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ON STATIONARITY AND SUPERPOSITION OF POINT PROCESSES

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This paper applies ideas from random set theory to simple point processes. We show stationarity of the hitting distributions suffices for the strict stationarity of a *simple* point process, but that in general all forms of stationarity differ. We compare and contrast the superposition operations of summation for random measures and union for random sets, specialized to point processes. Finally we consider completely random sets and their factors.

1. Basic definitions. There are various definitions of point processes in the literature; point processes can be viewed both as random measures and as random sets. In this paper we concentrate on the latter aspect. We need to allow 'sets' with multiple occurrences of points; we call such objects *multi-sets*.

Let X be a locally compact Hausdorff topological space with a countable base, $\mathscr{A}$ its Borel σ-field, $\mathscr{B}$ its class of bounded (relatively compact) sets, and $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$. (Those readers familiar with Ripley (1976) (which we will refer to as LFRS) will see we could generalize $(X, \mathscr{A}, \mathscr{B})$ to a standard bounded space. Indeed, for most of our results we only need $(X, \mathscr{A}, \mathscr{B})$ to be countably separated.)

Let N be the class of σ-additive functions $n: \mathscr{C} \to \mathbb{Z}_+$, the nonnegative integers, and $\mathscr{N}$ the smallest σ-field on N making the evaluation maps e_A measurable for all $A \in \mathscr{C}$. Each $n \in N$ is purely atomic (LFRS, Theorem 1) and corresponds to the multi-set of $n(\{x\})$ x's for each $x \in D(n) = \{x \mid n(\{x\}) > 0\}$. Such a multi-set is *locally finite*, i.e. its intersection with any bounded set is finite. We can identify N with the class of locally finite multi-sets, and its measurable subset $N_0 = \{n \mid n(\{x\}) \leqq 1 \ \forall\, x \in X\}$ with the class LF of locally finite *sets*. In particular D maps N_0 onto LF.

Every member of LF is closed, so D embeds N_0 in $\mathscr{F}$, the class of closed subsets of X. A specialization of the random set theory of Kendall (1974) gives $\mathscr{F}$ the σ-field $\mathscr{Y}$ generated by $\{\{F \mid F \cap G = \emptyset\} \mid G \in \mathscr{G}\}$, where $\mathscr{G}$ is the class of open sets, and Matheron (1975) gives F the σ-field $\mathscr{Y}'$ generated by $\{\{F \mid F \cap K = \emptyset \mid K \in \mathscr{K}\}$, $\mathscr{K}$ the class of compact sets. One can show $\mathscr{Y} = \mathscr{Y}'$ (cf. LFRS). Our key tool is the result that $D(N_0 \cap \mathscr{N}) = LF \cap \mathscr{Y}$ (for a proof see LFRS Section 5, cf. also Kallenberg (1973)). In words, a σ-field on LF containing the

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events of hitting open (or compact) sets makes measurable the maps counting the number of points in each bounded measurable set.

We define a *point process* to be a measurable map Z from a probability space to $(N, \mathscr{N})$, and a random set to be a measurable map to $(\mathscr{F}, \mathscr{Y})$. The *distribution* (or *name* (Kendall (1974))) of a point process or random set is the probability induced on $\mathscr{N}$ or $\mathscr{Y}$. The distribution P or a random set is uniquely specified by the avoidance function A defined on $\mathscr{G}$ (Kendall) or $\mathscr{K}$ (Matheron) by $A(E) = P(\{F \mid F \cap E = \emptyset\})$.

We say a point process is *simple* if its range is a.s. in N_0; this is a property of the distribution. A simple point process is precisely an a.s. locally finite random set. Let $\mathscr{P}$ and $\mathscr{P}_0$ be the classes of probabilities on $\mathscr{N}$ and $\mathscr{N}_0 = N_0 \cap \mathscr{N}$. We will identify $\mathscr{P}_0$ with $\{P \mid P \in \mathscr{P}, P(N_0) = 1\}$.

