_k\}} \frac{|x|}{a_n}\, dF(x) \\ &= \frac{1}{a_n}\sum_{k=1}^{n-1} k \int_{\{a_k<|x|\leqq a_{k+1}\}} |x|\, dF(x) + \frac{n}{a_n}\int_{\{|x|>a_n\}} |x|\, dF(x) \\ &\leqq \frac{1}{a_n}\int_{-a_n}^{a_n} |x| a^{-1}(|x|)\, dF(x) + \int_{\{|x|>a_n\}} \left(\frac{a^{-1}(|x|)}{|x|}\right) |x|\, dF(x) \\ &\leqq \frac{1}{\sqrt{a_n}} \int_{-\sqrt{a_n}}^{\sqrt{a_n}} a^{-1}(|x|)\, dF(x) + \int_{\{\sqrt{a_n}<|x|\leqq a_n\}} a^{-1}(|x|) \\ &\quad + \int_{\{|x|>a_n\}} a^{-1}(|x|)\, dF(x) \to 0 \qquad \text{as } n \to \infty\end{aligned}$$

since $Ea^{-1}(|X_1|) < \infty$ and $a_n \to \infty$, therefore

$$\frac{S_n}{a_n} \to 0 \quad \text{a.s.}$$

LEMMA 6. *Let $a(y)$ be a nonnegative continuous function such that for some $\alpha > \frac{1}{2}$, $a(y)/y^\alpha \nearrow \infty$. Then*

$$x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy \leqq \frac{a^{-1}(|x|)}{2\alpha - 1} \leqq \frac{|x|}{2\alpha - 1} \int_1^{(a^{-1}(|x|))\vee 1} (1/a(y))\, dy + \frac{1}{2\alpha - 1}.$$

PROOF.

$$\begin{aligned} x^2 \int_{a^{-1}(|x|)}^{\infty} \left(\frac{y^{2\alpha}}{a^2(y)}\right)\left(\frac{1}{y^{2\alpha}}\right) dy &\leqq x^2 \frac{[a^{-1}(|x|)]^{2\alpha}}{x^2} \int_{a^{-1}(|x|)}^{\infty} \frac{1}{y^{2\alpha}}\, dy \\ &= (a^{-1}(|x|))^{2\alpha} \frac{[a^{-1}(|x|)]^{1-2\alpha}}{2\alpha - 1} \\ &= \frac{(a^{-1}(|x|))}{2\alpha - 1} = \frac{1}{2\alpha - 1}\int_0^{a^{-1}(|x|)} dy \\ &\leqq \frac{1}{2\alpha - 1} + \left(\int_1^{(a^{-1}(|x|))\vee 1} (|x|/a(y))\, dy\right)/2\alpha - 1. \end{aligned}$$

THEOREM 6. *Let $\{X_n\}$ and $a(y)$ be as in Theorem 3. Assume also that for some $\alpha > \frac{1}{2}$, $a(y)/y^\alpha \nearrow \infty$.*

The following are equivalent:

$$(15)\qquad \int_{\{|x|>a_1\}} \left(|x| \int_1^{a^{-1}(|x|)} (1/a(y))\, dy\right) dF(x) = \infty$$

$$(16)\qquad \lim_{n\to\infty} E\left(\sum_{k=1}^n \frac{X_k}{a_k}\right)^+ = \infty$$

$$(17)\qquad E \sup_{n\geq 1} \frac{|S_n|}{a_n} = \infty$$

$$(18)\qquad \exists t \in T_\infty \ni E \frac{|S_t|}{a_t} = \infty$$

$$(19)\qquad E \sup_{n\geq 1} \frac{|X_n|}{a_n} = \infty$$

$$(20)\qquad \exists t \in T_\infty \ni E \frac{|X_t|}{a_t} = \infty .$$

PROOF. By Theorem 2, when (15) holds, we have (16), (17), and (19). Conversely, if (15) is false then we appeal to Lemma 6, obtaining

$$\int_{-\infty}^{\infty} x^2 \int_{a^{-1}(|x|)}^{\infty} \frac{1}{a^2(y)}\, dy \leq \frac{1}{2\alpha - 1} + \frac{1}{2\alpha - 1} \int_{\{|x|>a_1\}} \left(|x| \int_1^{a^{-1}(|x|)} (1/a(y))\, dy\right) dF(x) .$$

