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RANDOM OVERLAPPING INTERVALS—A GENERALIZATION OF ERLANG'S LOSS FORMULA

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Consider a queueing system with infinitely many servers, a general distribution of service times and an instantaneous rate α_k of new arrivals, where α_k depends only on the number of busy servers. This is called a generalized Erlang model (GEM) since if $\alpha_k = \alpha$ $(k < N)$, $\alpha_k = 0$ $(k \geqq N)$, then Erlang's model for a telephone exchange with N lines is recovered.

The synchronous and asynchronous stationary distributions of the GEM are determined and several interesting properties of the process are discussed. In particular the stationary GEM is shown to be reversible.

1. Introduction. Consider a queueing system with infinitely many servers, a general distribution F of service times and an instantaneous rate α_k of new arrivals which depends only on the number $K(t) = k$ of busy servers at time t. Call this a generalized Erlang model (GEM), since if $\alpha_k = \alpha$ $(k < N)$, $\alpha_k = 0$ $(k \geqq N)$ Erlang's model for a telephone exchange with N lines is recovered. The terminology of queueing theory is convenient and will be used in this paper. However, the GEM is a natural procedure for generating random ov rlapping intervals on the real line, and as such has other applications. For example, if $\alpha_k = a + kb$ then $K(t)$ becomes an immigration branching process. Oakes (1975) discusses an interesting connection with a process of stochastic backtrackings. With $\alpha_k = ab^k$ $(0 < b < 1)$ a generalization of the Type I counter is obtained, in which dead periods reduce but do not entirely eliminate new registrations.

If the service times are exponentially distributed, say $F(x) = 1 - e^{-\beta x}$, then $K(t)$ is a Markov birth and death process. It is easily seen that a stationary distribution $\{\pi_k\}$ of $K(t)$ must satisfy

$$k\beta\pi_k = \alpha_{k-1}\pi_{k-1}. \tag{1}$$

Thus

$$\pi_k = \prod_{i=1}^k \frac{\alpha_{i-1}}{i\beta} \Big/ \left\{1 + \sum_{j=1}^\infty \prod_{i=1}^j \frac{\alpha_{i-1}}{i\beta}\right\},$$

unless the sum diverges, when there is no stationary probability distribution.

It is known that in the specialization to the Erlang model the same stationary distribution applies for any distribution of service times which has finite mean β^{-1}. Let the customers being served at t be labelled in a random order $i = 1, 2, \cdots, K(t)$, and let the spent and residual service times of the ith customer

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be denoted by $U_i(t)$ and $V_i(t)$ respectively. Then it is also known that in the stationary Erlang model the $U_i(t)$ are mutually independent with the commom density $\beta\{1 - F(u)\}\,du$. The same result holds for the $V_i(t)$.

Khintchine (1963), Sevast'yanov (1957) and Takacs (1969) discuss these results from various viewpoints. Khintchine derives the specialization of (1) but appears to assume the independence of the $\{U_i(t)\}$ without proof. Sevast'yanov proves the ergodicity of the Markov process $Z(t) = \{K(t), U_1(t), U_2(t), \cdots, U_{K(t)}(t)\}$ and finds its stationary distribution by solving a set of integro-differential equations. Takacs studies the discrete parameter Markov process obtained by sampling $Z(t)$ at epochs of new arrivals, remarking that this approach considerably simplifies the discussion. More recently, Shanbhag and Tambouratzis (1973) show that the pooled process of departures and lost customers is itself a Poisson process.

We use a refinement of Takacs' approach to derive the stationary distribution for the GEM. The calculations become even simpler if $Z(t)$ is sampled at all transitions of $K(t)$—departures as well as arrivals. In Section 2 the stationary distribution of this discrete parameter skeleton of $Z(t)$ is obtained. In Section 3 the stationary distribution of $Z(t)$ itself is derived, using a result from the theory of semi-Markov processes. Four interesting properties of the stationary GEM are listed, and some simple moment relations given. In Section 4 a dual process $Z^*(t)$ is defined and used to show that $K(t)$ is a reversible process. This fact is used to generalize the result of Shanbhag and Tambouratzis.

