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ON THE OSCILLATION OF THE BROWNIAN MOTION RANDOM MEASURE[1]

BY SIAMAK KHALILI

IBM Corporation

We show that a Brownian motion random measure with q-dimensional parameter domain almost surely nowhere satisfies a Lipschitz condition of order greater than $\frac{1}{2}$.

Paley, Wiener and Zygmund [3] proved that with probability one the sample functions of a Brownian motion stochastic process nowhere satisfy a one-sided Lipschitz condition of order greater than $\frac{1}{2}$. In this paper we will extend this result to a Brownian motion random measure with q-dimensional parameter domain.

DEFINITION. Let $(\Omega, \mathscr{B}, P)$ be a probability space, $(T, \mathscr{S}, \mu)$ a measure space and $\mathscr{D} = \{D : D \in \mathscr{S} \text{ and } \mu(D) < \infty\}$. Then we say that ξ is a *Brownian motion random measure* over $(\Omega, \mathscr{B}, P)$ with parameter domain T iff it is a mapping from $\mathscr{D}$ into (jointly) Gaussian random variables with mean zero on Ω such that $\forall\, D_1, D_2 \in \mathscr{D}$, $\operatorname{Cov}(\xi(D_1), \xi(D_2)) = \mu(D_1 \cap D_2)$.

A Brownian motion random measure ξ has independent values on disjoint sets, since for jointly Gaussian random variables with mean 0, orthogonality implies independence.

Let N_+ denote the set of nonnegative integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+{}^q$ the set of all $t \in \mathbb{R}^q$ such that $t_j \geqq 0$ for $j = 1, \cdots, q$. If s, t and $t - s \in \mathbb{R}_+{}^q$, let $(s, t] = \times_{j=1}^{q} (s_j, t_j]$.

Let Leb. denote Lebesgue measures. For a Brownian motion random measure ξ with parameter domain $(\mathbb{R}_+{}^q, \text{Leb.})$ we may first define a stochastic process $\xi(0, t]$ for $t \in \mathbb{R}_+{}^q$. Then for any $(s, t]$ let

$$(1) \qquad \xi(s, t](\omega) = \sum_{j=1}^{2^q} (-1)^{k(j)} \xi(0, v_j](\omega) \qquad \text{for all} \quad \omega,$$

where v_j are the vertices of $(s, t]$ and $k(j)$ is the number of coordinates of v_j equal to those of s. Since (1) must hold almost surely for given $(s, t]$, we may define $\xi(s, t]$ consistently by (1) and then it is always finitely additive;

$$(2) \qquad \text{if} \quad (s, t] = \bigcup_{j=1}^{n} (s_j, t_j] \quad \text{and} \quad (s_j, t_j] \quad \text{are disjoint, then}$$
$$\xi(s, t](\omega) = \sum_{j=1}^{n} \xi(s_j, t_j](\omega) \quad \text{for all} \quad \omega.$$

THEOREM. *Let ξ be a Brownian motion random measure with parameter domain*

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($\mathbb{R}_+{}^q$, Leb.). *Then* $\forall\, \alpha > \frac{1}{2}$, *almost surely*, $\forall\, t \in \mathbb{R}_+{}^q$

$$\limsup_{h\to 0} |\xi(t, t+h](\omega)|/(\text{Leb. } (t, t+h])^\alpha = +\infty .$$

PROOF. It is clearly sufficient to consider only the subset $T = [0, 1]^q$. Let $\forall\, n, m \in N_+$,

$$A_m = \{\omega : \exists\, t \in T,\ \exists\, r > 0 \text{ such that for } h \in \mathbb{R}_+{}^q$$

$$\max_j h_j < r \Rightarrow |\xi(t, t+h](\omega)| \leqq m(\textstyle\prod_{j=1}^q h_j)^\alpha\} \tag{3}$$

$$C_n = \{\textstyle\times_{j=1}^q ((k_j - 1)/2^n, k_j/2^n] : k_j = 1, 2, \cdots, 2^n; j = 1, 2, \cdots, q\}$$

$$u(n, m) = 2^q m 2^{(1-n)q\alpha} .$$

For each $C \in C_n$, decompose C into a collection E_C of n^q cubes of side $1/n2^n$. Let $\forall\, n, m \in N_+$

$$B_n{}^m = \{\omega : \exists\, C \in C_n \text{ such that } \forall\, D \in E_C\ |\xi(D)(\omega)| \leqq u(n, m)\}$$

$$B = \textstyle\bigcup_{m=1}^\infty \liminf_{n\to\infty} B_n{}^m .$$

ASSERTION. *For each* $m \in N_+$, $A_m \subseteqq \liminf_{n\to\infty} B_n{}^m$.

PROOF. Let $\omega \in A_m$. Choose r and t from (3) and $p \in N_+$ such that $2^{1-p} < r$. Then $\forall\, n \geqq p$, $\exists\, C = (s_n, s_n + h_n] \in C_n$ such that $C \subseteqq \times_{j=1}^q (t_j, t_j + r]$ and $t \in (s_n - h_n, s_n]$. Let $D \in E_C$. Then

$$1_D = \textstyle\sum_{j=1}^{2^q} a_j 1_{(t, v_j]}$$

where v_j are the vertices of D and $a_j = \pm 1$ as in (1). Then by (2) we have

$$|\xi(D)(\omega)| \leqq \textstyle\sum_{j=1}^{2^q} |\xi((t, v_j])(\omega)| \leqq 2^q m 2^{(1-n)q\alpha} = u(n, m) .$$

Hence $\forall\, n \geqq p$, $\omega \in B_n{}^m$, proving the assertion.

Next, $\forall\, D \in E_C$, $\xi(D)$ is Gaussian with mean zero and variance $1/(n2^n)^q$. Therefore,

$$P\{|\xi(D)| \leqq u(n, m)\} \leqq (n2^n)^{q/2} u(n, m) .$$

Since ξ is finitely additive and independently scattered, $\forall\, n, m \in N_+$,

$$P(B_n{}^m) \leqq 2^{nq}\{(n2^n)^{q/2} u(n, m)\}^{n^q}$$

$$\leqq 2^{nq}\{m n^{q/2} 2^{q(1+\alpha)} 2^{nq(-\alpha+\frac{1}{2})}\}^{n^q} .$$

It is easy to show that $\forall\, m \geqq 1$, $\lim_{n\to\infty} P(B_n{}^m) = 0$ since the last factor is predominant and $\alpha > \frac{1}{2}$. Hence

$$P(B) \leqq \textstyle\sum_{m=1}^\infty P(\liminf_{n\to\infty} B_n{}^m) \leqq \sum_{m=1}^\infty \lim_{n\to\infty} P(B_n{}^m) = 0 . \qquad \square$$

Dvoretzky [1] has shown that the sample paths of a Brownian motion stochastic process, with probability one, nowhere satisfy a Lipschitz condition of order $\frac{1}{2}$ with a sufficiently small Lipschitz constant. As far as we know, Dvoretzky's result has not been extended to a Brownian motion random measure with parameter domain $R_+{}^q$, $q \geqq 2$. Logarithmic refinements of our result also seem to be open.

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DEPARTMENT OF MATHEMATICS
CLEVELAND STATE UNIVERSITY
CLEVELAND, OHIO 44115