2. Stationarity. Suppose G is a topological group acting continuously on X (i.e., there is a continuous map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $ex = x$). We define gn for $g \in G$ and $n \in N$ by $gn(A) = n(g^{-1}A)$ for $A \in \mathscr{C}$, and $P_g(A) = P(\{n \mid gn \in A\})$ for $A \in N$. We say $P \in \mathscr{P}$ is:

(a) *avoidance stationary* if for all $g \in G$ P_g and P agree on $\{\{n \mid n(E) = 0\} \mid E \in \mathscr{C}\}$,

(b) *m-stationary* if for all $g \in G$ P_g and P agree on sets of the form $\bigcap_1^m \{n \mid n(A_i) = k_i\}$, $A_i \in \mathscr{C}$, $k_i \in \mathbb{Z}_+$,

(c) *strictly stationary* if P is m-stationary for every m, or, equivalently, if for all $g \in G$ P_g and P agree on $\mathscr{N}$.

Obviously strict stationarity implies m- stationarity, and 1-stationarity implies avoidance stationarity.

PROPOSITION 1.[1] *For the distribution P of a simple point process all three forms of stationarity are equivalent; strict stationarity is implied by the agreement of P and P_g on $\{\{n \mid n(E) = 0\} \mid E \in \mathscr{T}\}$ for any class $\mathscr{T}$ to which the corollary of Theorem* 4 *of* LFRS *applies, in particular $\mathscr{G}$ and $\mathscr{K}$.*

PROOF. If $P \in \mathscr{P}$ is avoidance stationary then P_g and P agree on $\{\{n \mid n(E) = 0\}E \in \mathscr{T}\}$, so $P_g = P$ by the cited result.

This results can be very useful, especially for $X = G = \mathbb{R}$. Notice that stationarity in the random set sense, avoidance stationarity with $\mathscr{T} = \mathscr{G}$, is equivalent to strict stationarity as a point process for an a.s. locally finite random set.

Obviously a point process can be avoidance stationary but not even 1-stationary; take a Poisson point process on $\mathbb{R}$ and double each point in $(0, 1)$. The following example shows a point process can be 1-stationary but not 2-stationary or strictly stationary.

EXAMPLE 1. We define a point process on $\mathbb{R}$ with points only in $\{m/2 \mid m \in \mathbb{Z}\}$

[1] Proposition 1 is an extension of 3.1.8 of [9]. Matthes has told me that the general solution to his problem is negative; a counterexample will be published in the forthcoming English edition of [9].

points occurring independently in each interval $[m, m+1)$ with the following probabilities:

$n(\{m\})$	$n(\{m+\frac{1}{2}\})$	m odd 0	1	2	m even 0	1	2
0		$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0
1		$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	$\frac{1}{9}$	$\frac{2}{9}$
2		$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{1}{9}$.

This process has the property that the distribution of $n \to n(A)$ depends only on card $(A \cap \{m/2 \mid m \in \mathbb{Z}\})$. We form a 1-stationary process by translating this process by a random variable uniformly distributed on $(0, \frac{1}{2})$. Consider the event $A_m = \{n \mid n([m, m+\frac{1}{2})) = 2,\ n([m+\frac{1}{2}, m+1)) = 0\}$. Then $P(A_m) = \frac{1}{9}$ or 0 for m odd or even. Thus the process is not 2-stationary.

3. Superposition. The random measure "sum" operation corresponds to defining $(n_1 + n_2)(A) = n_1(A) + n_2(A)$. We may also define the union by $n_1 \cup n_2 = D^{-1}(Dn_1 \cup Dn_2)$, D^{-1} denoting the unique inverse in N_0. Then $(N, +)$ and $(N_0, \cup)$ are commutative semigroups with identities. Obviously $+$ is measurable, and D and D^{-1} are measurable, so $\cup$ is measurable. (A third operation, intersection of the corresponding sets, occurs in connection with infinitely divisible regenerative phenomena (Kendall (1967)).)

Suppose Z_1 and Z_2 are point processes defined on the same probability space. We define $Z_1 + Z_2$ and $Z_1 \cup Z_2$ by performing the indicated operation on each realization. Both are point processes, $Z_1 \cup Z_2$ being simple. Suppose Z_1 and Z_2 are independent with distributions P and Q. We define $P * Q$ and $P \vee Q$ as the distributions of $Z_1 + Z_2$ and $Z_1 \cup Z_2$. Note that $P \vee Q \in \mathscr{P}_0$.

For $P \in \mathscr{P}$ we define $\mu_P(E) = \sum rP(\{n \mid n(E) = r\}) \leqq \infty$ on $\mathscr{C}$. We call μ_P the *first moment measure* and say P is *first-order* if μ_P is finite. A lemma of Jagers (1973) shows that, if P and Q are *first-order*, $P * Q = P \vee Q$ if and only if $P, Q \in \mathscr{P}_0$ and μ_P and μ_Q have no common atoms.

