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SEQUENTIAL COMPACTNESS OF CERTAIN SEQUENCES OF GAUSSIAN RANDOM VARIABLES WITH VALUES IN $C[0, 1]$

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Let $(X_n(t): t \in [0, 1], n \geqq 1)$ be a sequence of Gaussian processes with mean zero and continuous paths on $[0, 1]$ a.s. Let $R_n(t, s) = EX_n(t)X_n(s)$ and suppose that $(R_n: n \geqq 1)$ is uniformly convergent, on the unit square, to a covariance function R. It is shown in this paper that under certain conditions the normalized sequence $(Y_n(t): t \in [0, 1], n \geqq 2)$ where $Y_n(t) = (2 \lg n)^{-\frac{1}{2}} X_n(t)$ is, with probability one, a sequentially compact subset of $C[0, 1]$ and its set of limit points coincides a.s. with the unit ball in the reproducing kernel Hilbert space generated by R. This is Strassen's form of the iterated logarithm in its intrinsic formulation and includes a special case studied by Oodaira in a recent paper.

1. Introduction. If $(X_n: n \geqq 1)$ is a sequence of independent $N(0, \sigma_n^2)$ random variables such that $\lim_{n\to\infty} \sigma_n^2 = 1$, then the normalized sequence $(Y_n: n \geqq 2)$ where $Y_n = (2\sigma_n^2 \lg n)^{-\frac{1}{2}} X_n$ is sequentially compact with probability one and its set of limit points is almost surely (a.s.) equal to the interval $[-1, +1]$. The extension of this fact to the case of a sequence of Gaussian vectors in $\mathbb{R}^N$ does not present special difficulties. A simple proof may be obtained by a suitable modification of the argument used by Finkelstein (1971) (Lemma 2, page 609). The study of this problem in a more general setting appears to be less immediate and a tentative solution will be presented here.

Throughout this paper we shall refer to a stochastic process $(X(t): t \in [0, 1])$ as a *centered* process if $EX(t) \equiv 0$ on $[0, 1]$. Suppose $(X(t): t \in [0, 1])$ is a centered Gaussian process. Then the reproducing kernel Hilbert space (RKHS) generated by the covariance function $R(t, s) = EX(t)X(s)$ plays a fundamental role in the characterization of its basic properties. It will be shown here that the unit ball in the RKHS generated by a continuous covariance function R identifies, under certain conditions, the set of limit points of a sequence of centered Gaussian processes, all having continuous paths on $[0, 1]$ a.s., that are *asymptotically independent* in some sense. To be more precise we may state the main result of our study in the following form.

Theorem 1.1. *Let $(X_n(t): t \in [0, 1], n \geqq 1)$ be a sequence of separable, centered Gaussian processes defined on $(\Omega, \mathscr{F}, P)$ and satisfying*

(1.1) (a) $$E[X_n(t) - X_n(s)]^2 \leqq g^2(|t - s|)$$

where g is a positive nondecreasing function on $[0, 1]$ such that $\int_1^\infty g(e^{-u^2})\, du < \infty$.

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If there exists a (covariance) function R on $[0, 1] \times [0, 1]$ such that

(1.2) $\quad \lim_{n\to\infty} EX_n(t)X_n(s) = R(t, s)$ *uniformly on the unit square,*

then $(Y_n : n \geqq 2)$, where $Y_n(t) = (2 \lg n)^{-\frac{1}{2}} X_n(t)$, is sequentially compact in $C[0, 1]$ a.s. *and its set of limit points is included in the unit ball K of the reproducing kernel Hilbert space generated by R. Letting $\mathscr{G}_n = (X_j(t) : t \in [0, 1], 1 \leqq j \leqq n)$ and $\mathscr{G} = \bigcup_{n\geqq 1} \mathscr{G}_n$, suppose furthermore that*

(b) *$\mathscr{G}$ is a Gaussian family of stochastic processes and*

(1.3) $\quad \lim_{r\to\infty} \max_{|m-n|>r} E[E(X_m(t) \mid \mathscr{F}_n)]^2 = 0$ *uniformly on* $[0, 1]$,

where $\mathscr{F}_n$ denotes the σ-algebra generated by $\mathscr{G}_n$.

Then the set of limit points of $(Y_n : n \geqq 2)$ in $C[0, 1]$ coincides with K a.s.

This theorem is modeled upon Theorem 1 of Lai (1974) and complements it in the sense that in Lai's paper only the case where:

(1.4) $$R_n(t, s) = EX_n(t)X_n(s)$$

equals $R(t, s)$ for all n has been considered. Equation (1.3) is an equivalent formulation of Lai's condition of asymptotic independence with the stronger requirement of uniform convergence.

We shall sketch hereafter the idea followed for the proof of the theorem. It is well known that a continuous covariance function $\Gamma(t, s)$ on $[0, 1] \times [0, 1]$ can be expressed as the sum of a uniformly convergent series $\sum \phi_j(t)\phi_j(s)$ where $(\phi_j : j \geqq 1)$ is any complete orthonormal sequence (CONS) in the (separable) RKHS $H(\Gamma)$ generated by Γ. Let $H(R_n)$ be the RKHS's generated by the functions R_n defined by equation (1.4). One basic point of our discussion consists in showing that under conditions (1.1) and (1.2) it is possible to find CONS's $(e_i^{(n)} : i \geqq 1)$ in $H(R_n)$ such that the *residual variability* $\sum_{N+1}^{\infty} [e_i^{(n)}(t)]^2$ of the processes $(X_n(t) : t \in [0, 1])$ can be made small, for sufficiently large N, uniformly with respect to n and t. This will allow, by applying a classical result of Fernique (1964), the reduction of the problem to the finite dimensional case. To this end we shall establish first, in Section 2, a lemma on the uniform convergence of the sets $(e_i^{(n)} : 1 \leqq i \leqq N)$ to $(e_i : 1 \leqq i \leqq N)$ where $(e_i : i \geqq 1)$ is a CONS in $H(R)$. In the same section a few definitions and some complementary facts will be given. Section 3 will be devoted to the proof of the theorem. In [5] Lai has shown that his Theorem 1 easily implies Theorem 4 of Oodaira (1972). In the same spirit we shall derive, in Section 4, Strassen's form of the law of the iterated logarithm ([10]) obtained by Oodaira (1973) for certain sequences of nonidentically distributed Gaussian processes. Finally we observe that our results can be extended with no essential modification to the case where $[0, 1]$ is replaced by an arbitrary interval of the real line.

