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# DISTRIBUTION ESTIMATES FOR FUNCTIONALS OF THE TWO-PARAMETER WIENER PROCESS[1]

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Bounds on absorption probabilities for Banach space-valued Brownian motion are obtained as expectations of estimates for the conditional probability given the endpoint of the path. The results are applied to the problem of computing the tail distributions of the supremum, $S$, of the two-parameter Wiener process and the supremum, $S'$, of its tied-down version. It is shown that for $\lambda \geqq 0$,

$$P\{S' \geqq \lambda\} \geqq (2\lambda^2 + 1)\exp[-2\lambda^2]$$

and

$$P\{S \geqq \lambda\} \geqq 4 \int_\lambda^\infty sN(-s)\, ds$$

where $N(s)$ denotes the standard normal distribution. A corollary is that $P(S \geqq \lambda) \approx 4N(-\lambda)$ as $\lambda \to +\infty$.

**1. Vector-valued Brownian motion and the $N$-parameter Wiener process.** A probability measure, $\mu$, defined on the Borel field of a real separable Banach space, $B$, is said to be mean-zero Gaussian if each element of $B^*$ has a Gaussian distribution with mean zero. One may define transition probabilities for a $B$-valued independent increment process, $\{W(t)\}_{t \geqq 0}$, with the formula

$$P\{x + W(t) \in E\} = \mu(t^{-\frac{1}{2}}(E - x)) \tag{1}$$

and as shown in [8] and more recently discussed in [11], there exists a separable process with continuous sample paths satisfying the above equation. The process is said to be Brownian motion in $B$ generated by $\mu$, and, in particular, we may consider the process $\{W(s)\}_{0 \leqq t \leqq 1}$ which we refer to as the Wiener process. The case $B = R^1$ with $\mu$ the standard normal measure gives the canonical Wiener process [7], [15].

The $N$-parameter Wiener process is defined to be a real-valued mean-zero Gaussian process $\{X(\mathbf{t}) : \mathbf{t} = (t_1, \cdots, t_N), t_j \geqq 0\}$ such that

$$E[X(\mathbf{s})X(\mathbf{t})] = \prod_{j=1}^N (s_j \wedge t_j)$$

and such that with probability one, $X(\mathbf{t})$ is continuous in $\mathbf{t}$, [12]. Throughout this paper, the parameter set will be restricted to $[0, 1]^N$ and thus the process $X(\mathbf{t})$ uniquely determines a probability measure on the Banach space, $C_0[0, 1]^N$, the space of all real-valued continuous functions defined on $[0, 1]^N$ which vanish if at least one of the coordinates vanish. A proof of existence of the above

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process is contained in the following theorem, due to Kuelbs [11], which relates vector-valued Brownian motion and the $N$-parameter Wiener process.

THEOREM 1 (Kuelbs). *A separable, continuous version of the $N$-parameter Wiener process is given by the $C_0[0, 1]^{N-1}$ valued Wiener process generated by the $N-1$ parameter process under the correspondence*

$$X(\mathbf{t}) = W(t_1)(t_2, \cdots, t_N) .$$

**2. Estimation of absorption probabilities for Brownian motion.**

DEFINITION 1. Let $\{W(t)\}_{t \geqq 0}$ denote the Brownian motion generated by a Gaussian probability measure on a Banach space, $B$. If $U \subset B$ is a Borel set the *transition probabilities for the Brownian motion absorbed by $U'$* are the subprobability measures $q_t(U, x, dy)$ given by

$$q_t(U, x, E) \equiv P\{x + W(t) \in E, x + W(s) \in U \quad \forall s \leqq t\}$$

where $t \geqq 0$, $x \in B$, and $E \subset B$ is Borel. The above measures form a Feller semigroup for a Markov process which has been studied by Gross [8]. Following the notation of [8], we write $p_t(x, dy)$ for the transition probability in the case $U = B$ and in those cases where the choice of $U$ is clear, we omit the parameter $U$ when referring to $q_t(U, x, dy)$.

DEFINITION 2. *The transition density of $q_t(U, x, dy)$ relative to Brownian motion* is a measurable function

$$g_t(U, x, y) \equiv \frac{dq_t(U, x, \cdot)}{dp_t(x, \cdot)}$$

given by the Radon–Nikodym derivative. Clearly, for each fixed $x$ in $B$, $g_t(U, x, y) \leqq 1$ a.s. relative to $p_t(x, dy)$.