Both $(\mathscr{P}, *)$ and $(\mathscr{P}_0, \vee)$ are commutative semigroups with identities.

4. Infinite divisibility. We denote by $\mathscr{I}(\mathscr{P})$ and $\mathscr{I}(\mathscr{P}_0)$ the classes of infinitely divisible elements of $(\mathscr{P}, *)$ and $(\mathscr{P}_0, \vee)$. (An element is infinitely divisible if it has an nth root for each n; such an nth root is unique). The structure of $\mathscr{I}(\mathscr{P})$ is well known (Lee (1967), Kerstan, Matthes and Mecke (1974)).

We will characterize $\mathscr{I}(\mathscr{P}_0)$. Suppose $P \in \mathscr{P}_0$. Let $F(P) = \{x \mid P(E_x) = 1\}$, where $E_x = \{n \mid n(\{x\}) > 0\}$. Thus $F(P)$ is the set of fixed points of a point process with distribution P. Let $n_P = D^{-1}(F(P))$, and DP be the unit mass at n_P. Define $RP \in \mathscr{P}_0$ by $RP(A) = P(\{n \mid n \geqq n_P,\ n - n_P \in A\})$. Then $P = DP \vee RP$. Thus we factor P into a fixed part and a part without fixed points. Obviously for $P \in \mathscr{I}(\mathscr{P})$, $F(P) = \emptyset$. Always $DP \in \mathscr{I}(\mathscr{P}_0)$, being its own nth root. Suppose $P \in \mathscr{I}(\mathscr{P}_0)$. Then $P = Q_n{}^{n\vee}$ for $Q_n \in P_0$, so $RP = R(Q_n{}^{n\vee}) = (RQ_n)^{n\vee} \in \mathscr{I}(\mathscr{P}_0)$. Let $\mathscr{I}' = \{P \mid P \in \mathscr{I}(\mathscr{P}_0),\ F(P) = \emptyset\}$. Thus $\mathscr{I}(\mathscr{P}_0) = \{\varepsilon_n \vee P \mid n \in N_0,\ P \in \mathscr{I}'\}$.

Let $\mathscr{S}$ denote the class of probabilities on $\mathscr{Y}$. We can define $\vee$ on $\mathscr{S}$ by

$(S_1 \vee S_2)(\{F \mid F \cap E = \emptyset\}) = S_1(\{F \mid F \cap E = \emptyset\}) \times S_2(\{F \mid F \cap E = \emptyset\})$ for $E \in \mathscr{G} \cup \mathscr{K}$, and so embed $(\mathscr{P}_0, \vee)$ in $(\mathscr{S}, \vee)$. Then $P \in \mathscr{P}_0$ is infinitely divisible in $(\mathscr{S}, \vee)$ if and only if $P \in \mathscr{I}(\mathscr{P}_0)$. Matheron (1975) has characterized the infinitely divisible members of $(\mathscr{S}, \vee)$ with no fixed points. Specializing his result to locally finite sets we see each member of $\mathscr{I}'$ is uniquely represented by a measure ν on $Y = N_0 \backslash \{0\}$. Let $\mathscr{Y} = Y \cap \mathscr{N}$ and $\mathscr{W}$ be the ideal of subsets of Y generated by $\{\{n \mid n(E) > 0\} \mid E \in \mathscr{C}\}$. Then $(Y, \mathscr{Y}, \mathscr{W})$ is a bounded space (LFRS), and we can define a Poisson probability on this space with mean measure ν if ν is finite on $\mathscr{Y} \cap \mathscr{W}$. We define maps $\phi: N(Y) \to N(X)$ and $\psi: N(Y) \to N_0(X)$ by $\phi(m) = \sum_{m(\{n\})>0} m(\{n\})n$ and $\psi = S \circ \phi$, where $S = D^{-1} \circ D$. Then both ϕ and ψ are measurable, and each $P \in \mathscr{I}'$ is the image under ψ of a Poisson probability on $N(Y)$. The converse is obvious. Thus an infinitely divisible member of $(\mathscr{P}_0, \vee)$ is the union of a fixed locally finite set, and locally finite sets selected by a Poisson process so that only a finite number meet each bounded set.

