WEAK CONVERGENCE TO BROWNIAN MEANDER
AND BROWNIAN EXCURSION[1]

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We show that (i) Brownian motion conditioned to be positive is
Brownian meander; (ii) tied-down Brownian meander is Brownian excursion; and (iii) Brownian bridge conditioned to be positive is Brownian
excursion. Using these results we derive the distribution of the suprema
of the meander and excursion.

1. Introduction and summary. Brownian meander and Brownian excursion
processes have recently appeared as the limit process of a number of conditional
functional central limit theorems. These results may be found in Belkin (1972),
Iglehart (1974, 1975), and Kaigh (1974, 1975, 1976).

Our purpose in this paper is to investigate relationships between Brownian
meander, Brownian excursion, Brownian motion and Brownian bridge. In particular, we present some conditioned families of Brownian processes which have
Brownian meander or Brownian excursion as their weak limit. (This is similar
in spirit to weak convergence to Brownian bridge of Brownian motion which
is conditioned to be close to 0 at time 1; see Billingsley (1968), pages 83–86.)
Employing the continuous mapping theorem, Durrett and Iglehart (1977) use
these results to determine the distributions of various functionals of Brownian
meander and Brownian excursion.

To describe our results in greater detail, we need to introduce the two processes mentioned above. Brownian meander, $W^+ = \{W^+(t) : 0 \leqq t \leqq 1\}$, can be
described as follows: Let $\{W(t) : t \geqq 0\}$ be standard Brownian motion, $\tau_1 = \sup \{t \in [0, 1] : W(t) = 0\}$, and $\Delta_1 = 1 - \tau_1$. Then

$$W^+(t) = \Delta_1^{-\frac{1}{2}} |W(\tau_1 + t\Delta_1)| , \qquad 0 \leqq t \leqq 1 .$$

In Belkin (1972), page 61, it is shown that W^+ is a continuous, nonhomogeneous
Markov process. If $n_s(x) = (2\pi s)^{-\frac{1}{2}} \exp(-x^2/2s)$ and $N_s(a, b) = \int_a^b n_r(x) \, dx$, then

W^+ has transition density given by

(1.1)
$$P\{W^+(t) \in dy\} = p^+(0, 0, t, y)\, dy$$
$$= 2t^{-\frac{3}{2}}y \exp(-y^2/2t)N_{1-t}(0, y)\, dy$$

for $0 \leq t \leq 1$ and $y > 0$; for $0 < s < t \leq 1$ and $x, y > 0$

(1.2)
$$P\{W^+(t) \in dy \mid W^+(s) = x\} = p^+(s, x, t, y)\, dy$$
$$= g(t - s, x, y)[N_{1-t}(0, y)/N_{1-s}(0, x)]\, dy\, ,$$

where

$$g(t, x, y) = n_t(y - x) - n_t(y + x)$$
$$= P\{W(t) \in dy; W(s) > 0, 0 \leq s \leq t \mid W(0) = x\}/dy\, .$$

The second equality follows from (11.10) of Billingsley (1968).

Note that $P\{W^+(1) \leq x\} = 1 - \exp(-x^2/2)$, $x \geq 0$, the Rayleigh distribution. Throughout the paper we will use $P\{Z \in dy\} = f(y)\, dy$ to mean that the distribution of Z has the density f with respect to Lebesgue measure dy.

Brownian excursion, $W_0^+ = \{W_0^+(t) : 0 \leq t \leq 1\}$, is also a continuous non-homogeneous Markov process. Let $\tau_2 = \inf\{t \geq 1 : W(t) = 0\}$ and set $\Delta_2 = \tau_2 - \tau_1$. Then

$$W_0^+(t) = \Delta_2^{-\frac{1}{2}}|W(\tau_1 + t\Delta_2)|\, , \qquad\qquad 0 \leq t \leq 1\, .$$