Estimates of the absorption probabilities $1 - q_t(U, x, U)$ will be made by estimating the corresponding transition densities. The technique is to replace the set $U$ by halfspaces so that the resulting transition density has an explicit form. For this purpose we introduce some notation.

An element $e \in B^*$ is said to be *normalized* relative to a Brownian motion $\{W(t)\}_{t \geqq 0}$ iff

$$E[\langle e, W(1)\rangle^2] = 1 .$$

Let

$$s(U, e) \equiv \sup\{\langle e, y\rangle : y \in U\} \quad \text{for a subset} \quad U \quad \text{of} \quad B .$$

PROPOSITION 1. *Let $e \in B^*$ be normalized for a Brownian motion $\{W(t)\}$. The transition density for the Brownian motion absorbed by $\{y : \langle e, y\rangle > \lambda\}$ is given by*

$$1 - \exp[-(2/t)(\langle e, y\rangle - \lambda)(\langle e, x\rangle - \lambda)]$$

*for $x, y$ such that $\langle e, x\rangle$, $\langle e, y\rangle \leqq \lambda$.*

PROOF. The result of Fernique [6] implies that $E[\|W(1)\|^2] < \infty$. Since the random vector $W(1)$ has separable range, the Bochner integral

$$z = E[\langle e, W(1)\rangle W(1)]$$

exists. Now the map $Px = \langle e, x\rangle z$ is a projection onto span $\{z\}$ since $P^2x = \langle e, z\rangle Px$ and $\langle e, z\rangle = E[\langle e, W(1)\rangle^2] = 1$. Moreover, $I - P$ is a projection onto the null space of $e$ since $\langle e, Px\rangle = \langle e, x\rangle$. Consider the stochastic process

$$W'(t) \equiv (I - P)W(t) .$$

It is clear that $\{W'(t)\}$ is a Gaussian independent increment process with continuous sample paths. Let $\{b(t)\}_{t\geqq 0}$ denote a one-dimensional Brownian motion which is independent of the process $\{W'(t)\}$. We claim that the process

$$W'(t) + b(t)z$$

is a version of the Brownian motion, $\{W(t)\}$. The above process is clearly a Gaussian independent increment process with continuous sample paths. Thus, it suffices to check that the distribution of the random vector

$$W'(1) + b(1)z$$

agrees with the distribution of $W(1)$. Since both distributions are mean zero Gaussian, it suffices to verify (see [11]) that the variance of each $f \in B^*$ is given by $E[\langle f, W(1)\rangle^2]$. This is a routine calculation which we leave to the reader.

We then have

$$\begin{aligned} q_t(x, E) &= P\{x + W(t) \in E, \langle e, x\rangle + \langle e, W(s)\rangle \leqq \lambda \ \ \forall s \leqq t\} \\ &= P\{(I - P)x + W'(t) + Px + b(t)z \in E, \langle e, x\rangle + b(s) \leqq \lambda \ \ \forall s \leqq t\} . \end{aligned}$$

The transition density for one-dimensional Brownian motion absorbed by the set $\{r : r > \lambda\}$ is found from a formula in Feller [5] to be

$$g_t(r, s) = 1 - \exp[-(2/t)(s - \lambda)(r - \lambda)]$$

for $r, s \leqq \lambda$. By choosing sets $E$ of the form $F + Gz$ where $F$ is a Borel subset of the null space of $e$ and $G \subset R^1$ one obtains the formula

$$q_t(x, E) = \int_{E \cap \{y:\langle e,y\rangle \leqq \lambda\}} \{1 - \exp[-(2/t)(\langle e, y\rangle - \lambda)(\langle e, x\rangle - \lambda)]\} p_t(x, dy) .$$

THEOREM 2. *If $\{e_n\} \subset B^*$ is a sequence of normalized elements for a Brownian motion, then the transition density $g_t(U, x, y)$ for a Borel absorbing set $U'$ satisfies*

$$g_t(U, x, y) \leqq [1 - \exp[-(2/t) \inf_n (\langle e_n, y\rangle - s(U, e_n))(\langle e_n, x\rangle - s(U, e_n))]] I_U(y)$$

a.s. *relative to $p_t(x, dy)$ for each $x$ in $U$.*

PROOF. Let $U_n = \{y : \langle e_n, y\rangle \leqq s(U, e_n)\}$. Then $U \subset U_n$ and hence for every Borel set $E \subset U$,

$$q_t(U, x, E) \leqq q_t(U_n, x, E) .$$

In view of Proposition 1, we may express the above inequality as

$$\begin{aligned} &\int_E g_t(U, x, y) p_t(x, dy) \\ &\qquad \leqq \int_E (1 - \exp[-(2/t)(\langle e_n, y\rangle - s(U, e_n))(\langle e_n, x\rangle - s(U, e_n))]) p_t(x, dy) . \end{aligned}$$

This yields the desired inequality since the integrands must satisfy the inequality a.s.