2. The skeleton. The skeleton $\hat{Z}_n$ of $Z(t)$ is defined by

$$\hat{Z}_n = \{\hat{\Delta}_n; \hat{K}_n; \hat{U}_{1n}, \hat{U}_{2n}, \cdots, \hat{U}_{\hat{K}(n),n}\}\,.$$

Here $\hat{\Delta}_n = 1$ or -1 according as the nth transition is an arrival or a departure, and $\hat{K}(n)$ is the number of busy servers at the time of the nth transition. The arriving or departing customer is not included. The $\hat{U}_{in}$ are the spent service times. To ensure that the particular labelling $i = 1, 2, \cdots, K(t)$ of the $K(t) = k$ customers being served at any time is uninformative, the following conventions are made.

(i) If the customer currently labelled j departs, customers $j + 1, j + 2, \cdots, k$ are then relabelled as $j, j + 1, \cdots, k - 1$ respectively.

(ii) An incoming customer is assigned label j $(1 \leqq j \leqq k + 1)$ with probability $1/(k + 1)$. Those customers previously labelled $j, j + 1, \cdots, k$ are relabelled as $j + 1, j + 2, \cdots, k + 1$ respectively.

If $\mathbf{u} = (u_1, u_2, \cdots, u_k)$ and $t > 0$ we write $(\mathbf{u}, t)_j = (u_1, u_2, \cdots, u_{j-1}, t, u_j, u_{j+1}, \cdots, u_k)$ for the vector $\mathbf{u}$ with the extra component t inserted in the jth position $(1 \leqq j \leqq k + 1)$, and $(\mathbf{u}, t) = (\mathbf{u}, t)_{k+1}$. Also $\mathbf{u} \pm t = (u_1 \pm t, u_2 \pm t, \cdots, u_k \pm t)$. The index of $\min_{1\leqq i\leqq k} u_i$ is denoted by m, so that $u_m = \min_{1\leqq i\leqq k} u_i$, and $\mathbf{u}^{(m)}$ denotes the $(k - 1)$-vector obtained by dropping the zero component from $\mathbf{u} - u_m$. Also,

$$d\mathbf{u} = \prod_{i=1}^k du_i\,, \qquad d\mathbf{u}^{(m)} = \prod_{i=1;i\neq m}^k du_i\,.$$

It is clear that $\hat{Z}_n$ is a Markov chain. We shall show that provided the normalizing factor

$$\nu = \sum_{k=0}^{\infty} \alpha_k \pi_k$$

is finite, there is a stationary distribution given by

$$q(\delta; k; \mathbf{u}) = \frac{\alpha_k \pi_k}{2\nu} \prod_{i=1}^{k} [\beta\{1 - F(u_i)\}]\, d\mathbf{u} \qquad \delta = -1, 1; k = 0, 1, \cdots.$$

Note that ν is the mean rate of new arrivals.

Let T_n denote the time of the nth transition and $X = X_n = T_{n+1} - T_n$ the intertransition interval. Take $\hat{K}_n = k$ and $\hat{U}_n = (u_1, u_2, \cdots, u_k)$. The transition probabilities of $\hat{Z}_n$ are determined by the (incomplete) conditional distributions $G_{i,1}(dx \mid k; \mathbf{u})$ and $G_{i,-1}(dx; j \mid k; \mathbf{u})$ of X given Z_n. The subscripts correspond to $\hat{\Delta}_n$ and $\hat{\Delta}_{n+1}$ respectively. The argument j refers to the label of a customer departing at the $(n+1)$th transition $(1 \leqq j \leqq k)$, $j = 0$ corresponding to a customer who arrives at the nth transition and is thus not included in $\hat{Z}_n$.

It is convenient to use the notation $F(x \mid u) = (F(x+u) - F(u))/(1 - F(u))$ for the conditional distribution of the residual service time given the spent service time. Noth that (as in renewal theory)

$$\int_{u=0}^{\infty} F(dx \mid u)\beta\{1 - F(u)\}\, du = \beta\{1 - F(x)\}\, dx.$$

Then it is easily seen that

$$G_{-1,1}(dx \mid k; \mathbf{u}) = \prod_{i=1}^{k} \frac{1 - F(u_i + x)}{1 - F(u_i)} \alpha_k e^{-\alpha_k x}\, dx,$$

$$G_{1,1}(dx \mid k; \mathbf{u}) = \frac{1}{k+1} \prod_{i=1}^{k} \frac{1 - F(u_i + x)}{1 - F(u_i)} \times \{1 - F(x)\}\alpha_{k+1} e^{-\alpha_{k+1} x}\, dx, \tag{2}$$

$$G_{-1,-1}(dx; j \mid k; \mathbf{u}) = \prod_{i \neq j; i=1}^{k} \frac{1 - F(u_i + x)}{1 - F(u_i)} e^{-\alpha_k x} F(dx \mid u_j) \qquad 1 \leqq j \leqq k,$$