Now by Theorem 3, $\lim_{n\to\infty} E \max_{1\leq k\leq n} |\sum_{j=1}^k (X_j)/a_j| < \infty$. Hence (16) is false. As in Kronecker's lemma, let $b_0 = 0$ and $b_n = \sum_{k=1}^n (X_k)/a_k$. $X_n = a_n(b_n - b_{n-1})$ so

$$\frac{S_n}{a_n} = \sum_{k=1}^n \frac{a_k(b_k - b_{k-1})}{a_n} = \sum_{k=1}^n \frac{a_k b_k}{a_n} - \sum_{k=1}^{n-1} \frac{a_{k+1} b_k}{a_n}$$
$$= b_n - \sum_{k=1}^{n-1} \frac{(a_{k+1} - a_k) b_k}{a_n} .$$

Hence $\max_{1\leq k\leq n} |S_k|/a_k \leq 2 \max_{1\leq k\leq n} |b_k|$, therefore

$$E \sup_{n\geq 1} \frac{|S_n|}{a_n} = \lim_{n\to\infty} E \max_{1\leq k\leq n} \frac{|S_k|}{a_k} \leq \lim_{n\to\infty} 2E \max_{1\leq k\leq n} |b_k| < \infty$$

so (17) fails and thus also (18).

$$\frac{|X_n|}{a_n} \leq \frac{|S_n|}{a_n} + \frac{|S_{n-1}|}{a_{n-1}} .$$

Hence (19) and (20) are also invalid. To complete the proof it suffices to verify that (17) $\Rightarrow$ (18) and (19) $\Rightarrow$ (20).

Suppose (17) holds. $\exists c_n \ni (c_n)/a_n \nearrow \infty$ and $E \sup_{n\geq 1} |S_n|/c_n = \infty$. (Take $c_n = a_n(E \max_{1\leq k\leq n} |S_k|/a_k)^{\frac{1}{2}}$.) Since (15) $\Leftrightarrow$ (17), extending c_n to a continuous function such that $c(y)/y^\alpha \nearrow$ we have

$$\int_{\{|x|>c_1\}} \left(|x| \int_1^{c^{-1}(|x|)} (1/c(y))\, dy\right) dF(x) = \infty .$$

Assume that $P(|S_n| \geq c_n \text{ i.o.}) = 1$. Choose integers $0 < k_1 < k_2 < \cdots$ such that $c_{k_n}/a_{k_n} \geq n^2$.

Let $t_1 = 1\text{st}\, j \geqq k_1 : |S_j|/c_j \geqq 1$. Having defined $t_1, \cdots, t_{n-1}$, let $t_n = 1\text{st}\, j \geqq k_n + t_{n-1} : |S_j|/c_j \geqq 1$. Each t_n is defined a.s.

Now introduce random variables $Y_1, Y_2, \cdots$ such that $X_1, Y_1, X_2, Y_2 \cdots$ are independent and

$$Y_n = 1 \quad \text{wp } 1/2n^2$$
$$= 0 \quad \text{otherwise.}$$

Let

$$\tau = 1\text{st}\, t_n : Y_n = 1 \quad \text{if such an } n \text{ exists}$$
$$= \infty \quad \text{otherwise.}$$

τ is a (randomized) extended-valued stopping rule.

$$E\frac{S_\tau}{a_\tau} = \sum_{n=1}^\infty E\left|\frac{S_{t_n}}{a_{t_n}} 1_{\{Y_n=1, Y_1=\cdots=Y_{n-1}=0\}}\right|$$
$$= \sum_{n=1}^\infty E\frac{|S_{t_n}|}{a_{t_n}} P(Y_n = 1)(\textstyle\prod_{j=1}^{n-1} P(Y_j = 0))$$
$$\geqq \prod_{j=1}^\infty \left(1 - \frac{1}{2j^2}\right) \sum_{n=1}^\infty E\frac{c_{t_n}}{a_{t_n}}\left(\frac{1}{2n^2}\right)$$
$$\geqq \prod_{j=1}^\infty \left(1 - \frac{1}{2j^2}\right) \sum_{n=1}^\infty \frac{c_{k_n}}{(a_{k_n})}\left(\frac{1}{2n^2}\right)$$
$$\geqq \prod_{j=1}^\infty \left(1 - \frac{1}{2j^2}\right) \sum_{n=1}^\infty \frac{n^2}{2n^2} = \infty .$$