The transition density is given by

(1.3)
$$P\{W_0^+(t) \in dy\} = p_0^+(0, 0, t, y)\, dy = \frac{2y^2 \exp(-y^2/2t(1 - t))}{(2\pi t^3(1 - t)^3)^{\frac{1}{2}}}\, dy$$

for $0 < t \leq 1$; for $0 < s < t < 1$ and $x, y > 0$

(1.4)
$$P\{W_0^+(t) \in dy \mid W_0^+(s) = x\}$$
$$= p_0^+(s, x, t, y)\, dy$$
$$= g(t - s, x, y)\left(\frac{1 - s}{1 - t}\right)^{\frac{3}{2}} \frac{y \exp(-y^2/2(1 - t))}{x \exp(-x^2/2(1 - s))}\, dy\, ;$$

see Itô-McKean (1965), page 76, for this result.

Next we introduce some notation. For $0 \leq t \leq 1$, let

$$m(t) = \inf\{W(s) : 0 \leq s \leq t\} \qquad \text{and} \qquad M(t) = \sup\{W(s) : 0 \leq s \leq t\}\, ,$$

where W is standard Brownian motion. Let $m = m(1)$ and $M = M(1)$. For Brownian bridge, W_0, Brownian meander, W^+, and Brownian excursion, W_0^+, we use corresponding notation for the infimum and supremum; e.g.

$$M^+(t) = \sup\{W^+(s) : 0 \leq s \leq t\}\, , \qquad m_0^+(t) = \inf\{W_0^+(s) : 0 \leq s \leq t\}\, .$$

Let $C = C[0, 1]$ be the space of continuous functions on $[0, 1]$, and let $\mathscr{C}$ be the Borel sets of C when it is endowed with the topology generated by the supremum metric, ρ. We shall need a concise definition and notation for conditioned processes. Suppose Y is a random function of $(C, \mathscr{C})$, i.e., Y is a measurable mapping from some probability space $(\Omega, \mathscr{F}, P)$ into $(C, \mathscr{C})$. The random

function induces a probability measure, $Q = PY^{-1}$, on $(C, \mathscr{C})$. Let Λ be a Borel subset of C with $Q(\Lambda) > 0$. Then let $(\Lambda, \Lambda \cap \mathscr{C}, Q_\Lambda)$ be the trace of $(C, \mathscr{C}, Q)$ on Λ: $\Lambda \cap \mathscr{C} = \{\Lambda \cap A : A \in \mathscr{C}\}$ and $Q_\Lambda(A) = Q(A)/Q(\Lambda)$ for $A \in \Lambda \cap \mathscr{C}$. Also let $(Y^{-1}(\Lambda), Y^{-1}(\Lambda) \cap \mathscr{F}, P_{Y^{-1}(\Lambda)})$ be the trace of $(\Omega, \mathscr{F}, P)$ on $Y^{-1}(\Lambda)$. Then we define the random function

$$Y \,|\, \Lambda : (Y^{-1}(\Lambda), Y^{-1}(\Lambda) \cap \mathscr{F}, P_{Y^{-1}(\Lambda)}) \to (\Lambda, \Lambda \cap \mathscr{C}, Q_\Lambda)$$

as the restriction of Y to $Y^{-1}(\Lambda)$. It follows that $P_{Y^{-1}(\Lambda)}((Y \,|\, \Lambda)^{-1}(\cdot)) = Q_\Lambda(\cdot)$. As discussed by Billingsley (1968), page 22, convergence in distribution of random functions is equivalent to weak convergence of the induced probability measures.

We now define the following conditioned random functions of $(C, \mathscr{C})$, for $\varepsilon > 0$:

$$W^\varepsilon = W \,|\, \{m > -\varepsilon\}\,,$$
$$W_\varepsilon^{\,+} = W^+ \,|\, \{W^+(1) \leqq \varepsilon\}\,,$$
$$W_0^{\,\varepsilon} = W_0 \,|\, \{m_0 > -\varepsilon\}\,.$$

The above processes are all Markov by virtue of the following lemma which we state without proof.