$$G_{1,-1}(dx; 0 \mid k; \mathbf{u}) = \prod_{i=1}^{k} \frac{1 - F(u_i + x)}{1 - F(u_i)} e^{-\alpha_{k+1} x} F(dx),$$

$$G_{1,-1}(dx; j \mid k; \mathbf{u}) = \frac{1}{k} \prod_{i \neq j; i=1}^{k} \frac{1 - F(u_i + x)}{1 - F(u_i)} e^{-\alpha_{k+1} x} \times \{1 - F(x)\} F(dx \mid u_j) \qquad 1 \leqq j \leqq k. \tag{3}$$

The factors $(k+1)^{-1}$ and k^{-1} in (2) and (3) arise from the need (at the second transition) to choose a label for the arrival at the first transition.

The one-step iterate q' of q is given by the equations

$$\begin{aligned} q'(1; k; \mathbf{u})\, d\mathbf{u} &= q(1; k-1; \mathbf{u}^{(m)})\, d\mathbf{u}^{(m)}\, G_{1,1}(du_m \mid k-1; \mathbf{u}^{(m)}) \\ &\quad + \int_{x=0}^{u_m} q(-1; k; \mathbf{u} - x)\, d\mathbf{u}\, G_{-1,1}(dx \mid k; \mathbf{u} - x), \\ q'(-1; k; \mathbf{u})\, d\mathbf{u} &= \int_{x=0}^{u_m} q(1; k; \mathbf{u} - x)\, d\mathbf{u}\, G_{1,-1}(dx; 0 \mid k; \mathbf{u} - x) \\ &\quad + \sum_{j=1}^{k} \int_{y=0}^{\infty} q(1; k; (\mathbf{u}^{(m)}, y)_j)\, d\mathbf{u}^{(m)}\, dy \\ &\quad \times G_{1,-1}(du_m; j \mid k; (\mathbf{u}^{(m)}, y)_j) \\ &\quad + \sum_{j=1}^{k+1} \int_{x=0}^{u_m} \int_{y=0}^{\infty} q(-1; k+1; (\mathbf{u} - x, \mathbf{y})_j)\, d\mathbf{u}\, dy \\ &\quad \times G_{-1,-1}(dx; j \mid k+1, (\mathbf{u} - x, y)_j). \end{aligned}$$

Note that if $\mathbf{U}_{n+1} = \mathbf{u}$, then $X_n = T_{n+1} - T_n$ satisfies $X_n = u_m$, or $0 \leqq X_n \leqq u_m$ according as $\hat{\Delta}_n = 1$ or -1. The three terms in the second equation arise according as the first transition is the arrival of the customer who departed at the second transition, the arrival of a different customer, or a departure.

It is easily shown by direct substitution, use of (1) and a single partial integration that

$$q'(\delta; k; \mathbf{u})\, d\mathbf{u} = q(\delta; k; \mathbf{u})\, d\mathbf{u}\,,$$

so that q is a stationary distribution for $\hat{Z}_n$.

Successive visits by $\hat{Z}_n$ to the state $E = \{1; 0\}$ form a persistent recurrent event in the sense of Feller (1967), Chapter XIII. For $q(1; 0) > 0$, and E may be reached with positive probability from any state in $\hat{Z}_n$. Following Takacs (1969) we may deduce that $\hat{Z}_n$ has the (Cesàro) limiting distribution q, independently of its initial state. Note that the chain $\hat{Z}_n$ has period 2.

3. The continuous time process. We now investigate the continuous time process $Z(t)$ using a standard construction from the theory of semi-Markov processes on abstract state spaces (see Orey, 1961; Kesten, 1974). Let $N(t) = \sup\{n : T_n \leqq t\}$ and $W(t) = t - N(t)$. Then

$$\tilde{Z}(t) = \{W(t), \hat{Z}_{N(t)}\}$$

is a Markov process in continuous time. It has a stationary distribution

$$2\nu q(dz) R(w \mid z)\, dw\,,$$

where $R(w \mid z)$ is the conditional survivor function

$$R(w \mid -1; k; \mathbf{u}) = e^{-\alpha_k w} \prod_{i=1}^{k} \frac{1 - F(u_i + w)}{1 - F(u_i)}\,,$$

$$R(w \mid 1; k; \mathbf{u}) = e^{-\alpha_{k+1} w}\{1 - F(w)\} \prod_{i=1}^{k} \frac{1 - F(u_i + w)}{1 - F(u_i)}\,.$$

It is easily seen that this distribution on $\tilde{Z}(t)$ induces on $Z(t)$ the stationary distribution

$$p(k; \mathbf{u})\, d\mathbf{u} = \pi_k \prod_{i=1}^{k} [\beta\{1 - F(u_i)\}]\, d\mathbf{u}\,.$$

It is convenient to refer to p as the asynchronous stationary distribution of $Z(t)$, in contrast to the synchronous stationary distribution q.