Now assume $P(|S_n| \geqq c_n \text{ i.o.}) = 0$. Then $P(|X_n| \geqq 2c_n \text{ i.o.}) = 0$. $c_n/n^\alpha \nearrow \infty$ so $P(|X_n| \geqq c_n \text{ i.o.}) = 0$ (Lemma 2). Choose N such that

$$\inf_{n \geqq N} P(|S_{n-1}| < c_n \textstyle\bigcap_{j=N}^\infty \{|X_j| < c_j\}) > \frac{1}{2} .$$

Let

$$t = 1\text{st}\, n \geqq N : |X_n| \geqq c_n \text{ and } |S_{n-1}| \leqq c_n \quad \text{if such } n \text{ exists}$$
$$= \infty \quad \text{otherwise.}$$

$$E\frac{|S_t|}{c_t} \geqq E\frac{|X_t|}{c_t} - 1$$
$$= \sum_{n=N}^\infty \frac{\int_{\{t\geqq n, |S_{n-1}|<c_n\}\{|X_n|\geqq c_n\}} |X_n|\, dP}{c_n} - 1$$
$$= \sum_{n=N}^\infty \frac{P\{t \geqq n, |S_{n-1}| < c_n\} \int_{\{|x|\geqq c_n\}} |x|\, dF(x)}{c_n} - 1$$
$$\geqq \sum_{n=N}^\infty \frac{P\{\bigcap_{j=N}^\infty \{|X_j| < c_j\}, |S_{n-1}| < c_n\}}{c_n} \int_{\{|x|\geqq c_n\}} |x|\, dF(x) - 1$$
$$\geqq \frac{1}{2} \sum_{n=N}^\infty \frac{\int_{\{|x|>c_n\}} |x|\, dF(x)}{c_n} - 1$$
$$= \infty \quad \text{by Corollary 1.}$$

Therefore (17) $\Rightarrow$ (18).

Lastly, assume (19) is true. Again $\exists c_n \ni c_n/a_n \nearrow \infty$ and $E \sup_{n\geq 1} |X_n|/c_n = \infty$. (Take $c_n = a_n(E \max_{1\leq k\leq n} |X_k|/a_k)^{\frac{1}{2}}$.)
Extending c_n to a continuous function in the appropriate manner we have that

$$\int_{\{|x|>c_1\}} \left(|x| \int_1^{c^{-1}(|x|)} (1/c(y))\, dy\right) dF(x) = \infty .$$

If $P(|X_n| \geq c_n \text{ i.o.}) = 1$, we proceed as in the S_τ/a_τ case, so suppose $P(|X_n| \geq c_n \text{ i.o.}) = 0$. Choose $N \ni P(\bigcap_{n=N}^{\infty} \{|X_n| < c_n\}) > \frac{1}{2}$.
Let

$$t = 1\text{st } n \geq N\colon |X_n| \geq c_n \quad \text{if such an } n \text{ exists}$$
$$= \infty \quad \text{otherwise.}$$

$$E\frac{|X_t|}{c_t} = \sum_{n=N}^{\infty} \frac{\int_{\{t\geq n, |X_n|\geq c_n\}} |X_n|\, dP}{c_n}$$
$$= \sum_{n=N}^{\infty} \frac{P(t \geq n) \int_{\{|x|\geq c_n\}} |x|\, dF(x)}{c_n}$$
$$\geq \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{\int_{\{|x|\geq c_n\}} |x|\, dF(x)}{c_n}\right) - \sum_{n=1}^{N-1} \frac{E|X_1|}{c_n}$$
$$= \infty \quad \text{by Corollary 1,}$$

completing the entire theorem.

THEOREM 7. *Let $\{X_n\}$ be a sequence of* i.i.d. *mean zero random variables. Let $a(y)$ be a nonnegative continuous function such that for some $\alpha > \frac{1}{2}$, $a(y)/y^\alpha \nearrow$. Assume also that* $\limsup_{y\to\infty} (a(y))/y > 0$.

Then (21)—(25) are equivalent.