2. Preliminaries. Let $C[0, 1] = \mathscr{C}$ denote the space of all continuous functions $[0, 1] \to \mathbb{R}$. If $x \in \mathscr{C}$, let $||x||_C = \sup(|x(t)| : t \in [0, 1])$. The space $\mathscr{C}$ with the topology induced by the norm $||\cdot||_C$ is a separable Banach space and the collection $\mathscr{C}^*$ of all linear and bounded functionals on $\mathscr{C}$ can be identified with the space of all (regular) measures ν on the Borel subsets of $[0, 1]$. If ν^+ and ν^- denote the Hahn decomposition of ν then $||\nu|| = \nu^+[0, 1] + \nu^-[0, 1]$ defines a norm on $\mathscr{C}^*$. We shall let $|\nu| = \nu^+ + \nu^-$. Now $\mathscr{C}$ can be endowed with the σ-algebra $\mathscr{B}$ generated by the open subsets of $\mathscr{C}$. Since it can be shown that $\mathscr{B}$ coincides with the smallest σ-algebra generated by the maps $x \to x(t)$, $x \in \mathscr{C}$, any stochastic process $(X(t): t \in [0, 1])$ on $(\Omega, \mathscr{F}, P)$ with continuous paths on $[0, 1]$ a.s. can be interpreted as a $(\mathscr{F}, \mathscr{B})$ measurable map $X: \Omega \to \mathscr{C}$ if we let $X(t) = \delta_t(X)$ where δ_t is the evaluation function at t. We may then say that X is Gaussian if and only if $\nu(X)$ is a real-valued Gaussian random variable for each $\nu \in \mathscr{C}^*$. Suppose X is centered. Then $R(t, s) = EX(t)X(s)$ is a symmetric, positive-definite form on $[0, 1] \times [0, 1]$ and it generates a Hilbert space $H(R)$ of real-valued functions on $[0, 1]$. (Neveu (1968), Chapter 3.) Such a Hilbert space, that is obtained as the completion of $sp(R(t, \cdot): t \in [0, 1])$, has an inner product $\langle \cdot, \cdot \rangle_R$ with the following *reproducing* property:

$$\text{(2.1)} \qquad \text{if} \quad h \in H(R) \quad \text{then} \quad \langle h, R(t, \cdot) \rangle_R = h(t), \quad t \in [0, 1].$$

Equation (2.1) implies that if R is continuous, as is always the case for Gaussian processes with continuous paths on $[0, 1]$ a.s., then $H(R) \subset \mathscr{C}$ set-theoretically and convergence on $H(R)$ implies convergence on $\mathscr{C}$. Our first lemma gives a useful representation of certain elements of $H(R)$, their norms and inner products, in terms of an element of $\mathscr{C}^*$. (Unless otherwise indicated, all integrals will be henceforth evaluated on $[0, 1]$.)

LEMMA 2.1. *Let $\nu \in \mathscr{C}^*$ and R be a continuous covariance function. Define $h(t) = \int R(t, s)\nu(ds)$. Then*

$$\text{(2.2)} \quad \text{(a)} \qquad h \in H(R),$$

$$\text{(2.3)} \quad \text{(b)} \qquad \textit{for any} \quad f \in H(R), \quad \langle h, f \rangle_R = \int f(t)\nu(dt),$$

$$\text{(2.4)} \quad \text{(c)} \qquad ||h||_R^2 = \langle h, h \rangle_R = \int\int R(t, s)\nu(dt)\nu(ds).$$

PROOF. (a) For $m = 1, 2, \cdots$ and $t \in [0, 1]$ let $h_m(t) = \sum_{i=1}^m R(t, s_i)\nu_i$ denote the Riemann–Stieltjes sums corresponding to a partition of $[0, 1]$ at the points $s_1, \cdots, s_m$. Then $h_m(t) \to h(t)$, pointwise, as $m \to \infty$. Now h_m belongs to $sp(R(t, \cdot): t \in [0, 1])$ for each m. Using the property

$$\text{(2.5)} \qquad R(t, s) = \langle R(t, \cdot), R(s, \cdot) \rangle_R$$

we have

$$\text{(2.6)} \qquad \langle h_m, h_n \rangle_R = \sum_{i=1}^m \sum_{j=1}^n R(s_i, s_j)\nu_i \nu_j \to I = \int\int R(t, s)\nu(dt)\nu(ds)$$

as m and n tend to infinity. Consequently

$$||h_m - h_n||_R^2 = \langle h_m, h_m \rangle_R + \langle h_n, h_n \rangle_R - 2\langle h_n, h_m \rangle_R \to I + I - 2I = 0$$

for $m \to \infty$, $n \to \infty$. Let h' denote the limit of $(h_m, m \geqq 1)$ in $H(R)$ that exists because of completeness. Since convergence in $||\cdot||_R$-norm implies pointwise convergence, $h'(t) = \lim_{m\to\infty} h_m(t) = h(t)$, $t \in [0, 1]$ and $h \in H(R)$.

