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LAST EXIT TIMES FROM THE BOUNDARY OF A CONTINUOUS TIME MARKOV CHAIN

BY MICHAEL W. CHAMBERLAIN

U. S. Naval Academy

Probabilities of events involving the jump of a Markov chain to the state space immediately after the last exit before a given time from a boundary atom are determined, for the most part, by the initial time value of the canonical entrance law corresponding to that atom. Three of these probabilities are calculated in terms of canonical quantities in order to attach a probabilistic meaning to an entrance law decomposition of Reuter's and to improve an analytical condition of Chung's for when the Kolmogorov forward equations are satisfied by the chain's transition matrix.

In recent years, the study of last exit times has provided considerable insight into the probabilistic and analytic structures of Markov chains. Quite naturally, the entrance laws for a chain play a pivotal role in these results. In this note, we will consider the interrelationship between the initial time vector of these laws and certain sample path behavior for the type of chain whose boundary theory is developed by Chung in [1], [2] and [3]. Briefly, the chain $X(t)$ is Borel measurable and right separable relative to the rational numbers, with a standard stochastic transition matrix $P(t) = (p_{ij}(t))$ indexed by the denumerable state space $\mathbf{I}_\theta = \mathbf{I} \cup \{\theta\}$. The initial derivative matrix $Q = P'(0)$ is conservative, the associated minimal chain with transition matrix $\Phi(t)$ has only one recurrent state θ, if any at all, and the passable part $\mathbf{A}$ of the Martin exit boundary induced on $\mathbf{I}_\theta$ by the minimal chain contains only finitely many distinguishable boundary atoms.

Chung's canonical decomposition of $P(t)$ involves looking at how the chain first reaches a boundary atom a at time

$$\alpha = \inf\{u : u > 0,\ X(u) \in \mathbf{A}\} \tag{1}$$

and then noticing what happens until time

$$\beta^a = \inf\{u : u > 0,\ X(u) \in \mathbf{A} - \{a\}\} \tag{2}$$

when a "switch of banners" occurs. Here we will be concerned with what happens when the chain last leaves a before time t at

$$\gamma_t^{\,a} = \sup\{u : 0 \leqq u \leqq t,\ X(u) = a\}\,. \tag{3}$$

The next theorem provides three related results which supplement those of

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Section 19 in [2] and Theorem 4 in [6]. Indeed, (6) below can be seen as a special case of equation (41) in [6].

(4) THEOREM. *For* $t > 0$

$$\mathbf{P}^a(\beta^a > \gamma_t^a, X(\gamma_t^a+) = j) = \int_0^t \eta_j^a(0)(1 - L_j^a(t-s))\, dE^a(s) \tag{5}$$

$$\mathbf{P}^a(\beta^a > t, X(\gamma_t^a+) = j) = \int_0^t \eta_j^a(0)(1 - L_j(t-s))\, dE^a(s) \tag{6}$$

$$\mathbf{P}^a(\beta^a > t, X(u) = j \text{ for } \gamma_t^a < u \leqq t) = \int_0^t \eta_j^a(0) e^{-q_j(t-s)}\, dE^a(s)\,. \tag{7}$$

PROOF. For positive integer n, let $m = 2^n$, $t_k = kt/m$, and

$$\Lambda_n = \bigcup_{k=1}^{m-1} \{t_{k-1} \leqq \gamma_t^a < t_k, \beta^a > t_{k+1}, X(t_{k+1}) = j\}\,.$$

From the entrance law definition (13.1) and Theorem 14.4 of [2], we have for $0 < v < u \leqq t$

$$\begin{aligned}\mathbf{P}^a(\gamma_t^a < v, \beta^a > u, X(u) = j) &= \sum_i \rho_i^a(v) f_{ij}(u-v)(1 - L_j^a(t-u)) \\ &= \int_0^v \eta_j^a(u-s)(1 - L_j^a(t-u))\, dE^a(s)\,,\end{aligned}$$

enabling us to write

$$\mathbf{P}^a(\Lambda_n) = \sum_{k=1}^{m-1} \int_{t_{k-1}}^{t_k} \eta_j^a(t_{k+1}-s)(1 - L_j^a(t - t_{k+1}))\, dE^a(s)\,. \tag{8}$$

On the one hand, it can be shown by Theorem II.7.4 of [3] that as n tends to infinity Λ_n converges to the event inside the parentheses on the left side of (5), while on the other hand the sum in (8) converges to the integral on the right of (5), thus proving that equality. With a few obvious modifications, both (6) and (7) follow in the same way.