(21) $$\int_{a_1}^{\infty} x \int_1^{a^{-1}(x)} (1/a(y))\, dy\, dF(x) = \infty$$

(22) $$E \sup_{n\geq 1} \frac{S_n^+}{a_n} = \infty$$

(23) $$\exists t \in T_\infty \ni E\frac{S_t}{a_t} = \infty$$

(24) $$E \sup_{n\geq 1} \frac{X_n^+}{a_n} = \infty$$

(25) $$\exists t \in T_\infty \ni E\frac{X_t^+}{a_t} = \infty .$$

PROOF. Assume (21). Let

$$t = 1\text{st } k\colon X_k \geq a_k \quad \text{if such } k \text{ exists}$$
$$= \infty \quad \text{otherwise.}$$

$$
\begin{aligned}
E\frac{X_t}{a_t} &= \sum_{n=1}^{\infty} \frac{\int_{\{t=n\}} X_n\, dP}{a_n} \\
&= \sum_{n=1}^{\infty} \frac{\int_{\{t\ge n, X_n \ge a_n\}} X_n\, dP}{a_n} = \sum_{n=1}^{\infty} \frac{P(t \ge n) \int_{a_n}^{\infty} x\, dF(x)}{a_n} \\
&\ge P(t=\infty) \sum_{n=1}^{\infty} \frac{\int_{a_n}^{\infty} x\, dF(x)}{a_n} \\
&\ge P(t=\infty)\left(-\frac{EX_1^+}{a_1} + \int_{a_1}^{\infty} \left(x \int_1^{a^{-1}(x)} (1/a(y))\, dy\right) dF(x)\right) \qquad \text{(by Lemma 1)} \\
&= \infty\,.
\end{aligned}
$$

Therefore (21) $\Rightarrow$ (25) $\Rightarrow$ (24).
For $j < n$,

$$
\begin{aligned}
\int_{\{t=n\}} X_j\, dP &= \frac{P(t=n)}{P(X_j < a_j)} \int_{-\infty}^{a_j} x\, dF(x) \\
&= P(t=n)E(X_j \mid X_j < a_j) \ge P(t=n)E(X_1 \mid X_1 < a_1)\,.
\end{aligned}
$$

As in McCabe and Shepp [7] and Klass [6],

$$
\begin{aligned}
E\frac{S_{t-1}}{a_t} &= \sum_{n=1}^{\infty} \frac{\sum_{j=1}^{n-1} \int_{\{t=n\}} X_j\, dP}{a_n} \ge \sum_{n=1}^{\infty} \frac{P(t=n)E(X_1 \mid X_1 < a_1)(n-1)}{a_n} \\
&\ge E(X_1 \mid X_1 < a_1) \sup_{n\ge 1} \frac{n}{a_n} > -\infty\,.
\end{aligned}
$$

Therefore $E(S_t/a_t) = \infty$ and (21) $\Rightarrow$ (23) $\Rightarrow$ (22).

Conversely, we verify that if (21) fails, then (22)—(25) are also false.

Let $Y_n = X_n^+ - EX_n^+$. $\{Y_n\}$ is a sequence of i.i.d. mean zero random variables which is bounded below. Hence

$$
EX_1^+ \int_1^{a^{-1}(X_1^+)\vee 1} (1/a(y))\, dy < \infty \Rightarrow E|Y_1| \int_1^{a^{-1}(|Y_1|)\vee 1} (1/a(y))\, dy < \infty\,.
$$

Let $T_n = Y_1 + \cdots + Y_n$.

Invoking Theorem 6, $E\sup_{n\ge 1} |T_n|/a_n < \infty$ and $E\sup_{n\ge 1} |Y_n|/a_n < \infty$.

$$
\begin{aligned}
E\sup_{n\ge 1} \frac{S_n^+}{a_n} &\le E\sup_{n\ge 1} \frac{\sum_{j=1}^n X_j^+}{a_n} \le E\sup_{n\ge 1} \frac{\sum_{j=1}^n Y_j}{a_n} + E\sup_{n\ge 1} \frac{\sum_{j=1}^n EX_j^+}{a_n} \\
&= E\sup_{n\ge 1} \frac{T_n}{a_n} + \sup_{n\ge 1} \frac{nEX_1^+}{a_n} < \infty\,,
\end{aligned}
$$

so (22) and (23) fail.

$$
E\sup_{n\ge 1} \frac{X_n^+}{a_n} \le E\sup_{n\ge 1} \frac{Y_n + EX_n^+}{a_n} \le E\sup_{n\ge 1} \frac{Y_n}{a_n} + \sup_{n\ge 1} \frac{EX_1^+}{a_n} < \infty\,.
$$

Hence (24) and (25) are also invalid, thereby completing the entire proof.