(b) Let $f \in H(R)$ be of the form $f(t) = R(t, t_0)$, for some $t_0 \in [0, 1]$. Using (2.5) again we obtain

$$\langle h, f \rangle_R = \int R(t, t_0)\nu(dt) = \int f(t)\nu(dt) .$$

Hence (2.3) holds, by taking linear combinations for all f in $sp(R(t, \cdot) : t \in [0, 1])$. By passing to the limit, if necessary, one can see that the same conclusion holds for all f in $H(R)$.

(c) This is a consequence of (2.6).

Let $L_2(\Omega, \mathscr{F}, P)$ be the Hilbert space of (classes of) random variables on $(\Omega, \mathscr{F}, P)$ with finite second moments. If $L(X)$ denotes the closed, in $L_2(\Omega, \mathscr{F}, P)$, linear manifold spanned by a centered Gaussian process $X = (X(t) : t \in [0, 1])$ on $(\Omega, \mathscr{F}, P)$, then $L(X)$ and $H(R)$ are congruent. Let $\theta : L(X) \to H(R)$ denote such a congruence that can be defined as $R(t, \cdot) = \theta(X(t))$ for each t (*natural* congruence). Suppose that X has continuous paths on [0, 1] a.s. Sato (1969) has shown that $L(X)$ is separable; furthermore for some CONS $(\xi_i : i \geq 1)$ in $L(X)$ one can write $\xi_i = \nu_i(X)$ where $\nu_i \in \mathscr{C}^*$ have the following properties:

$$\nu_i = \mu_i/\sigma_i ; \qquad \mu_i \in \mathscr{C}^* , \qquad ||\mu_i|| = 1 ; \tag{2.7}$$

$$\sigma_i = [\int\int R(t, s)\mu_i(dt)\mu_i(ds)]^{\frac{1}{2}} > 0 ; \qquad \int\int R(t, s)\mu_i(dt)\mu_j(ds) = 0 , \quad i \neq j . \tag{2.8}$$

We now have

LEMMA 2.2. *Let $(X_n(t) : t \in [0, 1], n \geq 1)$ be a sequence of centered Gaussian processes on $(\Omega, \mathscr{F}, P)$ with continuous paths on* [0, 1] a.s. *and covariance functions $R_n(t, s)$. Suppose there exists a probability space $(\Omega', \mathscr{F}', P')$ and a centered Gaussian process $(X(t) : t \in [0, 1])$ on it with continuous sample paths on* [0, 1] a.s. *and covariance function $R(t, s)$. Let N be a positive integer. If*

$$R(t, s) = \lim_{n\to\infty} R_n(t, s) \quad \textit{uniformly on the unit square,} \tag{2.9}$$

then there exist orthonormal sets

$$(e_i^{(n)} : 1 \leq i \leq N) \tag{2.10}$$

in $H(R_n)$ such that

$$e_i(t) = \lim_{n\to\infty} e_i^{(n)}(t) \qquad i = 1, \cdots, N \tag{2.11}$$

uniformly on [0, 1], *where $(e_i : i \geq 1)$ is a* CONS *in $H(R)$.*

PROOF. Let $(\xi_i : i \geq 1)$ be a CONS in $L(X)$ where $\xi_i = \nu_i(X)$ with $\nu_i \in \mathscr{C}^*$ and satisfying (2.7) and (2.8). Let θ be the natural congruence from $L(X)$ onto $H(R)$ and $e_i = \theta(\xi_i)$. Then $(e_i : i \geq 1)$ is a CONS in $H(R)$ and by the reproducing property

$$e_i(t) = \langle e_i, R(t, \cdot) \rangle_R = E\xi_i X(t) = \int R(t, s)\nu_i(ds) .$$

Because of (2.9)

$$E[\nu_i(X_n)]^2 = \int\int R_n(t, s)\nu_i(dt)\nu_i(ds) > 0 \tag{2.12}$$

for all $n \geqq n_0 = n_0(N)$ and each $i = 1, \cdots, N$. Now for each $n \geqq n_0' \geqq n_0$ and $i = 1, \cdots, N$ we may define $\nu_i^{(n)} \in \mathscr{C}^*$ recursively as follows:

$$\nu_1^{(n)} = \nu_1/\sigma_1^{(n)}; \qquad \nu_i^{(n)} = m_i^{(n)}/\sigma_i^{(n)}; \qquad i = 2, \cdots, N$$

where

$$\sigma_1^{(n)} = [E(\nu_1(X_n))^2]^{\frac{1}{2}}; \qquad \sigma_i^{(n)} = [E(m_i^{(n)}(X_n))^2]^{\frac{1}{2}}, \qquad i = 2, \cdots, N$$

and

$$m_i^{(n)} = \nu_i - \sigma_{i,i-1}^{(n)} \cdot \nu_{i-1}^{(n)} - \cdots - \sigma_{i,1}^{(n)} \cdot \nu_1^{(n)},$$

$$\sigma_{i,j}^{(n)} = \int\int R_n(t, s)\nu_i(dt)\nu_j^{(n)}(ds), \qquad j < i.$$

Obviously $||\nu_1^{(n)} - \nu_1|| \to 0$ as $n \to \infty$ which in turn implies $\sigma_{2,1}^{(n)} \to 0$ and $\sigma_2^{(n)} \to 1$ as $n \to \infty$. Hence we have $||\nu_2^{(n)} - \nu_2|| \to 0$ as $n \to \infty$ as well as, by a simple induction

$$(2.13) \qquad ||\nu_i^{(n)} - \nu_i|| \to 0 \quad \text{as} \quad n \to \infty, \qquad i = 1, \cdots, N.$$