**3. Distribution inequalities for functionals of the two parameter Wiener process.** Let $X(s_1, s_2)$ denote the two parameter Wiener process. We obtain estimates for the distribution of the functionals

$$S \equiv \sup_{0 \leqq s_i \leqq 1} X(s_1, s_2)\,, \qquad S' \equiv \sup_{0 \leqq s_i \leqq 1} [X(s_1, s_2) - s_1 s_2 X(1, 1)]\,.$$

The distributions arise as asymptotic distributions in certain limit problems. In fact, the distribution of $S'$ is the asymptotic distribution for a multivariate analog of the Smirnov statistic; this is discussed in detail in [14].

Let $\{W(t)\}$ denote a $C_0[0, 1]$-valued Brownian motion as in Theorem 1 so that $W(t)(s)$ is a version of $X(t, s)$. Then for $\lambda > 0$,

$$\begin{aligned} P\{S \leqq \lambda\} &= P\left\{\sup_{0 \leqq s_i \leqq 1} \frac{1}{\lambda} W(s_1)(s_2) \leqq 1\right\} \\ &= P\{\sup_{0 \leqq s_i \leqq 1} W(s_1/\lambda^2)(s_2) \leqq 1\} \\ &= P\{W(s) \in U \ \ \forall\, s \leqq 1/\lambda^2\} \end{aligned}$$

where

$$U \equiv \{y \in C_0[0, 1] : \sup_{0 \leqq s \leqq 1} y(s) \leqq 1\}\,. \tag{2}$$

Thus, the calculation of the distribution function for $S$ is equivalent to the calculation of the absorption probabilities $q_t(U, 0, U)$ for the above set, $U$. We obtain the following estimates which are expressed in terms of the standard normal distribution, $N(\lambda)$.

THEOREM 3. *For* $\lambda \geqq 0$,

$$4 \textstyle\int_\lambda^\infty sN(-s)\, ds \leqq P\{S \geqq \lambda\} \leqq 4[N(-\lambda)]$$

*and*

$$(2\lambda^2 + 1) \exp[-2\lambda^2] \leqq P\{S' \geqq \lambda\}\,.$$

PROOF. The upper bound on the tail distribution of $S$ follows from a comment of Kiefer [10]. We derive the lower bounds. Let $\{r_n\}$ denote an enumeration of the rationals in the unit interval. Then since the distribution of $W(1)$ in the absorption problem above is the canonical Wiener measure, the linear functionals $e_n \equiv r_n^{-\frac{1}{2}} \delta_{r_n}$ are normalized and $s(U, e_n) = r_n^{-\frac{1}{2}}$. Theorem 2 implies that the transition density $g_t(0, y)$ satisfies

$$\begin{aligned} g_s(0, y) &\leqq [1 - \exp[-(2/t) \inf_n (1 - y(r_n))/r_n]] I_U(y) \\ &= [1 - \exp[-(2/t) \inf_{0 \leqq s \leqq 1} (1 - y(s))/s]] I_U(y) \\ &= 1 - \exp[(2/t) \sup_{0 \leqq s \leqq 1} [0 \wedge (y(s) - 1)/s]] \end{aligned}$$

a.s. $p_t(0, dy)$. Hence, a lower bound on the absorption probability $1 - q_t(U, 0, C_0[0, 1])$ is obtained by integrating the above estimate with respect to $p_t(0, dy)$. For this, it suffices to compute the conditional distribution of

$$\sup_{0 \leqq s \leqq 1} [0 \wedge (y(s) - 1)/s]$$

with respect to $p_t(0, dy)$ restricted to the $\sigma$ field, $\sigma(y(1))$, generated by $y(1)$.