Before we put the above theorem to use, let us recall that if a is not a recurrent trap, then $E^a(t) = \mathbf{P}^a(\gamma^a \leqq t)$ by (17.1) of [2], where γ^a is the last exit from a before a switch of atoms occurs; that is, γ^a is defined by the right-hand side of (3) with β^a in place of t. Then in this case, the integral in (5) is the probability that starting from a, a last exit from a before β^a occurs some time before t, followed by a jump to the state space governed by $\eta^a(0)$, with no more visits to a until after time t. Corresponding interpretations can also be made for the integrals in (6) and (7).

A well-known decomposition

$$\eta^a(t) = \eta^a(0)\Phi(t) + \bar{\eta}^a(t)$$

for entrance laws (see, e.g., [7]) provides us with the following zero-or-one type law about whether the chain is in the state space immediately after γ_t^a, knowing that a switch of boundary atoms has not yet occurred.

(9) COROLLARY.

$$\mathbf{P}^a(X(\gamma_t^a+)\varepsilon\mathbf{I}_\theta \mid \beta^a > \gamma_t^a) = 0 \tag{10}$$

for all $t > 0$ *if* $\eta^a(\cdot) = \bar{\eta}(\cdot)$;

$$\mathbf{P}^a(X(\gamma_t^a+)\varepsilon\mathbf{I}_\theta \mid \beta^a > \gamma_t^a) = 1 \tag{11}$$

for all $t > 0$ *if* $\eta^a(\cdot) = \eta^a(0)\Phi(\cdot)$.

PROOF. Simply sum both sides of (5) over j to get (10). To prove (11) assume for the present that

$$\mathbf{P}^a(\beta^a > \gamma_t^a) = \int_0^t \lim_{u\downarrow 0} \langle \eta^a(u), 1 - L^a(t-u-s)\rangle \, dE^a(s)\,. \tag{12}$$

If $\eta^a(\cdot) = \eta^a(0)\Phi(\cdot)$, then by the definitions and exit law properties of $L(\cdot)$ and $L^a(\cdot)$ in (2.15), (4.2) and (4.3) of [1], the limit under the integral sign in (12) equals

$$\lim_{u\downarrow 0} \langle \eta^a(0), 1 - L^a(t-s) - \textstyle\sum_{b\neq a} L^b(u)\rangle\,,$$

which is just $\langle \eta^a(0), 1 - L^a(t-s)\rangle$ by the monotone convergence theorem. This, with (5), yields (11).

Now to establish (12), rewrite the limit under the integral sign as

$$\lim_{u\downarrow 0} (\eta_*{}^a(u) - \sigma^{aa}(u)) + \sigma^{aa}(t-s)$$

by (14.20) and (14.21) of [2], which can be shown to equal

$$\delta^a \rho_*{}^a(\infty) + \textstyle\sum_{b\neq a} \sigma^{ab}(0) + \sigma^{aa}(t-s) \qquad \text{or}$$
$$\delta^a + \sigma^{aa}(t-s) - \textstyle\sum_{b\neq a} d^{ab}$$

by (14.29), Theorem (14.5), Theorem (14.8), (14.2) and (17.9) of [2]. Hence by Corollary 1 to Theorem 14.7 and by (17.7) of [2], the value of the integral in (12) equals $1 - \sum_{b\neq a} d^{ab}E^a(t)$ or $\mathbf{P}^a(\beta^a > \gamma_t^a)$, upon observing that

$$\{\gamma_t^a \geqq \beta^a\} = \{\gamma^a = \beta^a \leqq t\}\,.$$

It may be that the converse of (11) as it now stands is true. Whatever the case, the interested reader can show that upon replacing the conditioning event $\{\beta^a > \gamma_t^a\}$ by $\{\beta^a > t\}$, both (10) and (11) and their converses follow in a straightforward manner from (6).

