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HAUSDORFF DIMENSION AND GAUSSIAN FIELDS

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Let $X(t)$ be a Gaussian process taking values in R^d and with its parameter in R^N. Then if X_j has stationary increments and the function $\sigma^2(t) = E\{|X_j(s+t) - X_j(s)|^2\}$ behaves like $|t|^{2\alpha}$ as $|t| \downarrow 0$, $0 < \alpha < 1$, the graph of X has Hausdorff dimension $\min\{N/\alpha, N + d(1-\alpha)\}$ with probability one. If X is also ergodic and stationary, and if $N - d\alpha \geqq 0$, then the dimension of the level sets of X is a.s. $N - d\alpha$.

1. Introduction. In previous papers (e.g. [1], [2]) we have studied excursion and level sets of N-dimensional Gaussian fields possessing "smooth" sample paths (i.e., almost surely (a.s.) continuous, continuously differentiable, etc.). When some of these smoothness conditions are relaxed it is clear by analogy with the well-researched one-dimensional case that the topological properties of these sets that we have been studying are no longer appropriate concepts, as the sample paths become wildly erratic. However, in these situations it becomes interesting to somehow measure the "size" of the sample paths and level sets. The appropriate concept in this regard is that of *Hausdorff dimension*. A set $E \subset R^N$ is said to have Hausdorff dimension α if

$$\alpha = \inf_\beta \{\beta : \lim\inf_{\varepsilon\to 0} \sum d_i^{\,\beta} = 0\}$$

where the infimum within the brackets is taken over all collections of closed balls in R^N, each with radius $d_i < \varepsilon$, whose union covers E.

In this paper we shall be concerned with random processes $X(t)$ that we shall call (N, d) Gaussian fields, i.e.,

$$X(t, \omega) = (X_1(t, \omega), \cdots, X_d(t, \omega)) \in R^d\,, \qquad \text{where} \quad t = (t_1, \cdots, t_N) \in R^N\,,$$

and the coordinate functions X_j are mutually independent, separable, Gaussian fields with mean zero and identical covariance function $R(s, t) = E\{X_j(s)X_j(t)\}$. In the following section we shall obtain the dimension of the graph of such a process, and in Section 3 we shall consider the main results of this paper, the dimension of the level sets. The results of Section 2 are obtained by a relatively straightforward application to the Gaussian case of techniques used by Yoder [13] for (N, d) Brownian motion, and contain the corresponding result for (1, 1) Gaussian fields (Orey [9], Theorem 1). In Section 3 we apply, in a more direct fashion, a technique used initially by Kahane [5]. The results of this section generalize to (N, d) fields similar results for the (1, 1) case due to Berman [3]. A referee has pointed out that results for the (1, 1) case are also due to Marcus

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[7], but at the time of writing I have not seen this paper. The paper concludes with a sequence of remarks.

2. Dimension of the graph. In this section we shall assume that each coordinate function X_j of X has stationary increments. Thus there exists a function $\sigma(t): R^N \to R^1$ for which for all j and s

$$E\{|X_j(s+t) - X_j(s)|^2\} = \sigma^2(t)\,. \tag{2.1}$$

We shall throughout this paper use $|\cdot|$ to denote the Euclidean norm in whatever dimension is appropriate at each use. Following [9] we introduce the notations:

$$\alpha^* = \sup\{\alpha : \sigma(t) = o(|t|^\alpha),\ |t| \downarrow 0\}\,, \qquad \alpha_* = \inf\{\alpha : |t|^\alpha = o(\sigma(t)),\ |t| \downarrow 0\}\,.$$

Then $0 \leqq \alpha_* \leqq \alpha^* \leqq \infty$. When $\alpha^* = \alpha_* = \alpha$ we shall say that $\sigma(t)$ has *index* α. We shall be interested in the case in which σ has index lying (strictly) between zero and one.