Several useful properties of the stationary process $Z(t)$ follow immediately. These are

(i) the *robustness* of $K(t)$ to F. The one dimensional marginal synchronous and asynchronous distributions of $K(t)$ depend on the distribution F of service times only through the mean of F.

(ii) the *conditional independence* of the spent holding times $U_i(t)$, given $K(t)$. Note also that the distribution of each $U_i(t)$ is the same as the recurrence time distribution of an equilibrium renewal process with interval distribution F.

(iii) *truncation*. If, for some N, the process is modified by setting $\alpha_N = 0$, then the stationary distribution of the modified process is the same as the conditional distribution of the original process given $K(t) \leqq N$.

(iv) *synchroneity*. The distribution of the number $\hat{K}_n$ of busy servers just before an arbitrary arrival, or just after an arbitrary departure is that obtained by a naive application of the Bayes formula $P(B_j \mid A) \alpha P(A \mid B_j) P(B_j)$ with $A = \{\text{arrival at } t\}$ and $B_j = \{K(t) = j\}$. Moreover, (ii) still holds.

A fifth property, reversibility, is discussed in Section 4 below. Notice that these properties do not generally hold unless the process is in a stationary state.

An easy consequence of (iv) is the relation $\nu \mathbb{E}(\hat{K}_n{}^r) = \beta \mathbb{E}[K(t)\{K(t) - 1\}^r]$ between the moments of the synchronous and asynchronous distributions of $K(t)$ (when these moments exist). It follows from (ii) that the expected number of customers whose service times completely cover the interval $[t, t + l]$ is $\nu \int_{x=l}^{\infty} \{1 - F(x)\}\, dx$, whereas the expected number whose service times are completely included in $[t, t + l]$ is $\nu \int_0^l F(x)\, dx$.

4. Reversibility. Consideration of residual rather than spent service times leads to a new Markov process

$$Z^*(t) = \{K(t); V_1(t), V_2(t), \cdots, V_k(t)\}$$

with skeleton

$$\hat{Z}_n{}^* = \{\hat{\Delta}_n; \hat{K}_n : \hat{V}_{1n}, \hat{V}_{2n}, \cdots, \hat{V}_{kn}\},$$

where $\hat{\Delta}_n$ and $\hat{K}_n$ have the same meaning as before, and the $V_i(t)$, $\hat{V}_{in}$ are the residual service times of the customers in the system at t, T_n respectively. As before the currently departing or arriving customer is not included.

The transition distributions for $\hat{Z}_n{}^*$ are determined by the (incomplete) conditional distributions of X_n given $\hat{Z}_n{}^*$. These are

$$G^*_{-1,1}(dx \mid k; \mathbf{v}) = \alpha_k e^{-\alpha_k x}\, dx \quad (x \leqq v_m),$$
$$G^*_{-1,-1}(v_m \mid k; \mathbf{v}) = e^{-\alpha_k v_m},$$
$$G^*_{1,1}(dx; ds; j \mid k; \mathbf{v}) = \frac{1}{k+1}\, \alpha_{k+1} e^{-\alpha_{k+1} x}\, dx\, F(ds) \quad (x \leqq s, v_m),$$
$$G^*_{1,-1}(dx; 0 \mid k; \mathbf{v}) = e^{-\alpha_{k+1} x} F(dx) \quad (x < v_m),$$
$$G^*_{1,-1}(ds; j \mid k; \mathbf{v}) = \frac{1}{k} e^{-\alpha_{k+1} v_m} F(ds) \quad (s \geqq v_m).$$

Here s is the service time of the earlier arrival and j is its assigned label. Note that $X_n = v_m$ when the second transition is a departure.