(1.5)   LEMMA. *Let Y be a Markov random function of $C[0, 1]$. Let Λ be a Borel subset of C with $Q(\Lambda) > 0$. Let $\pi_{[0,t]}$ $(\pi_{[t,1]})$ be the projection map of $C[0, 1]$ onto $C[0, t]$ $(C[t, 1])$. If for all $0 \leqq t \leqq 1$ there exist sets A_t and B_t such that $\Lambda = \pi_{[0,t]}^{-1} A_t \cap \pi_{[t,1]}^{-1} B_t$, then $Y \,|\, \Lambda$ is Markov.*

The main results of the paper are organized as follows. In Section 2 we show that $W^\varepsilon \Rightarrow W^+$ as $\varepsilon \downarrow 0$. In Section 3 we present an alternate proof of this result. In Section 4 we show that $W_\varepsilon^{\,+} \Rightarrow W_0^{\,+}$ as $\varepsilon \downarrow 0$. In Section 5, $W_0^{\,\varepsilon} \Rightarrow W_0^{\,+}$ as $\varepsilon \downarrow 0$. (Throughout this paper when considering tightness and convergence, "$\varepsilon \downarrow 0$" and "$\varepsilon > 0$," shall in reality correspond to a fixed sequence of numbers tending to 0; see footnote, Billingsley (1968), page 84.) In Section 6 the results of Sections 2 and 5 are used to obtain the distributions of M^+ and $M_0^{\,+}$.

2. Convergence of conditioned Brownian motion to Brownian meander. This section is devoted exclusively to proving

(2.1)   THEOREM. $W^\varepsilon \Rightarrow W^+$ *as $\varepsilon \downarrow 0$.*

Let $\{W(t) : t \geqq 0\}$ be standard Brownian motion defined on the probability triple $(\Omega, \mathscr{F}, P)$ with σ-fields $\mathscr{F}_t = \sigma\{W(s) : s \leqq t\}$ and shift operators $\{\theta_t : t \geqq 0\}$. The symbol $P^x\{A\}$ means $P\{A \,|\, W(0) = x\}$ and E^x the expectation with respect to P^x. When $x = 0$, the superscript is omitted. To facilitate the proof of (2.1) we break it into three lemmas. Define for $\varepsilon \geqq 0$, $s_\varepsilon = \inf\{s > 0 : W(s) = \varepsilon, W(u) > 0$ for $s < u \leqq s + 1\}$ and for $0 \leqq t \leqq 1$ $Z_\varepsilon(t) = W(s_\varepsilon + t)$. The first step is

(2.2)   LEMMA. *For $\varepsilon > 0$, $s_\varepsilon < \infty$ a.s.; $s_\varepsilon \downarrow s_0$ and $\rho(Z_\varepsilon, Z_0) \to 0$ a.s. as $\varepsilon \downarrow 0$.*

PROOF.   For $\varepsilon > 0$, let $t_\varepsilon^{\,0} = -1$ and $t_\varepsilon^{\,k} = \inf\{t \geqq t_\varepsilon^{\,k-1} + 1 : W(t) = \varepsilon\}$. Since

$P^y\{t_\varepsilon^{-1} < \infty\} = 1$ for all y, an induction argument shows

$$P\{t_\varepsilon^{k+1} < \infty\} = E[P^{W(t_\varepsilon^{k}+1)}(t_\varepsilon^{-1} < \infty); t_\varepsilon^{k} < \infty] = P\{t_\varepsilon^{k} < \infty\} = 1 .$$