Define $\xi_i^{(n)} = \nu_i^{(n)}(X_n)$. Then $\xi_i^{(n)} \in L(X_n)$ and $E\xi_i^{(n)}\xi_j^{(n)} = \delta_{ij}$ where δ_{ij} equals 1 or 0 according as $i = j$ or $i \neq j$. Consequently, if $e_i^{(n)} = \theta_n(\xi_i^{(n)})$, where θ_n is the natural congruence from $L(X_n)$ onto $H(R_n)$, then $(e_i^{(n)}: 1 \leqq i \leqq N)$ is an orthonormal set in $H(R_n)$ and $e_i^{(n)}(t) = E\xi_i^{(n)}X_n(t) = \int R_n(t, s)\nu_i^{(n)}(ds) \to \int R(t, s)\nu_i(ds) = e_i(t)$ as $n \to \infty$, uniformly on $[0, 1]$ for each $i = 1, \cdots, N$.

The following lemma will allow us to deduce an important consequence from Lemma 2.2.

LEMMA 2.3. *Let $(R_n: n \geqq 1)$ and R be covariance functions corresponding to centered Gaussian processes as in Lemma* 2.2. *Then for any $\varepsilon > 0$ it is possible to find a positive integer $N = N(\varepsilon)$ such that if condition* (2.9) *is satisfied*

(a) *for some* CONS'*s $(e_i^{(n)}: i \geqq 1)$ in $H(R_n)$, $\lim_{n\to\infty} e_i^{(n)}(t) = e_i(t)$, $i = 1, \cdots, N$, uniformly on* $[0, 1]$ *where $(e_i: i \geqq 1)$ is a* CONS *in $H(R)$*;

(b) $\sup_{n\geqq 1} ||R_N^{(n)}||_C < \varepsilon$, *where* $R_N^{(n)}(t) = \sum_{N+1}^{\infty} [e_i^{(n)}(t)]^2$.

PROOF. Let $(e_i: i \geqq 1)$ be a CONS in $H(R)$ defined by $e_i(t) = \int R(t, s)\nu_i(ds)$ where the measures ν_i satisfy conditions (2.7) and (2.8). Let us set $R'(t) = R(t, t)$, $R_n'(t) = R_n(t, t)$. Then for any $\varepsilon > 0$ there exists an integer $N_1 = N_1(\varepsilon)$ such that $||R' - \sum_{i=1}^{N_1} e_i^2||_C < \varepsilon/10$. For $n \geqq n_0' \geqq n_0$, where n_0 is determined by (2.12), we write $H(R_n) = H_{N_1,n} \oplus H_{N_1,n}^{\perp}$ where $H_{N_1,n} = sp(e_1^{(n)}, \cdots, e_{N_1}^{(n)})$ and the set $(e_i^{(n)}: 1 \leqq i \leqq N_1)$ satisfies (2.10) and (2.11). Now if $(f_i^{(n)}: i \geqq 1)$ is an orthonormal basis for $H_{N_1,n}^{\perp}$ and we set $e_{N_1+i}^{(n)} = f_i^{(n)}$, $i \geqq 1$, then $(e_i^{(n)}: i \geqq 1)$ are CONS's in $H(R_n)$ and (a) holds. Furthermore we can write $R_n'(t) = \sum_{i=1}^{\infty} [e_i^{(n)}(t)]^2$ uniformly on $[0, 1]$; hence,

$$||\sum_{N_1+1}^{\infty} [e_i^{(n)}]^2||_C \leqq ||R_n' - R'||_C + ||\sum_{i=1}^{N_1} [e_i^{(n)}]^2 - \sum_{i=1}^{N_1} e_i^2||_C + ||R' - \sum_{i=1}^{N_1} e_i^2||_C \leqq \tfrac{3}{10}\varepsilon$$

for all $n \geqq n_1 = n_1(N_1, \varepsilon)$. Since $\sum_{N_1+1}^{\infty} [e_i^{(n)}]^2$ is a series of positive functions one can choose an $N > N_1$, if necessary, so that the conclusion claimed under (b) will follow.

3. **Proof of the theorem.** Let E be a topological space and $A \subset E$. The set A is said to be sequentially compact if every sequence in A has at least a subsequence converging to some point of E. The set of limit points of A in E will be denoted by $\mathscr{L}_E(A)$ or simply by $\mathscr{L}(A)$ if E is understood.

Proof of the relation $\mathscr{L}(Y_n) \subset K$ a.s. Under assumption (a) of the theorem all processes $(X_n(t): t \in [0, 1], n \geqq 1)$ have continuous paths on $[0, 1]$ a.s. by a classical result of Fernique (1964). Moreover, (1.2) combined with (1.1) imply the existence of a centered Gaussian process $(X(t): t \in [0, 1])$ on some probability space $(\Omega', \mathscr{F}', P')$ also having continuous paths on $[0, 1]$ a.s. Let $\varepsilon > 0$ be chosen and $(e_i^{(n)}: i \geqq 1)$, $(e_i: i \geqq 1)$ be CONS's in $H(R_n)$ and $H(R)$ respectively, determined according to Lemma 2.3 where we take N so large that

$$\sup_{n\geqq 1} ||R_N^{(n)}||_C^{\frac{1}{2}} \leqq \varepsilon/4 . \tag{3.1}$$