It may be verified from these equations or (more simply) from the results of Sections 2 and 3 that the stationary distributions of $\hat{Z}_n{}^*$ and $Z^*(t)$ are given by

$$q^*(\delta; k; \mathbf{v})\, d\mathbf{v} = \frac{\alpha_k \pi_k}{2\nu} \prod_{i=1}^{k} [\beta\{1 - F(v_i)\}]\, d\mathbf{v},$$
$$p^*(\delta; k; \mathbf{v})\, d\mathbf{v} = \pi_k \prod_{i=1}^{k} [\beta\{1 - F(v_i)\}]\, d\mathbf{v}.$$

The main purpose of this section is to show that the distributions of $\hat{Z}_n{}^*$ are the same as the reversed-time distributions of $\hat{Z}_n$ when the roles of arrivals and residual service times are exchanged with, respectively, departures and spent service times. The corresponding result relating $Z(t)$ and $Z^*(t)$ follows immediately. This duality principle will be used to prove that $K(t)$ is a reversible stochastic process, and hence to generalize Shanbhag and Tambouratzis' result concerning the output process.

The following equations, which are easily checked by direct substitution, are the analogues of the familiar "$p_i p_{ij} = p_j p_{ji}$" reversibility conditions for discrete Markov chains. We have

$$q(-1; k; \mathbf{u} - x)\, d\mathbf{u}\, G_{-1,1}(dx \mid k; \mathbf{u} - x) = q^*(-1; k; \mathbf{u})\, d\mathbf{u}\, G^*_{-1,1}(dx \mid k, \mathbf{u}) ,$$
$$q(-1; k - 1; \mathbf{u}^{(m)})\, d\mathbf{u}^{(m)}\, G_{1,1}(du_m \mid k - 1; \mathbf{u}^{(m)}) = q^*(-1; k; \mathbf{u})\, d\mathbf{u}\, G^*_{-1,-1}(u_m \mid k, \mathbf{u}) ,$$
$$q(-1; k + 1; (\mathbf{u} - y, s - y)_j)\, d\mathbf{u}\, ds\, G_{1,-1}(dy; j \mid k + 1; (\mathbf{u} - y, s - y)_j) ,$$
$$= q^*(1; k; \mathbf{u})\, d\mathbf{u}\, G^*_{1,1}(dy; ds, j \mid k, u) ,$$
$$q(1; k; \mathbf{u} - y)\, d\mathbf{u}\, G_{1,-1}(dy; 0 \mid k; \mathbf{u} - y) = q^*(1; k; \mathbf{u})\, d\mathbf{u}\, G^*_{-1,1}(dy; 0 \mid k; \mathbf{u}) ,$$
$$q(1; k; (\mathbf{u}^{(m)}, s - u_m)_j)\, d\mathbf{u}^{(m)}\, ds\, G_{1,-1}(du_m; j \mid k; (\mathbf{u}^{(m)}, s - u_m)_j) ,$$
$$= q^*(1; k; \mathbf{u})\, d\mathbf{u}\, G^*_{1,-1}(ds; j \mid k; \mathbf{u}) .$$

As the consequence of these equations we have *property* (v) *reversibility*. The GEM is invariant under the transformation which reverses time, exchanges spent with residual holding times, and arrivals with departures.

Thus $K(t)$ is itself a reversible stochastic process. With exponential service times, $K(t)$ is a Markovian birth-death process and the reversibility is immediate. However it is easily seen that non-Markovian birth-death processes need not be reversible, so the reversibility of $K(t)$ proved here is a nontrivial result.

Another consequence of property (v) sheds some light on the results of Shanbhag and Tambouratzis concerning the output process of the Erlang model. Consider a pure loss queueing system with infinitely many servers, general service time distribution and Poisson arrivals. Suppose however that the number of busy servers acts as a deterrent to a new customer joining the queue. In terms of the GEM, this means taking $\alpha_i \leqq \alpha$ for all i, where α is the rate of arrivals. The probability that an arrival will enter the system when there are already k busy servers is α_k/α. The rejected arrivals are termed losses.

Conditionally on the process $Z(t)$, the losses form a nonstationary Poisson process whose rate function is given by $\lambda(t) = \alpha - \alpha_{K(t)}$. This process is unaltered by the transformation defined by property (v). Since the superposition of the process of accepted arrivals and the process of losses is a Poisson process of rate α, (v) ensures that the superposition of the departures and the losses is also a Poisson process of rate α. This result was given by Shanbhag and Tambouratzis (1973) for the Erlang model with $\alpha_k = \alpha$ $(k < N)$ $\alpha_N = 0$.

In conclusion, we note that the arrivals and departures from a GEM form a bivariate point process in the sense of Cox and Lewis (1972) and Wisniewski (1972). However, this aspect of the GEM will not be explored further here.

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