In Section 12 of [2], the first hitting time α_s of $\mathbf{A}$ after time s is defined by the right-hand side of (1) with s in place of 0. For the proof of the next theorem, let us extend this notion to some other variables by defining

$$\alpha_s{}^a = \inf\{u: u > s, X(u) = a\}$$

and, similarly, $\beta_s{}^a$ and $\gamma_{s,t}^a$ through the substitution of s for 0 in (2) and (3). As in Section 12 of [2], we will write $i \rightsquigarrow a$ to mean that the chain, starting at i, has positive probability of hitting the boundary atom a in finite time, or, equivalently, that $K_i{}^a(t) = \mathbf{P}_i(\alpha_0{}^a \leqq t)$ is positive for all positive t.

Since Q is conservative, the matrix $P(t)$ satisfies the Kolmogorov backward equations $P'(t) = QP(t)$ automatically. As to the forward equations, Theorem II.17.4 of [3] gives a probabilistic answer while Theorem 6.8 of [1] gives an analytic answer, but only for transient chains. With Theorem (4), it is now a simple matter to extend the analytic result to the general case.

(13) THEOREM. *The forward equation*

$$\frac{d}{dt} p_{ij}(t) = \textstyle\sum_k p_{ik}(t) q_{kj} \tag{14}$$

holds if and only if $i \rightsquigarrow a$ *for some* $a \in \mathbf{A}$ *implies* $\rho_j{}^a(0) = \eta_j{}^a(0) = 0$.

PROOF. By Theorem II.17.4 of [3] we need to show that for any fixed $t > 0$, with probability 0 the chain begins at i, is at j at time t, and has as the last discontinuity before t a pseudo-jump from ∞. Now if this were to happen for a sample function $X(\cdot, \omega)$, then the pseudo-jump must occur at the last exit time $\gamma_t^a(\omega)$ from some boundary atom a. Under the assumption that the boundary atoms are distinguishable, by Theorem 4.6 of [1] we know that (almost surely) boundary atoms other than a are not visited in an interval before $\gamma_t^a(\omega)$. Hence, there exists a rational number $r(\omega)$ less than $\gamma_t^a(\omega)$ so that the chain is in the state space at time $r(\omega)$, can visit only a at the boundary until $\gamma_t^a(\omega)$, and subsequently remains at j until time t.

More precisely, if we define the event

$$\Gamma_j^a(s, t) = \{\alpha_s^a \leqq t < \beta_s^a, X(u) = j \text{ for } \gamma_{s,t}^a < u \leqq t\}$$

for $0 \leqq s < t$ and $a \in \mathbf{A}$, then (14) will hold if and only if $\mathbf{P}_i(\Gamma_j^a(r, t)) = 0$ for all boundary atoms a and all rational numbers r between 0 and t. But

$$\mathbf{P}_i(\Gamma_j^a(r, t)) = \textstyle\sum_l p_{il}(r)\mathbf{P}_l(\Gamma_j^a(0, t - r)) , \tag{15}$$

where the second factor in the series can be written, with v in place of $t - r$, as the integral

$$\textstyle\int_0^v \mathbf{P}^a(\beta^a > v - s, X(u) = j \text{ for } \gamma_{v-s}^a < u \leqq v - s)\, dK_l^a(s) . \tag{16}$$

Hence, by (15), (16) and (7), $\mathbf{P}_i(\Gamma_j^a(r, t)) = 0$ if and only if $i \rightsquigarrow l$ and $l \rightsquigarrow a$ (or $i \rightsquigarrow a$) implies $\eta_j^a(0) = 0$ and so $\rho_j^a(0) = \eta_j^a(0)\, E^a(0) = 0$.

We know that $P(t)$ satisfies the entire system of backward equations if and only if the unit vector is a regular function for the embedded jump chain. Since Theorem (6.7) of [1] can easily be extended to give $\eta^a(0) = -e^aQ$, where e^a is the canonical entrance sequence corresponding to $\eta^a(t)$, we can prove an analogous result from Theorem (13).

(17) COROLLARY. *The forward equations*

$$P'(t) = P(t)Q$$

hold if and only if each canonical entrance sequence for $P(t)$ is a regular measure for the embedded jump chain.

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Mathematics Department
U. S. Naval Academy
Annapolis, Maryland 21402