Since $e_i^{(n)} = \theta_n(\xi_i^{(n)})$, where $\xi_i^{(n)} = \nu_i^{(n)}(X_n) \in L(X_n)$ are independent $N(0, 1)$ random variables and θ_n is the natural congruence from $L(X_n)$ onto $H(R_n)$, we can write

$$X_n(t) = \sum_{i=1}^{\infty} e_i^{(n)}(t)\xi_i^{(n)} \quad \text{a.s.} \tag{3.2}$$

and the series is uniformly convergent on $[0, 1]$ (Jain and Kallianpur (1970), Kuelbs (1971). Let

$$X_n^N(t) = \sum_{i=1}^{N} e_i^{(n)}(t)\xi_i^{(n)} , \qquad U_N^{(n)}(t) = X_n(t) - X_n^N(t) .$$

Then

$$E[U_N^{(n)}(t) - U_N^{(n)}(s)]^2 \leqq g^2(|t - s|)$$

so that if $R_N^{(n)}$ denotes the covariance function of $(U_N^{(n)}(t): t \in [0, 1])$ we obtain by using again Fernique's theorem

$$\begin{aligned} &P(||U_N^{(n)}||_C \geqq \varepsilon(2 \lg n)^{\frac{1}{2}}) \\ &\quad = P(||U_N^{(n)}||_C \geqq c_{N,n}(||R_N^{(n)}||_C^{\frac{1}{2}} + 4 \textstyle\int_1^\infty g(p^{-u^2})\, du)) \\ &\quad \leqq 4p \textstyle\int_{c_{N,n}}^\infty e^{-u^2/2}\, du \end{aligned} \tag{3.3}$$

for all positive integers p, whenever

$$c_{N,n} = \frac{\varepsilon(2 \lg n)^{\frac{1}{2}}}{||R_N^{(n)}||_C^{\frac{1}{2}} + 4 \int_1^\infty g(p^{-u^2})\, du} \geqq (1 + 4p)^{\frac{1}{2}} .$$

If we now choose $p = p(\varepsilon)$ so large that

$$4 \int_1^\infty g(p^{-u^2})\, du = 4(\lg p)^{-\frac{1}{2}} \int_{(\lg p)^{\frac{1}{2}}}^\infty g(e^{-u^2})\, du \leqq \varepsilon/4$$

then by using (3.1) we have

$$c_{N,n} \geqq 2(2 \lg n)^{\frac{1}{2}} \geqq (1 + 4p)^{\frac{1}{2}}$$

for all large n. By the usual estimate of the tail of the normal distribution we obtain from (3.3)

$$P(\|U_N^{(n)}\|_C \geqq \varepsilon(2\lg n)^{\frac{1}{2}}) \leqq (\text{const.})n^{-4}$$

for all $n \geqq n_1 = n_1(N, p, \varepsilon)$. Hence by the Borel–Cantelli lemma

$$P(\|Y_n - Y_N^{(n)}\|_C \geqq \varepsilon, \text{ i.o.}) = 0 \tag{3.4}$$

where $Y_N^{(n)}(t) = (2\lg n)^{-\frac{1}{2}} X_n^N(t)$ and i.o. is relative to the index n. Let $x_i^{(n)} = (2\lg n)^{-\frac{1}{2}}\xi_i^{(n)}$ and $x^{(n)} = (x_1^{(n)}, \cdots, x_N^{(n)})$. It is easy to check that $(x^{(n)}: n \geqq 2)$ is with probability one a sequentially compact subset of $\mathbb{R}^N$ and $\mathscr{L}_{\mathbb{R}^N}(x^{(n)}) \subset B_N$ a.s. where B_N is the unit ball in $\mathbb{R}^N$. Thus for each $\omega \in \Omega_0$, where $\Omega_0 \in \mathscr{F}$ and $P\Omega_0 = 1$, any subsequence $(x^{(m)}: m \geqq 2)$ of $(x^{(n)}: n \geqq 2)$ has a further subsequence $(x^{(m')}: m' \geqq 2)$, depending on ω, such that $x_i^{(m')} \rightarrow_{m'} x_i$, $i = 1, \cdots, N$ and $\sum_{i=1}^N x_i^2 \leqq 1$. Moreover, since $e_i^{(n)} \rightarrow_n e_i$ in $\mathscr{C}$, we also have:

$$Y_N^{(m')} \rightarrow_{m'} \sum_{i=1}^N e_i x_i \in K_N \quad \text{in} \quad \mathscr{C}, \tag{3.5}$$

where $K_N = (h \in H(R): h \in sp(e_1, \cdots, e_N), \|h\|_R \leqq 1)$. It follows from (3.5) that $\mathscr{L}(Y_N^{(n)}) \subset K_N \subset K$ a.s. and by taking into account (3.4) we may conclude that $\mathscr{L}(Y_n)$ is a.s. contained in an ε-neighborhood of K. This finishes the proof of the first part of the theorem.

Proof of the relation $K \subset \mathscr{L}(Y_n)$ a.s. Let us choose first ε in (0, 1). From now on, for sake of simpler exposition we shall assume, with no loss of generality, the following two conditions:

(a) $\sigma = \sup(R^{\frac{1}{2}}(t, t): t \in [0, 1]) = 1$;

(b) the index n is taken larger than the integer n^* defined as follows: $n^* = \sup(n: |\sigma_n - 1| \geqq \varepsilon)$ where $\sigma_n = \sup(R_n^{\frac{1}{2}}(t, t): t \in [0, 1])$.