Let $\{b(s)\}$ denote the canonical one-dimensional Brownian motion. Then it is immediate from equation (1) that the distribution of the process $t^{\frac{1}{2}}b(\cdot)$ on $C_0[0, 1]$ is $p_t(0, dy)$. Hence,

$$p_t(0, \{\sup_{0\leq s\leq 1} [0 \wedge (y(s) - 1)/s] \leqq r\} \mid \sigma(y(1))) = P\{\sup_{0\leq s\leq 1} [0 \wedge (t^{\frac{1}{2}}b(s) - 1)/s] \leqq r \mid \sigma(t^{\frac{1}{2}}b(1))\}\,.$$

Without loss of generality assume that $r < 0$. The above conditional probability may then be written as

$$P\{b(s) \leqq rt^{-\frac{1}{2}}s + t^{-\frac{1}{2}} \quad \forall s \leqq 1 \mid \sigma(t^{\frac{1}{2}}b(1))\}\,.$$

Such probabilities are well known. From [9], page 284, we have

$$\begin{aligned} P\{b(s) \leqq rts^{-\frac{1}{2}} + t^{-\frac{1}{2}} \quad \forall\, s \leqq 1 \mid t^{\frac{1}{2}}b(1) = a\} & \\ = 1 - \exp[-2t^{-1}(r + 1 - a)] \quad & a - 1 \leqq r < 0 \\ = 0 \quad & r < a - 1\,. \end{aligned}$$

Thus, the conditional density, $p(r, a)$, of the random variable $\sup_{0\leq s\leq 1} [0 \wedge (y(s) - 1)/s]$ as a function of $a = y(1)$ is found by differentiating the above formula. Then

$$\begin{aligned} p(r, a) &= 2t^{-1}\exp[-2t^{-1}(r + 1 - a)] \quad & a - 1 \leqq r < 0 \\ &= 0 & r < a - 1 \end{aligned} \tag{3}$$

and we have

$$\begin{aligned} & q_t(U, 0, C_0[0, 1] \mid y(1) = a) \\ & \leqq \textstyle\int_{a-1}^{0} \{1 - \exp[2t^{-1}r]\} p(r, a)\, dr \\ & = 1 - \exp[2t^{-1}(a - 1)] + 2t^{-1}(a - 1)\exp[2t^{-1}(a - 1)] \end{aligned} \tag{4}$$

a.s. in $a$, relative to the distribution of $y(1)$.

An argument in Billingsley [1], page 84, with an obvious reformulation, yields the result the family of measures

$$P\{X \in A \mid 0 \leqq X(1, 1) \leqq \varepsilon\} \qquad \varepsilon > 0$$

converges weakly on $C_0[0, 1]^2$ as $\varepsilon \to 0$ to the measure $P'$ for the process $X(s_1, s_2) - s_1 s_2 X(1, 1)$. Since $A \equiv \{y : \sup_{0\leq s_i\leq 1} y(s_1, s_2) < \lambda\}$ is open in $C_0[0, 1]^2$, we have

$$P'(A) \leqq \lim_{\varepsilon\to 0} \inf P\{X \in A \mid 0 \leqq X(1, 1) \leqq \varepsilon\}\,,$$

and in view of (4), we may take $a = 0$ and $t = \lambda^{-2}$ in the right-hand side of (4) to obtain

$$P\{S' \leqq \lambda\} \leqq 1 - (2\lambda^2 + 1)\exp[-2\lambda^2]\,.$$

To obtain the lower bound on the tail distribution of $S$ we use the fact that $y(1)$ has a mean 0 Gaussian distribution with variance $t$. The integration of the estimate in (4) against this distribution yields

$$1 - q_t(U, 0, C_0[0, 1]) \geqq (2/\pi)^{\frac{1}{2}} \textstyle\int_{1/t^{\frac{1}{2}}}^{\infty} \{1 + t^{-1}(t^{\frac{1}{2}}a - 1)\} \exp[-a^2/2]\, da\,.$$

If the substitution $t = \lambda^{-2}$ is made in the right-hand expression, one finds that the derivative with respect to $\lambda$ is equal to the derivative of

$$4 \int_\lambda^\infty [sN(-s)\,ds]\,.$$

Hence, the above expression is a lower bound for $P(S \geqq \lambda)$.

COROLLARY. $\lim_{\lambda\to+\infty} P\{S \geqq \lambda\}/4[N(-\lambda)] = 1$.

PROOF. The elementary inequality

$$(2\pi)^{-\frac{1}{2}}[s^{-1} - s^{-3}]\exp[-s^2/2] \leqq N(-s)$$

combined with Theorem 3 yields

$$1 \geqq P(S \geqq \lambda)/4N(-\lambda) \geqq N(-\lambda)^{-1}(2\pi)^{-\frac{1}{2}} \int_\lambda^\infty [1 - s^{-2}]\exp[-s^2/2]\,ds$$
$$\geqq 1 - \lambda^{-2}\,.$$

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