Let I_n denote the unit cube in R^n. Then, without any loss of generality, we shall now consider the Hausdorff dimension of the graph of an (N, d) Gaussian field $X(t)$ as t varies over I_N, (i.e., $\{(t, X(t)) : t \in I_N\} \subset R^{N+d}$) which we write as dim (gr X). (It is easy to see that in what follows I_N may be replaced by any compact subset of R^N). We shall establish

THEOREM 1. *Suppose* $\sigma(t)$, *as defined by* (2.1), *has index* α, $0 < \alpha < 1$. *Then with probability one,*

$$\dim(\mathrm{gr}\,X) = \min\{N/\alpha,\ N + d(1-\alpha)\}\,. \tag{2.2}$$

PROOF. The proof is not very different than that used by Yoder [13] to establish the same result for (N, d) Brownian motion ($\alpha = \frac{1}{2}$), so that we shall not give it in full detail. The proof falls into three parts. First we note that from Corollary 4 of Theorem 2 of Yadrenko [12] it easily follows that $X(t)$ is a.s. Lipschitz of order α, so that by statement I in the proof in [13] we have

$$\dim(\mathrm{gr}\,(X)) \leqq \min\{N/\alpha,\ N + d(1-\alpha)\} \quad \text{a.s.} \tag{2.3}$$

It remins to establish the opposite inequality, which is proven in two stages.

Consider first the case $N \leqq d\alpha$. We shall show that in this case dim (gr(X)) $\geqq N/\alpha$ a.s. Let $\beta < N/\alpha$. Then since the dimension of the graph of X must be at least that of its range it follows that it is sufficient to show that the β-capacity of the range, $C_\beta(\mathrm{ra}\,(X))$, is a.s. positive. For this it sufficient to show that

$$\int_{I_N \times I_N} |X(s) - X(t)|^{-\beta}\, ds\, dt < \infty \quad \text{a.s.} \tag{2.4}$$

(cf. Theorem B, Taylor [11]). By the independence and identical distribution of the X_j we have

$$\begin{aligned} &E\{|X(t) - X(s)|^{-\beta}\} \\ &\quad = [2\pi\sigma^2(t-s)]^{-d/2} \int_{R^d} |x|^{-\beta} \exp\left(-\frac{x_1^2 + \cdots + x_d^2}{2\sigma^2(t-s)}\right) dx\,. \end{aligned} \tag{2.5}$$

By changing to spherical coordinates and then letting $r = \sigma(t-s)x$ this becomes

$$K_1(\sigma(t-s))^{-\beta} \int_0^\infty x^{d-1-\beta} \exp(-\tfrac{1}{2}x^2)\, dx .$$

The integral is finite since $\beta < d$, so that

$$E\{|X(t) - X(s)|^{-\beta}\} \leqq K_2(\sigma(t-s))^{-\beta} .$$

Thus

$$E\{\int_{I_N \times I_N} |X(t) - X(s)|^{-\beta}\, dt\, ds\} \leqq \int_{I_N \times I_N} K_2(\sigma(t-s))^{-\beta}\, dt\, ds .$$

Since σ has index α the last integral is finite when $\alpha\beta < N$, and then (2.4) must hold by Fubini's theorem. This suffices to establish the appropriate result for $N \leqq d\alpha$.

When $N > d\alpha$ we can follow III of [13] to show $\dim(\mathrm{gr}(X)) \geqq N + d(1-\alpha)$ a.s. Combining this with the previous case and (2.3) establishes the theorem.

3. Dimension of the level sets. In this section we make the further assumption that the coordinate functions X_j are also stationary, so that their common covariance function R is a function of $t - s$ only. Furthermore, for the sake of simplicity, we set $R(0) = 1$. We define $X^{-1}(u)$ to be the u-level set of X over I_N, i.e.,

(3.1) $$X^{-1}(u) = \{t \in I_N : X(t) = u\} .$$

We are interested in the dimension of $X^{-1}(u)$, and shall prove

THEOREM 2. *If the conditions of Theorem* 1 *hold, X is stationary, and $N - d\alpha \geqq 0$, then for almost every u*

(3.2) $$\dim X^{-1}(u) = N - d\alpha$$

with positive probability.