Suppose $P \in \mathscr{I}(\mathscr{P})$. Then $SP \in \mathscr{I}(\mathscr{P}_0)$ (since $S(Q^{n*}) = (SQ)^{n\vee}$) and $S(\mathscr{I}(\mathscr{P})) \subset \mathscr{I}'$. Conversely, suppose $P \in \mathscr{I}'$. Then the image Q under ϕ of the corresponding Poisson process on Y is a member of $\mathscr{I}(\mathscr{P})$, and $SQ = P$. Thus $\mathscr{I}' = S(\mathscr{I}(\mathscr{P}))$, so a simple point process, infinitely divisible under union, differs by a fixed part from a point process infinitely divisible under summation, viewed as a random *set*.

5. Convergence of unions. Suppose (P_{ni}) is a triangular array from $\mathscr{P}$ which is infinitesimal, i.e., $\lim_n \max_i P_{ni}(\{n \mid n(E) > 0\}) = 0$ for all $E \in \mathscr{C}$. Let $P_n = *_i P_{ni}$ and $Q_n = \bigvee_i P_{ni}$ be the row sum and union. We will give $\mathscr{P}$ the topology of finite-dimensional convergence, so $P_n \to P$ if $P_n(A) \to P(A)$ for all sets of the form $\bigcap_1^m \{n \mid n(A_i) = k_i\}$, $A_i \in \mathscr{C}$, $m, k_i \in \mathbb{Z}_+$. It follows from the corresponding result for random vectors that the class of limits of row sums of infinitesimal triangular arrays is $\mathscr{I}(\mathscr{P})$ (the "central limit theorem" for $(\mathscr{P}, *)$). It is not obvious that this theorem holds for $(\mathscr{P}_0, \vee)$. We do have the following half of this result.

PROPOSITION 2. *Every infinitely divisible member of* $(\mathscr{P}_0, \vee)$ *is the limit of the row unions of an infinitesimal triangular array from* $\mathscr{P}_0$.

PROOF. We have $P = DP \vee RP$. Let S_n be the nth root of RP, and let T_n be the probability making each point of $F(P)$ occur independently with probability $a_n = 1/\log n$. Let $P_{ni} = S_n \vee T_n$, $i = 1, \cdots, n$. Then $Q_n = RP \vee T_n^{n\vee}$ and so has the same finite-dimensional distributions as P for sets disjoint from $F(P)$. Since $F(P)$ is locally finite it suffices to show that $T_n^{n\vee}(E_x) \to 1$ for each $x \in F(P)$, which follows from the choice of a_n. Thus $Q_n \to P$. The array is infinitesimal since $P_{ni}(\{n \mid n(E) > 0\}) \leqq T_n(\{n \mid n(E) > 0\}) + (1 - P(\{n \mid n(E) = 0\})^{1/n}) \to 0$.

The following result enables us to transfer results from $(\mathscr{P}, *)$ to $(\mathscr{P}_0, \vee)$, in particular convergence to a Poisson process (cf. Çinlar (1972)).

THEOREM 1. *Suppose* $P \in \mathscr{P}_0$ *and* μ_P *is nonatomic and finite. Suppose* (P_{ni}) *is an* *infinitesimal triangular array of first-order members of* $\mathscr{P}_0$. *Then* $Q_n \to P$ *if and only if* $P_n \to P$.

PROOF. Let $C = \{x \mid P_{ni}(E_x) > 0$ for some $n, i\}$. Then C is locally countable. By the remarks in Section 3 the finite-dimensional distributions of P_n and Q_n coincide for sets disjoint from C. Now $P_n(\{n \mid n(E \cap C) = 0\}) = Q_n(\{n \mid n(E \cap C) = 0\}) \to 1$ if $P_n \to P$ or $Q_n \to P$, so $P_n \to P$ if and only if $Q_n \to P$.

Notice that $(\mathscr{P}_0, \vee)$ cannot be Delphic (Kendall (1968)) because the circuitous proof of Proposition 2 shows there is no suitable homomorphism. I do not know whether $(\mathscr{P}, *)$ is Delphic.

6. Completely random sets. We say a random set Z is completely random if $(Z \cap E_i = \varnothing)$ are independent whenever (E_i) is a disjoint subclass of $\mathscr{K}$ (and hence for $\mathscr{K}_\sigma$). If A is the avoidance function of a completely random set we define $\psi(E) = -\log A(E)$ for $E \in \mathscr{K}_0$. Then ψ is additive and monotone on $\mathscr{K}$. If $E_1, E_2 \in \mathscr{K}$, $E_3 = E_2 \backslash E_1 \in \mathscr{K}_0$, so $\psi(E_1 \cup E_2) = \psi(E_1 \cup E_3) = \psi(E_1) + \psi(E_3) \leqq \psi(E_1) + \psi(E_2)$, so ψ is also subadditive.