Let now f be in K. Then for some $\nu \in \mathscr{C}^*$ the element h in $H(R)$ defined by $h(t) = \int R(t, s)\nu(ds)$ satisfies the condition $\|f - h\|_R < \varepsilon/2$. This in turn implies

$$\|f - h\|_C < \varepsilon. \tag{3.6}$$

Given such an element h we construct a sequence (h_n) with $h_n \in H(R_n)$ for each n as follows

$$h_n(t) = \int R_n(t, s)\nu(ds). \tag{3.7}$$

By the uniform convergence of (R_n) to R on the unit square we observe that

$$\|h_n - h\|_C \rightarrow_{(n\to\infty)} 0, \tag{3.8}$$

$$\|h\|_R = \lim_{n\to\infty} \|h_n\|_n. \tag{3.9}$$

In (3.9), that is a consequence of Lemma 2.1, we have set $\|\cdot\|_n = \|\cdot\|_{R_n}$. Let us give R_n the representation $R_n(t, s) = \sum_{i=1}^\infty e_i^{(n)}(t)e_i^{(n)}(s)$, where $(e_i^{(n)}: i \geqq 1)$ are CONS's in $H(R_n)$ defined at the beginning of this section. Then each function h_n defined by (3.7) can be written in the form $h_n(t) = \sum_i e_i^{(n)}(t)h_i^{(n)}$, where

$$h_i^{(n)} = \int e_i^{(n)}(t)\nu(dt). \tag{3.10}$$

For each integer k set $h_{k,n}(t) = \sum_{i=1}^{k} e_i^{(n)}(t) h_i^{(n)}$. Then

$$||h_n - h_{k,n}||_C = ||\sum_{i=k+1}^{\infty} e_i^{(n)} h_i^{(n)}||_C \leqq ||\sum_{i=k+1}^{\infty} e_i^{(n)} h_i^{(n)}||_n \cdot \sigma_n$$
$$\leqq (\sum_{i=k+1}^{\infty} [h_i^{(n)}]^2)^{\frac{1}{2}}(1 + \varepsilon) .$$

Furthermore,

$$\sum_{i=k+1}^{\infty} [h_i^{(n)}]^2 = \sum_{i=k+1}^{\infty} [\int e_i^{(n)}(t)\nu(dt)]^2$$
$$\leqq ||\nu|| \sum_{i=k+1}^{\infty} [\int [e_i^{(n)}(t)]^2 |\nu(dt)|]$$
$$= ||\nu|| \int \sum_{i=k+1}^{\infty} [e_i^{(n)}(t)]^2 |\nu(dt)|$$
$$\leqq ||\nu||^2 ||\sum_{i=k+1}^{\infty} [e_i^{(n)}]^2||_C .$$

Hence there exists an integer N, independent of n, such that

$$(3.11) \qquad ||h_n - h_{N,n}||_C < \varepsilon$$

and

$$(3.12) \qquad P(||Y_n - Y_N^{(n)}||_C \geqq \varepsilon, \text{ i.o.}) = 0$$

where as before $Y_N^{(n)} = (2 \lg n)^{-\frac{1}{2}} \sum_{i=1}^{N} \xi_i^{(n)} e_i^{(n)}$. Such an N will be kept fixed from now on. We claim that the theorem is proved if we can show

$$(3.13) \qquad P(||Y_N^{(n)} - (1 - \varepsilon)h_{N,n}||_C < 2\varepsilon, \text{ i.o.}) = 1 .$$

In fact

$$||Y_n - f||_C \leqq ||Y_N^{(n)} - (1 - \varepsilon)h_{N,n}||_C$$
$$+ [||Y_n - Y_N^{(n)}||_C + ||(1 - \varepsilon)h_{N,n} - h_{N,n}||_C$$
$$+ ||h_{N,n} - h_n|| + ||h_n - h||_C + ||h - f||_C] .$$

Of the terms in brackets, in the above expression, $||h - f||_C$ and $||h_{N,n} - h_n||_C$ are already each less than ε by (3.6) and (3.11). By (3.8) $||h_n - h||_C < \varepsilon$ for all $n \geqq n_2$ and by (3.9)

$$||(1 - \varepsilon)h_{N,n} - h_{N,n}||_C = \varepsilon||\sum_{i=1}^{N} e_i^{(n)} h_i^{(n)}||_C$$
$$\leqq \varepsilon(\sum_{i=1}^{\infty} [h_i^{(n)}]^2)^{\frac{1}{2}}(1 + \varepsilon)$$
$$\leqq \varepsilon(1 + \varepsilon)^2 \qquad \text{for all } n \geqq n_3 .$$

To prove (3.13) we shall follow Lai's paper. Recall that the random variables $\xi_i^{(n)}$ in (3.2) can be written in the form $\xi_i^{(n)} = \nu_i^{(n)}(X_n)$ where $\nu_i^{(n)} \in \mathscr{C}^*$ and by (2.13) $||\nu_i^{(n)}|| \leqq A_i$, $i = 1, \cdots, N$. Consequently:

$$E[(\xi_i^{(m)} \mid \mathscr{F}_n)]^2 \leqq ||\nu_i^{(m)}||^2 \sup_{t \in [0,1]} E[E(X_m(t) \mid \mathscr{F}_n)]^2 ,$$

so that $\max_{|m-n|>r} E[E(\xi_i^{(m)} \mid \mathscr{F}_n)]^2 \to 0$ as $r \to \infty$ for $i = 1, \cdots, N$, if (1.3) holds. Therefore there exists an integer $r = r(\varepsilon, N)$ such that

$$E[E(\xi_i^{(rn)} \mid \mathscr{F}_{r(n-1)})]^2 < v^2 \qquad i = 1, \cdots, N$$

where

$$(3.14) \qquad v = \varepsilon/3N(1 + \varepsilon) .$$

Let us take $n' = \max(n_2, n_3, n^*)$ and set for $n \geqq n'$

$$U_i^{(rn)} = E[\xi_i^{(rn)} \mid (\xi_1^{(rj)}, \cdots, \xi_N^{(rj)}), j = 1, \cdots, n-1].$$