PROOF. Our proof of this result will be based on a technique used by Kahane [5] to prove a similar result for Gaussian Fourier series. However we require firstly some results from potential theory, which can all be found in the first two chapters of Landkof [6].

Let μ be a measure with compact support in R^N. We say that μ has finite β-energy if

$$I_\beta(\mu) = \int_{R^N \times R^N} |t-s|^{-\beta}\, d\mu(t)\, d\mu(s) < \infty .$$

A compact set $E \subset R^N$ has positive β-capacity $C_\beta(E)$ if it carries a nonzero positive measure of finite β-energy. We shall use this fact to show that the β-capacity of $X^{-1}(u)$ is positive for all $\beta < N - d\alpha$, so that $\dim X^{-1}(u) \geqq N - d\alpha$. This is Lemma 2. Combining this with the following lemma then establishes Theorem 2.

LEMMA 1. *Let $F: I_N \to R^d$ be a continuous function that is Lipschitz of order α on I_N. Then if $N - d\alpha \geqq 0$, $\dim F^{-1}(u) \leqq N - d\alpha$ for almost every u in R^d.*

PROOF. This lemma generalizes a known result in the case where $N = 1$ ([5], page 142) and we follow the original proof. For given $h > 0$ let $n = (n_1, \cdots, n_N)$ be a lattice point for which the n_j are one of the numbers $0, 1, \cdots, [h^{-1}]$, and let J_n be the rectangle $\{t \in R^N : n_j h \leqq t_j \leqq (n_j + 1)h, j = 1, \cdots, N\}$. Let $G(h)$ be the union of the rectangles $J_n \times F(J_n) \subset R^{N+d}$. Then the graph of F is contained in $G(h)$ for any h, and the Lebesgue measure of $G(h)$ in R^{N+d} is $O(h^{d\alpha})$. Given $u \in R^d$, let $E(u, h)$ denote the union of rectangles J_n for which $u \in F(J_n)$. This set clearly contains $F^{-1}(u)$. Let us now write λ_r for Lebesgue measure in R^r. Then

$$\lambda_{N+d}(G(h)) = \int_{R^d} \lambda_N(E(u, h))\, du .$$

Now let $h_\nu = 2^{-\nu}$. Then given $\varepsilon > 0$ we have

$$\sum_{\nu=1}^{\infty} h_\nu^{\varepsilon - d\alpha} \int_{R^d} \lambda_N(E(u, h))\, du < \infty ,$$

implying

$$\sum_{\nu=1}^{\infty} h_\nu^{\varepsilon - d\alpha} \lambda_N(E(u, h)) < \infty$$

for (λ_N) almost every u. Thus

$$\lambda_N(E(u, h)) = o(h^{d\alpha - \varepsilon})$$

for almost every u. Thus, using the fact that $N - d\alpha \geqq 0$, it follows that $F^{-1}(u)$ has measure zero in the dimension $N - d\alpha + \varepsilon$ for almost every u, which proves the lemma.

LEMMA 2. *If the conditions of Theorem* 2 *hold and* $\beta < N - d\alpha$ *then*

$$C_\beta(X^{-1}(u)) > 0 \tag{3.3}$$

with positive probability.

PROOF. As we noted earlier, it is sufficient to show that with positive probability $X^{-1}(u)$ carries a nonzero positive measure of finite β-energy. For the moment, let us consider only the case $u = 0$. For each $\varepsilon > 0$ define

$$\begin{aligned} T_\varepsilon(t) &= (2\pi/\varepsilon)^{d/2} \exp(-|X(t)|^2/2\varepsilon) \\ &= \int_{R^d} \exp(-\tfrac{1}{2}\varepsilon|u|^2 + iu \cdot X(t))\, du . \end{aligned}$$

Then T_ε is the density of a (positive) measure, μ_ε say, on R^N. Note T_ε may be written as $T_\varepsilon = \delta_\varepsilon(X)$, where as ε tends to zero the measure corresponding to δ_ε tends to the Dirac measure in R^d. (In [1] and [2] such a measure was used in obtaining expressions for the mean values of the excursion characteristics of $(N, 1)$ Gaussian fields.)

