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CONVERGENCE OF SOME EXPECTED FIRST PASSAGE TIMES

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We discuss the convergence of the expected times until the partial sums of a sequence of independent, identically distributed random variables with zero means and unit variances first rise a height h above their previous minimum as $h \to \infty$. We also consider the convergence as $r \to \infty$ of the expected times until the range of these partial sums exceeds a value r. Applications of these results to a quality control procedure and to queueing theory are mentioned.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables with zero means and unit variances. Let $S_0 = 0$, $S_n = X_1 + X_2 + \cdots + X_n$, $W_0 = 0$, $W_n = S_n - \min_{0 \leq i < n} S_i$, and $R_n = \max_{0 \leq i \leq n} S_i - \min_{0 \leq i \leq n} S_i$. Define N_h as the smallest n such that $W_n \geqq h > 0$ and N_r^* as the smallest n such that $R_n \geqq r$. We prove the following two theorems.

THEOREM 1. $E(N_h/h^2) \to 1$ *as* $h \to \infty$.

THEOREM 2. $E(N_r^*/r^2) \to \frac{1}{2}$ *as* $r \to \infty$.

Let $Y(t)$, $0 \leqq t < \infty$, denote the standard Wiener process, $W(t) = Y(t) - \inf_{0 \leq t' < t} Y(t')$, τ_a the smallest t such that $W(t) \geqq a$, $R(t) = \sup_{0 \leq t' \leq t} Y(t') - \inf_{0 \leq t' \leq t} Y(t')$, and τ_a^* the smallest t such that $R(t) \geqq a$. It is shown in Robbins (1971) that $E(\tau_h/h^2) = E(\tau_1) = 1$, and in Nadler and Robbins (1971) that $E(\tau_r^*/r^2) = E(\tau_1^*) = \frac{1}{2}$. Therefore, these theorems show that with suitable norming the expectations of N_h and N_r^* converge to the expectations of the corresponding functionals of the Wiener process. The two problems are quite similar. In fact, R_{N_h} can be considered a two-sided version of W_{N_h}, since it is shown in Nadler and Robbins (1971) that $R_{N_h} \geqq h$ is equivalent to $S_{N_h} - \min_{0 \leq i < N_h} S_i \geqq h$ or $\max_{0 \leq i < N_h} S_i - S_{N_h} \geqq h$.

Page (1954) suggested the use of W_n for detecting a change in one direction only in the location of a parameter for quality control applications. To detect a change in either direction he suggested the use of the procedure equivalent to rejecting when $R_n \geqq r$. The average run lengths (the expected number of articles sampled at a given quality level before action is taken) for Page's one- and two-sided procedures are given by EN_h and EN_r^*, respectively. When there is no change in the parameter these average run lengths can be approximated by the expected time until a Wiener process rises a height h above its previous

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minimum, and the expected time until the range of a Wiener process exceeds r. Therefore, Theorems 1 and 2 show that the average run length functions, suitably normed, actually converge to the approximations to them when there is no change. These approximations were used in Robbins (1971) and Nadler and Robbins (1971). Note that $W_n \geqq h$ is equivalent to $S_n' \geqq h$ where $S_0' = 0$ and $S_k' = \max(S_{k-1}' + X_k, 0)$. This equivalence is pointed out in Page (1954). In this form it is clear that $\max_{0 \leqq i \leqq n} W_i$ represents the maximum waiting time of the first n customers in a single-server first-come-first-served queue, where the waiting time of the nth customer refers to the time from his entry to the system until the time his service commences. Therefore, Theorem 1 discusses the convergence to an approximation of the expected number of customers until the maximum waiting time exceeds a given value.

2. Proof of Theorem 1. Let $Z_n = \max_{0 \leqq k \leqq n}(S_k - \min_{0 \leqq i < k} S_i)$ and $Z(t) = \sup_{0 \leqq t' \leqq t}(Y(t') - \inf_{0 \leqq s < t'} Y(s))$. We first show that $N_h/h^2 \to_d \tau_1$. $Z_n/n^{\frac{1}{2}} \to_d Z(1)$ as $n \to \infty$ since the same functional applied to both the X_n and W of Billingsley's Donsker theorem in Billingsley (1968), page 68, yields the indicated random variables. For each $x > 0$,

$$\begin{aligned} P\{N_h/h^2 \leqq x\} &= P\{N_h \leqq xh^2\} \\ &= P\{N_h \leqq [xh^2]\} = P\{Z_{[xh^2]} \geqq h\} \\ &= P\{Z_{[xh^2]}/([xh^2])^{\frac{1}{2}} \geqq h/([xh^2])^{\frac{1}{2}}\} \to P\{Z(1) \geqq 1/x^{\frac{1}{2}}\} \end{aligned}$$

since the distribution of $Z(1)$ is shown to be continuous in Robbins (1971). But $P\{Z(1) \geqq 1/x^{\frac{1}{2}}\} = P\{\tau_{1/x^{\frac{1}{2}}} \leqq 1\} = P\{\tau_1 \leqq x\}$.

We next prove that N_h/h^2 is uniformly integrable. Since N_h is an increasing function of h it suffices to prove this for integral h. Let $A_j^{(h)} = \max_{(j-1)h^2 \leqq k \leqq jh^2}(S_k - \min_{(j-1)h^2 \leqq i < k} S_i)$. Note that $A_1^{(h)}, A_2^{(h)}, \cdots$ form a sequence of independent, identically distributed random variables. Let $p_h = P\{A_1^{(h)} \geqq h\}$ and define T_h to be the smallest n such that $A_n^{(h)} \geqq h$. Then T_h is a geometric random variable with probability of success p_h, $E(T_h) = 1/p_h$, and $\mathrm{Var}(T_h) = q_h/p_h^2$ where $q_h = 1 - p_h$. As $h \to \infty$, $p_h \to p$, where $p = P\{Z(h^2) \geqq h\} = \sum_{k=-\infty}^{\infty}(-1)^k\{\Phi(2k+1) - \Phi(2k-1)\}$. This follows from the well known equivalence of $Y(t) - \inf_{0 \leqq t' < t} Y(t')$ and $|Y(t)|$ which is shown in Chapter 6 of Lévy (1965). For any $\varepsilon > 0$ choose h_0 such that for all $h > h_0$, $p_h > \varepsilon$. For all $h > h_0$ the T_h form a uniformly integrable sequence of random variables, since $\sup_{h > h_0} E(T_h^2) < \infty$. But $N_h/h^2 \leqq T_h$. Therefore, for all $h > h_0$ the N_h/h^2 are uniformly integrable, so that $E(N_h/h^2) \to E(\tau_1) = 1$.

3. Proof of Theorem 2. By Donsker's theorem $R_n/n^{\frac{1}{2}} \to_d R(1)$ as $n \to \infty$. Then $N_r^*/r^2 \to_d \tau_1^*$ by the same argument as used in Theorem 1. The continuity of $R(1)$ follows from Nadler and Robbins (1971). Uniform integrability follows by noting that $N_r^*/r^2 \leqq N_r/r^2$.

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