Then $E[U_i^{(rn)}]^2 < v^2$, $i = 1, \cdots, N$. Defining $V_i^{(rn)} = \xi_i^{(rn)} - U_i^{(rn)}$ for $i = 1, \cdots, N$ we observe that the sequence $(\{V_1^{(rn)}, \cdots, V_N^{(rn)}\} : n \geqq n')$ has independent components. If we set

$$S_N^{(rn)}(t) = (2 \lg rn)^{-\frac{1}{2}} \sum_{i=1}^N V_i^{(rn)} e_i^{(rn)}(t)$$

and

$$G_{N,n} = (\|Y_N^{(rn)} - (1-\varepsilon)h_{N,rn}\|_C < \varepsilon/2),$$
$$F_{N,n} = (\|S_N^{(rn)} - (1-\varepsilon)h_{N,rn}\|_C < \varepsilon),$$
$$B_{i,n} = (|U_i^{(rn)}| \|e_i^{(rn)}\|_C < (2 \lg rn)^{\frac{1}{2}}\varepsilon/2N)$$

one can easily check that

$$P(F_{N,n}) \geqq P(G_{N,n}) - \sum_{i=1}^N P(B_{i,n}^c). \tag{3.15}$$

(If A is an event, A^c denotes its complement.) We now claim that

$$\sum_n P(G_{N,n}) = \infty, \tag{3.16}$$

$$\sum_n P(B_{i,n}^c) < \infty \qquad \text{for} \quad i = 1, \cdots, N, \tag{3.17}$$

$$P(\|S_N^{(rn)} - Y_N^{(rn)}\|_C \geqq \varepsilon, \text{ i.o.}) = 0. \tag{3.18}$$

Recalling (3.10), (3.14) and using standard estimates (cf. Lai (1974), Oodaira (1972) and (1973)), we obtain for all $n \geqq n'' \geqq n'$

$$\begin{aligned} P(G_{N,n}) &\geqq \prod_{i=1}^N P(|\xi_i^{(rn)} - (2 \lg rn)^{\frac{1}{2}}(1-\varepsilon)h_i^{(rn)}| < (2 \lg rn)^{\frac{1}{2}} 3v/2) \\ &\geqq (\text{const.})(2 \lg rn)^{-\frac{1}{2}}(rn)^{-\varepsilon_{N,n}}, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_{N,n} &= (1-\varepsilon)^2 \sum_{i=1}^N [h_i^{(rn)}]^2 \\ &\leqq (1-\varepsilon)^2 \sum_{i=1}^\infty [h_i^{(rn)}]^2 \\ &= (1-\varepsilon)^2 \|h_{rn}\|_{rn}^2 \\ &\leqq (1-\varepsilon)^2(1+\varepsilon)^2 = (1-\varepsilon^2)^2 < 1. \end{aligned}$$

(The last inequality follows from (3.9) and the fact that $\|f - h\|_R < \varepsilon/2$.) Thus (3.16) holds. Now let $D_i^{(rn)} = (E[U_i^{(rn)}]^2)^{\frac{1}{2}}$. Then $P(B_{i,n}^c) = 0$ for all n such that $D_i^{(rn)} = 0$. On the other hand

$$\varepsilon/2N \|e_i^{(rn)}\|_C D_i^{(rn)} \geqq \tfrac{3}{2} \qquad i = 1, \cdots, N$$

if $D_i^{(rn)} > 0$. Hence if Z denotes a $N(0, 1)$ random variable

$$P(B_{i,n}^c) \leqq P(|Z| \geqq \tfrac{3}{2}(2 \lg rn)^{\frac{1}{2}})$$

and (3.17) follows. Finally we observe that

$$P(\|S_N^{(rn)} - Y_N^{(rn)}\|_C \geqq \varepsilon) \leqq \sum_{i=1}^N P(B_{i,n}^c).$$

Thus by using the Borel–Cantelli lemma, (3.18) follows from (3.17). Since

$(F_{N,n}: n \geqq n')$ is a sequence of independent events, we obtain from (3.15), (3.16) and (3.17) that $P(F_{N,n}, \text{ i.o.}) = 1$. This, coupled with (3.18), obviously implies (3.12), and the proof is complete.

4. An application to the law of the iterated logarithm. Let $(X(t): t \geqq 0)$ be a centered Gaussian process with continuous paths on $[0, \infty)$ a.s. and $P[X(0) = 0] = 1$. Set $\sigma^2(n) = EX^2(n)$ and $f_n(t) = (2\sigma^2(n) \lg \lg n)^{-\frac{1}{2}} X(nt)$, $t \in [0, 1]$. Oodaira (1973) has shown that $\mathscr{L}_{\mathscr{C}}(f_n) = K$ a.s. where, as in the statement of the theorem, K is the unit ball in the RKHS generated by a covariance function that is the uniform limit of a sequence of covariance functions. The conditions imposed by Oodaira to establish such a result are the following:

CONDITION (a). There exists a strictly positive covariance function Γ on $[0, 1] \times [0, 1]$ such that

(a-1) $\lim_{r\to\infty} R(rt, rs)/v(r) = \Gamma(t, s)$ uniformly on $[0, 1] \times [0, 1]$, where $v(r) \to \infty$, as $r \to \infty$, and $R(t, s) = EX(t)X(s)$;

(a-2) $|R(rt, rt) - 2R(rt, rs) + R(rs, rs)| \leqq v(r)g(|t - s|)$ where g is a nondecreasing, positive function on $[0, 1]$ such that $\int_1^\infty g(e^{-u^2})\, du < \infty$;

(a-3) $\Gamma(t, t)$ is strictly increasing and $\Gamma(1, 1) = 1$.