A couple of examples are in order to illustrate some of the results.

EXAMPLE 2. Let $a_n = n^{1/\alpha}$ for some $1 < \alpha < 2$.

$$\int_{\{|x|>1\}} |x| \int_1^{|x|^\alpha} \frac{1}{y^{1/\alpha}}\, dy\, dF(x) = \int_{\{|x|>1\}} |x| \left.\frac{y^{1-1/\alpha}}{1-1/\alpha}\right|_1^{|x|^\alpha} dF(x)$$
$$= \frac{\alpha}{\alpha-1} \int_{\{|x|>1\}} (|x|^\alpha - |x|)\, dF(x)\,.$$

Hence $E|X|^\alpha < \infty \Leftrightarrow E \sup_{n\geq 1} |S_n|/n^{1/\alpha} < \infty$. Also,

$$E|X|^\alpha < \infty \Rightarrow \lim_{n\to\infty} E\left|\sum_{k=1}^n \frac{X_k}{k^{1/\alpha}}\right| < \infty$$
$$\Rightarrow \sum_{k=1}^\infty \frac{X_k}{k^{1/\alpha}} \quad \text{converges a.s.}$$
$$\Rightarrow \frac{S_n}{n^{1/\alpha}} \to 0 \quad \text{a.s.}$$

Whenever $E|X|^\alpha < \infty\ \exists\, \tau \in T_\infty \ni$

$$E\frac{S_\tau}{\tau^{1/\alpha}} = \sup_{t\in T_\infty} E\frac{S_t}{t^{1/\alpha}} < \infty\,. \tag{26}$$

Theorem 6 also shows that $E|X|^\alpha < \infty$ is necessary for (26).

EXAMPLE 3. Let $a_n = (n \log n)$ for $n \geqq 3$ and extend it appropriately for $n = 1$ and 2.

$$a^{-1}(y) > \frac{y}{\log y} \qquad \text{for } y \geqq e$$

$$\lim_{x\to\infty} \frac{\int_e^{a^{-1}(x)} \left(\frac{1}{a(y)}\right) dy}{\int_{a^{-1}(x)}^x \left(\frac{1}{a(y)}\right) dy} \geqq \lim_{x\to\infty} \frac{\int_e^{x/\log x} \left(\frac{1}{y\log y}\right) dy}{\int_{x/\log x}^x \left(\frac{1}{y\log y}\right) dy}$$
$$= \lim_{x\to\infty} \frac{\log\log \frac{x}{\log x}}{\log\log x - \log\log\frac{x}{\log x}}$$
$$= \lim_{x\to\infty} \frac{\log\log x}{(\log\log x)/\log x} = \infty \qquad \text{therefore,}$$

$$\int_{a_1}^\infty x \int_1^{a^{-1}(x)} (1/a(y))\, dy\, dF(x) < \infty$$
$$\Leftrightarrow \int_e^\infty \left(x \int_e^x \frac{1}{y\log y} dy\right) dF(x)$$
$$= \int_e^\infty x \log\log x\, dF(x) < \infty\,.$$

Recalling the appropriate theorems, it is apparent that $EX_1^+ \log^+ \log X_1 < \infty$ is necessary and sufficient for there to exist $\tau \in T_\infty \ni$

$$E\frac{S_\tau}{(\tau\log\tau)\vee 1} = \sup_{t\in T_\infty} E\frac{S_t}{(t\log t)\vee 1} < \infty\,.$$

More generally, let $\log_{(0)} x = x$ and $\log_{(n)} x = \log (\log_{(n-1)} x)$. It is easy to verify that $\exists \tau \in T_\infty \ni$

$$E \frac{S_\tau}{(\log_{(0)} \tau \log_{(1)} \tau \cdots \log_{(n)} \tau) \vee 1} = \sup_{t \in T_\infty} E \frac{S_t}{(\log_{(0)} \tau \log_{(1)} t \cdots \log_{(n)} t) \vee 1} < \infty$$

iff $EX_1^+ \log^+_{(n+1)} X_1^+ < \infty$.

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