Now it is known that the set of positive measures of finite β-energy form a complete metric space within the larger Hilbert space of signed measures of finite β-energy with the inner product

$$(\mu, \nu) = \int_{R^N \times R^N} |t - s|^{-\beta}\, d\mu(t)\, d\nu(s)$$

([6], page 90). We shall now show that under the conditions of the lemma a particular sequence of measures μ_ε is a.s. Cauchy with respect to the derived metric.

The expected value of the squared norm of the difference $\mu_\varepsilon - \mu_\eta$ is

$$E(\|\mu_\varepsilon - \mu_\eta\|^2) = \int_{I_N \times I_N} \int_{R^d \times R^d} E\{[\exp(-\tfrac{1}{2}\varepsilon|u|^2 + iuX(t)) - \exp(-\tfrac{1}{2}\eta|u|^2 + iuX(t))] \times [\exp(-\tfrac{1}{2}\varepsilon|\hat{u}|^2 + iuX(s)) - \exp(-\tfrac{1}{2}\eta|\hat{u}|^2 + i\hat{u}X(s))]\}|t - s|^{-\beta}\, du\, d\hat{u}\, dt\, ds\,. \tag{3.4}$$

Consider the expectation in this expression. This equals

$$E\{\exp(iuX(t) + i\hat{u}X(s)) \times [\exp(-\tfrac{1}{2}\varepsilon|u|^2) - \exp(-\tfrac{1}{2}\eta|u|^2)][\exp(-\tfrac{1}{2}\varepsilon|\hat{u}|^2) - \exp(-\tfrac{1}{2}\eta|\hat{u}|^2)]\}\,. \tag{3.5}$$

Only the first expression here is random. Consider it:

$$E\{\exp(iuX(t) + \hat{u}X(s))\} = \exp(-\tfrac{1}{2}(|u|^2 + |\hat{u}|^2 + 2R(t - s)u \cdot \hat{u}))\,. \tag{3.6}$$

Here (3.6) is a simple consequence of the fact that the components X_j of X are all independent Gaussian variates with covariance function R. On comparing (3.4)—(3.6) it is clear from dominated convergence that $\lim_{\varepsilon,\eta\to 0} E(\|\mu_\varepsilon - \mu_\eta\|^2) = 0$ if the following integral is bounded:

$$\int_{I_N \times I_N} \int_{R^d \times R^d} \exp(-\tfrac{1}{2}(|u|^2 + |\hat{u}|^2 + 2R(t - s)u \cdot \hat{u}))|t - s|^{-\beta}\, du\, d\hat{u}\, dt\, ds\,.$$

However, a little careful manipulation of multivariate normal integrals gives us that except for a constant factor this integral is equal to

$$\int_{I_N \times I_N} [1 - (1 - \sigma^2(t - s))^2]^{-d/2}|t - s|^{-\beta}\, dt\, ds \leqq \int_{I_N \times I_N} |t - s|^{-\beta}\sigma^{-d}(t - s)\, dt\, ds\,. \tag{3.7}$$

Since σ has index α and $\beta < N - d\alpha$ this integral is finite. Thus we have established that

$$\lim_{\varepsilon,\eta\to 0} E(\|\mu_\varepsilon - \mu_\eta\|^2) = 0\,.$$

Now choose a sequence ε_n tending to zero so that $E(\|\mu_{\varepsilon_{n+1}} - \mu_{\varepsilon_n}\|^2) < 2^{-n}$. Then it can be readily seen that

$$\sum_{n=1}^{\infty} \|\mu_{\varepsilon_{n+1}} - \mu_{\varepsilon_n}\| < \infty \quad \text{a.s.}$$

so that for this sequence of ε_n the measures form a Cauchy sequence, and there exists a limit. As in [5], page 148, it is now easy to show that this limit measure is carried by $X^{-1}(u)$, and is nonzero with positive probability. This completes the proof of the lemma for the case $u = 0$.