Let $F_0 = (\bigcup \{E \mid E \in \mathscr{G}, A(E) > 0\})^c$, which is closed. Suppose $x \in F_0$. Then $A(\{x\}) = \sup \{A(E) \mid x \in E \in \mathscr{G}\} = 0$ (Matheron (1975) 2-1-1) so F_0 is contained in the class of fixed points. Suppose $K \in \mathscr{K}$ and $K \cap F_0 = \varnothing$. Then $K \subset E_1 \cup \cdots \cup E_n$, $E_i \in G$, so $A(K) > 0$, and F_0 is the class of fixed points. Replacing X by $X \backslash F_0$ we may assume A is positive, so ψ is finite. Thus ψ is a content which has an extension (by the cited equation) to a Borel measure ν (Halmos (1950) 53). Thus $A(K) = \exp -\nu(K)$ for all $K \in \mathscr{K}$, and A is the avoidance function of the locally finite random set generated by the (not necessarily simple) Poisson point process with mean measure ν and distribution P_ν. Thus $P = DP_\nu \vee \varepsilon_{F_0}$.

We say a point process is completely random if $(Z(A_i))$ are independent whenever (A_i) is a finite disjoint subclass of $\mathscr{C}$. If Z is a completely random simple point process, $D \circ Z$ is a completely random set with distribution $DP \vee \varepsilon_F$ for $F \in LF$. Thus Z is completely random if and only if $D \circ Z$ is a completely random set. Thus Z differs from a simple point process on a locally countable set in which points occur independently with positive probability, a result which can also be derived by random measure theory (Kingman (1967), Jagers (1974)).

Suppose a completely random point process Z is the sum $Z_1 + Z_2$ of independent point processes. Matthes (cf. Daley (1971)) asked whether Z_1 and Z_2 are necessarily completely random. We show this is so if Z is simple.

THEOREM 2. *The distributions of completely random simple point processes form a hereditary subclass of* $(\mathscr{P}_0, *)$.

PROOF. Let Z, Z_1 and Z_2 be given as above, Z being simple. Let $C = \{x \mid \Pr (Z(\{x\}) > 0) > 0\}$, and let C_1 and C_2 be the corresponding sets for C_1 and C_2. Then C is countable and Z restricted to C^c is a Poisson process with mean measure ν finite on $\mathscr{C}$. Suppose $B \in \mathscr{C}$ and $B \cap C = \varnothing$. Then $Z(B) = Z_1(B) + Z_2(B)$ is Poisson, so $Z_1(B)$ and $Z_2(B)$ are Poisson (Raikov's theorem, Loève (1963), Section 19.2). In particular $\Pr(Z_2(B) = 0) > 0$. Now suppose $A \subset C_1$. Then $\Pr(Z(A \cup B) = 0) = \Pr(Z_1(A \cup B) = 0)\Pr(Z_2(A \cup B) = 0) = \Pr(Z_1(A \cup B) = 0)\Pr(Z_2(B) = 0)$ since $Z_2(C_1) = 0$ a.s. Also $\Pr(Z(A \cup B) = 0) = \Pr(Z(A) = 0)\Pr(Z(B) = 0) = \Pr(Z_1(A) = 0)\Pr(Z_1(B) = 0)\Pr(Z_2(B) = 0)$. Thus $\Pr(Z_1(A \cup B) = 0) = \Pr(Z_1(A) = 0)\Pr(Z_1(B) = 0)$. Now the restrictions of Z and Z_1 to C_1 agree a.s., and the restriction of Z_1 to $C\backslash C_1$ is null a.s., so Z_1 restricted to C is completely random. Also Z_1 restricted to C^c is Poisson (LFRS, Theorem 4, cf. Rényi (1967)) and so is completely random. By the last equation $D \circ Z_1$ is a completely random set and so Z_1 is completely random.

The following example shows this theorem is false for $(\mathscr{P}_0, \vee)$ on any space with two or more points.

EXAMPLE 2. Let $X = \{a, b\}$. We define P and Q by the following tables. Neither is completely random, but $P \vee Q$ is.

	P		Q		$P \vee Q$	
a \ b	0	1	0	1	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{64}$	$\frac{7}{64}$
1	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{7}{64}$	$\frac{49}{64}$

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