Now if $W(t)$ has no zero in $[t_\varepsilon^{k}, t_\varepsilon^{k} + 1]$, then $s_\varepsilon \leqq t_\varepsilon^{k}$ so $P\{s_\varepsilon \leqq t_\varepsilon^{k} \,|\, s_\varepsilon > t_\varepsilon^{k-1}\} \geqq$ $P^\varepsilon\{W(t) > 0$ for all $0 \leqq t \leqq 1\} > 0$ and hence $P\{s_\varepsilon < \infty\} = 1$. For $0 \leqq \delta < \varepsilon$, $s_\delta \leqq \sup\{t < s_\varepsilon : W(t) = \delta\}$ so $s_\varepsilon \downarrow$ as $\varepsilon \downarrow 0$. To see that $s_\varepsilon \downarrow s_0$, note that $\gamma = \inf\{t - s_0 - 1 : t > s_0, W(t) = 0\} > 0$, so $s_{W(s_0+\xi\gamma)} - s_0 \leqq \xi\gamma$, for all $0 < \xi < 1$. Since W has continuous paths and $Z_\varepsilon(t) = W(s_\varepsilon + t)$, $s_\varepsilon \downarrow s_0$ implies $\rho(Z_\varepsilon, Z_0) \to 0$.

(2.3)   LEMMA.   *For $\varepsilon > 0$ the random functions $Z_\varepsilon - \varepsilon$ and W^ε have the same finite-dimensional distributions and hence induce the same probability measures on $(C, \mathscr{C})$.*

PROOF.   Let $0 \leqq t_1 < \cdots < t_k \leqq 1$; let $A_1, \cdots, A_k$ be Borel sets of $[0, \infty)$, and $F = \{x \in C : x(t_i) \in A_i; i = 1, \cdots, k\}$. Define the first entrance time $t_\varepsilon = \inf\{t > 0 : W(t) = \varepsilon\} \leqq s_\varepsilon$. Decompose $\{Z_\varepsilon \in F\}$ to obtain

$$(2.4) \qquad P\{Z_\varepsilon \in F\} = P\{Z_\varepsilon \in F, s_\varepsilon = t_\varepsilon\} + P\{Z_\varepsilon \in F, s_\varepsilon > t_\varepsilon\} .$$

Since t_ε is a stopping time and W a strong Markov process

$$\begin{aligned}
P\{Z_\varepsilon \in F, s_\varepsilon = t_\varepsilon\} &= E\{P\{Z_\varepsilon \in F, s_\varepsilon = t_\varepsilon \,|\, \mathscr{F}_{t_\varepsilon}\}\} \\
&= E\{P^{W(t_\varepsilon)}\{W \in F, m > 0\}\} \\
&= P^\varepsilon\{W \in F, m > 0\} = P\{W \in F - \varepsilon, m > -\varepsilon\} ,
\end{aligned}$$

where $F - \varepsilon = \{f - \varepsilon : f \in F\}$. If $s_\varepsilon > t_\varepsilon$, then $W(s) = 0$ for some $s \in (t_\varepsilon, t_\varepsilon + 1]$. Let $\tau_\varepsilon = \inf\{s : s \in (t_\varepsilon, t_\varepsilon + 1], W(s) = 0\}$, where $\tau_\varepsilon = +\infty$ if the last set is empty. Clearly, $s_\varepsilon > \tau_\varepsilon$ on $\{s_\varepsilon > t_\varepsilon\}$. Proceeding now with the second term of (2.4) we have

$$(2.5) \qquad \begin{aligned}
P\{Z_\varepsilon \in F, s_\varepsilon > t_\varepsilon\} &= P\{Z_\varepsilon \in F, \tau_\varepsilon < \infty\} \\
&= E\{E\{1_{\{Z_\varepsilon \in F\}} \,|\, \mathscr{F}_{\tau_\varepsilon}\}; \tau_\varepsilon < \infty\} .
\end{aligned}$$

On the set $\{\tau_\varepsilon < \infty\}$, $1_{\{Z_\varepsilon \in F\}} = \phi \cdot \theta_{\tau_\varepsilon}$ for some $\mathscr{F}_\infty$ measurable ϕ. Thus from (2.5) and the strong Markov property we get

$$(2.6) \qquad \begin{aligned}
P\{Z_\varepsilon \in F, s_\varepsilon > t_\varepsilon\} &= E\{E^{W(\tau_\varepsilon)}\{\phi\}; \tau_\varepsilon < \infty\} \\
&= E^0\{\phi\} P\{\tau_\varepsilon < \infty\} \\
&= P\{Z_\varepsilon \in F\}[1 - P\{m > -\varepsilon\}] .
\end{aligned}$$