For more general u the same proof goes through, except that in the definition of $T_\varepsilon(t)$ we replace $|X(t)|$ by $|X(t) - u|$. This change makes no essential difference to the remainder of the argument.

Thus we have now succeeded in establishing Theorem 2 and know that (3.2) holds with positive probability, δ say, which from the form of the proof clearly does not depend on either u or the fact that we are considering level sets of X over I_N, rather than any other compact subset of R^N. It is of interest to know conditions under which this probability will be one, and to study this it is necessary to introduce the concept of ergodicity for multidimensional parameter spaces. To do this we introduce N commuting shift transformations $\tau^{(j)}$, $j = 1, \cdots, N$ on the probability space of functions $F: R^N \to R^d$, given by $\tau^{(j)}F(t_1, \cdots, t_j, \cdots, t_N) = F(t_1, \cdots, t_j + \tau, \cdots, t_N)$. Then an (N, d) random field is said to be *ergodic* if the σ-field of measurable sets which are invariant under these transformations contains only sets of probability zero or one. With this definition it is easy to extend the one-dimensional arguments of [4] or [8] to obtain the following result (cf. Rosenblatt [10], Lemma 3.1, where a corresponding result is presented for fields defined on a lattice).

LEMMA 3. *In order that a stationary (N, d) Gaussian field with a continuous covariance function be ergodic, it is necessary and sufficient that its spectral distribution function be continuous.*

We are now in a position to state our final result.

THEOREM 3. *If the conditions of Theorem 2 hold, and, moreover, X has a continuous spectral distribution function, then with probability one, for almost every u, $\dim \{t \in R^N : X(t) = u\} = N - d\alpha$.*

PROOF. Suppose (3.2) holds with probability δ, $0 < \delta \leqq 1$. Denoting the direct sum of sets in R^N by $\oplus$, let $I^{(j)} = (j, j, \cdots, j) \oplus I_N$, and $S^{(n)} = \bigcup_{j=0}^n I^{(j)}$. Then for all j and n

$$\begin{aligned} \delta &= P\{\dim (t \in I^{(j)} : X(t) = u) = N - d\alpha\} \\ &= P\{\dim (t \in S^{(n)} : X(t) = u) = N - d\alpha\}\,. \end{aligned} \tag{3.8}$$

Now let $Y^{(j)}$ be the indicator variable for the event $\{\dim (t \in I^{(j)} : X(t) = u) = N - d\alpha\}$. By Lemma 3 we have ergodicity, so that $n^{-1} \sum_{j=0}^n Y^{(j)} \to P\{I^{(0)}\}$ a.s. as $n \to \infty$. But this means that, with probability one, for large enough n at least one of the sets $(t \in I^{(j)} : X(t) = u)$, $0 \leqq j \leqq n$, must have dimension $N - d\alpha$, which implies $\dim (t \in S^{(n)} : X(t) = u) = N - d\alpha$ a.s., for large enough n. This proves the theorem.

4. Remarks. (A) There seem two obvious directions in which our results could be extended, both, however, requiring considerable awkward algebra. The first direction would be to drop the requirement that the increments of the component processes X_j have identical "variance function," $\sigma^2(t)$, and allow each X_j its own variance function σ_j^2. However, when this is done problems arise at around equation (2.5), and these become rapidly compounded shortly thereafter if we do not insist that the σ_j^2 are of the same index. In this case it is not at all clear what the correct analogues of our results become.

(B) The second condition which could possibly be dropped is that of stationarity in Section 3. However, we then face problems at around (3.6), and the statement of Theorems 2 and 3 would need to contain unpleasant assumptions in relation to the covariance function $R(s, t)$, sufficient to ensure convergence in the new integrals that would replace those in (3.7).

(C) As we noted earlier, Theorem 1 contains the corresponding result for

(N, d) Brownian motion ($\alpha = \frac{1}{2}$) obtained in [13]. There are no results for Brownian motion corresponding to our results in Section 3 however, perhaps because of the difficulties raised in (B) above.

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