It follows from (a-1) and (a-3) that $R(n, n)/v(n) \to \Gamma(1, 1) = 1$, as $n \to \infty$; consequently $\mathscr{L}(f_n) = \mathscr{L}(Z_n)$ where $Z_n(t) = (2v(n) \lg \lg n)^{-\frac{1}{2}} X(nt)$. Thus if we let

$$X_n(t) = [v(n)]^{-\frac{1}{2}} X(nt) \tag{4.1}$$

then $(X_n(t): t \in [0, 1], n \geqq 1)$ is a sequence of centered Gaussian processes on the same $(\Omega, \mathscr{F}, P)$ such that, by letting $R_n(t, s) = EX_n(t)X_n(s)$ we have $R_n \to \Gamma$, as $n \to \infty$, uniformly on the unit square. Set $n_k = [c^k]$, $c > 1$, $k \geqq 3$; then $\lg \lg n_k \sim \lg k$ and if $Y_k(t) = (2 \lg k)^{-\frac{1}{2}} X_{n_k}(t)$, it follows from our theorem that

$$\mathscr{L}(Y_k) = \mathscr{L}(f_{n_k}) \subset K \quad \text{a.s.} \tag{4.2}$$

By defining two-dimensional time Gaussian processes

$$(X_n(t_1, t_2): (t_1, t_2) \in [0, 1] \times [0, 1], |t_2 - t_1| < \delta)$$

as $X_n(t_1, t_2) = X_n(t_1) - X_n(t_2)$, it is an easy consequence of Lemma 2 of Lai (1974) and a standard argument that the family (f_n) is equicontinuous a.s. so that for any $\varepsilon > 0$ there exists a c sufficiently close to one to get

$$P(\sup_{n_k \leq n < n_{k+1}} \|f_n - f_{n_k}\|_C \geqq \varepsilon, \text{ i.o.}) = 0\,. \tag{4.3}$$

Equations (4.2) and (4.3) together imply $\mathscr{L}(f_n) \subset K$ a.s., that is Theorem 2 of [8]. To insure that $K \subset \mathscr{L}(f_n)$ a.s. the following additional conditions are given:

CONDITION (b). For any $0 \leqq \delta \leqq 1$ let $L^*(X, r\delta)$ be the smallest closed (in $L_2(\Omega, \mathscr{F}, P)$) linear manifold spanned by $(X(t): t \in [0, r\delta])$ and $L'(X, r\delta) = L(X, r) \ominus L^*(X, r\delta)$. Let $X^*_{r\delta}(t)$ and $X'_{r\delta}(t)$ be the projections of $X(t)$, $t \in [0, r]$, on $L^*(X, r\delta)$ and $L'(X, r\delta)$ respectively. Set

$$R^*_{r\delta}(t, s) = EX^*_{r\delta}(t)X^*_{r\delta}(s) \qquad \text{for} \quad t, s \in [0, r]$$
$$R'_{r\delta}(t, s) = EX'_{r\delta}(t)X'_{r\delta}(s) \qquad \text{for} \quad t, s \in [0, r]$$

and supppose that for each $\delta \in [0, 1)$ there exist covariance functions $\Gamma_\delta^*(t, s)$ and $\Gamma_\delta'(t, s)$ on $[0, 1] \times [0, 1]$ such that

(b-1) $|v^{-1}(r)R^*_{r\delta}(rt, rs) - \Gamma_\delta^*(t, s)| = o((\lg r)^{-1})$, $|v^{-1}(r)R'_{r\delta}(rt, rs) - \Gamma_\delta'(t, s)| = o((\lg r)^{-1})$, uniformly on $[0, 1] \times [0, 1]$;

(b-2) $H(\Gamma) = H(\Gamma_\delta^*) \oplus H(\Gamma_\delta')$ where $\Gamma(t, s) = \Gamma_\delta^*(t, s) + \Gamma_\delta'(t, s)$;

(b-3) $\Gamma_\delta^*(t, t) \to 0$ uniformly on $[0, 1]$ as $\delta \downarrow 0$;

(b-4) $\Gamma_\delta'(t, s)$, $s, t \in [\delta, 1]$, is strictly positive definite and $\Gamma_\delta'(t, t)$ is strictly monotone increasing on $[\delta, 1]$.

To obtain Theorem 3 of Oodaira, asserting $K \subset \mathscr{L}(f_n)$ a.s., we first observe that we can write

$$X^*_{r\delta}(rt) = E[X(rt) \mid \mathscr{F}_{r\delta}], \qquad t \in [0, 1], \tag{4.3}$$

where $\mathscr{F}_{r\delta} = \bigcap_{h>0} L(X, r\delta + h)$. Next, an inspection of the proof presented in Section 3 shows that the conclusion $K \subset \mathscr{L}(Y_n)$ a.s. can still be deduced if (1.3) is replaced by the following, somewhat weaker condition.

For every $\varepsilon > 0$ there exists a positive, increasing function d, from the positive integers into the positive integers, such that $n - d(n) \to \infty$, as $n \to \infty$ and

$$\limsup_{n\to\infty} \{\sup_{t\in[0,1]} E[E(X_n(t) \mid \mathscr{F}_{d(n)})]^2\} \leqq \varepsilon . \tag{4.4}$$

One may now notice that $\mathscr{F}_{[r\delta]} = \overline{sp}(X(it) : t \in [0, 1], i = 1, \cdots, [r\delta])$ so that by using (4.3) and conditions (b-1), (b-3) one obtains

$$\lim_{\delta\downarrow 0} \lim_{r\to\infty} E[E(X_r(t) \mid \mathscr{F}_{[\delta n]})]^2 = 0 \tag{4.5}$$

uniformly on $[0, 1]$. Obviously (4.4) holds by choosing δ in (4.5) sufficiently small and then letting $d(n) = [\delta n]$.

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