Combining (2.4), (2.5) and (2.6) yields

$$P\{Z_\varepsilon \in F\} = P\{W \in F - \varepsilon, m > -\varepsilon\} + P\{Z_\varepsilon \in F\}[1 - P\{m > -\varepsilon\}]$$

or

$$P\{Z_\varepsilon \in F\} = P\{W \in F - \varepsilon \,|\, m > -\varepsilon\} .$$

(2.7)   LEMMA.   *The finite-dimensional distributions of W^ε converge to those of W^+ as $\varepsilon \downarrow 0$.*

PROOF. We shall compute the transition probabilities of W^ε. From the reflection principle $P\{m > -\varepsilon\} = 2N_1(0, \varepsilon) \sim \varepsilon(2/\pi)^{\frac{1}{2}}$ as $\varepsilon \downarrow 0$. Using the Markov property for W we obtain, for $0 < t \leq 1$,

$$P\{W(t) \in dy, m > -\varepsilon\} = P\{W(t) \in dy, m(t) > -\varepsilon\}P^y\{m(1 - t) > -\varepsilon\}$$

(2.8)
$$= g(t, \varepsilon, y + \varepsilon) \cdot 2N_{1-t}(0, y + \varepsilon)\, dy$$

$$\sim (2\pi t)^{-\frac{1}{2}}[2\varepsilon(y/t)e^{-y^2/2t}] \cdot 2N_{1-t}(0, y)\, dy$$

as $\varepsilon \downarrow 0$. Hence

$$P\{W^\varepsilon(t) \in dy\} = p^\varepsilon(0, 0, t, y)\, dy = g(t, \varepsilon, y + \varepsilon)\frac{N_{1-t}(0, y + \varepsilon)}{N_1(0, \varepsilon)}\, dy$$

(2.9)
$$\to t^{-\frac{3}{2}}ye^{-y^2/2t}2N_{1-t}(0, y)\, dy$$

$$= P\{W^+(t) \in dy\}\,,$$

as $\varepsilon \downarrow 0$. For $0 < s < t \leq 1$ and $x > 0$, by the Markov property of W

$$P\{W^\varepsilon(t) \in dy \mid W^\varepsilon(s) = x\}$$

$$= [P\{W(s) \in dx, m(s) > -\varepsilon\}P^x\{W(t - s) \in dy, m(t - s) > -\varepsilon\}$$

(2.10)
$$\times P^y\{m(1 - t) > -\varepsilon\}]/[P\{W(s) \in dx, m > -\varepsilon\}]$$

$$= \frac{g(s, \varepsilon, x + \varepsilon)g(t - s, x + \varepsilon, y + \varepsilon)2N_{1-t}(0, y + \varepsilon)\, dx\, dy}{g(s, \varepsilon, x + \varepsilon)2N_{1-s}(0, x + \varepsilon)\, dx}$$

$$\to g(t - s, x, y)\frac{N_{1-t}(0, y)}{N_{1-s}(0, x)}\, dy$$

$$= P\{W^+(t) \in dy \mid W^+(s) = x\}\,,$$

as $\varepsilon \downarrow 0$. (These transition probabilities are derived in [13] from distribution functions rather than using differentials, dx's and dy's.)

Convergence of the transition densities imply convergence of the finite-dimensional densities, which in turn implies convergence of f.d.d.'s, completing the proof of (2.7).

Lemmas 2.2 and 2.3 imply that $W^\varepsilon \Rightarrow Z_0$ as $\varepsilon \downarrow 0$. The finite-dimensional sets are a determining class (Billingsley (1968), page 15); therefore, convergence of the finite-dimensional distributions of W^ε to W^+ implies $W^\varepsilon \Rightarrow W^+$ as $\varepsilon \downarrow 0$; see also Billingsley (1968), page 35.

3. An alternative proof of Theorem 2.1. Verification of weak convergence of probability measures on function spaces usually involves two steps: namely, (i) convergence of finite-dimensional distributions and (ii) tightness. In this section another proof of (sequential) tightness of $\{W^\varepsilon, \varepsilon > 0\}$ is presented; when combined with convergence of f.d.d.'s (Lemma 2.7) this will provide an alternate proof of Theorem 2.1. The proof is based on a characterization of tightness of random elements of $C[0, 1]$, namely

(3.1)   THEOREM. *Let* $\{X_n, n = 1, 2, \cdots\}$ *be a sequence of random elements of* $C[0, 1]$. *Define random elements* $Y_{n,s}$ *of* $C[s, 1]$ *as* $Y_{n,s} = \pi_{[s,1]} \circ X_n$. *If* (i) *for any*

$s > 0$, $\{Y_{n,s}, n = 1, 2, \cdots\}$ *induces a tight family of measures on* $C[s, 1]$, *and* (ii) $\lim_{s \downarrow 0} \lim_{n \to \infty} P\{\sup_{0 \leq u \leq s} |X_n(u)| > \xi\} = 0$ *for all* $\xi > 0$, *the family of measures induced on* $C[0, 1]$ *by* $\{X_n, n = 1, 2, \cdots\}$ *is tight.*

PROOF. The modulus of continuity of an element x of C is $w_x(\delta, a, b) = \sup_{a \leq s, t \leq b; |s-t| < \delta} |x(s) - x(t)|$. By Theorem 8.2 of Billingsley (1968) it will suffice to show that for each positive ε, $\lim_{\delta \downarrow 0} \lim_{n \to \infty} P_n\{x : w_x(\delta, 0, 1) \geq \varepsilon\} = 0$. Clearly $\{w_x(\delta, 0, 1) \geq \varepsilon\} \subset \{w_x(\delta, 0, s) \geq \varepsilon/2\} \cup \{w_x(\delta, s, 1) \geq \varepsilon/2\}$. Thus it suffices to show that, given $\eta > 0$, there exist $s_0 > 0$ and $\delta > 0$ such that

$$(3.2) \qquad \lim_{n \to \infty} P_n\{x : w_x(\delta, 0, s_0) \geq \varepsilon/2\} < \eta/2$$

and

$$(3.3) \qquad \lim_{n \to \infty} P_n\{x : w_x(\delta, s_0, 1) \geq \varepsilon/2\} < \eta/2 .$$

Clearly $w_x(s, 0, s) \leq \sup_{0 \leq u \leq s} |x(u)|$; consequently by assumption

$$\lim_{s \downarrow 0} \lim_{n \to \infty} P_n\{x : w_x(s, 0, s) \geq \varepsilon/2\} = 0 .$$

Let s_0 be a value such that $\lim_{n \to \infty} P_n\{x : w_x(s, 0, s) \geq \varepsilon/2\} < \eta/2$ for $s \leq s_0$. Now pick $\delta < s_0$ such that (3.3) is satisfied. This is possible, again using Billingsley (1968), Theorem 8.2, because by assumption $\{Y_{n,s_0}, n = 1, 2, \cdots\}$ is tight for $s_0 > 0$. This completes the proof.

In Sections 4 and 5 we shall have occasion to use the following variations of Theorem 3.1:

(3.4)   THEOREM. *The random elements* $\{X_n, n = 1, 2, \cdots\}$ *of* $C[0, 1]$ *when restricted to* $C[0, 1 - s]$ *form a tight family and* $\lim_{s \downarrow 0} \lim_{n \to \infty} P\{\sup_{1-s \leq u \leq 1} |X_n(u)| > \xi\} = 0$, *for* $\xi > 0$. *Then* $\{X_n, n = 1, 2, \cdots\}$ *is tight.*

(3.5)   THEOREM. *The random elements* $\{X_n, n = 1, 2, \cdots\}$ *of* $C[0, 1]$ *when restricted to* $C[s, 1 - s]$ *form a tight family and* $\lim_{s \downarrow 0} \lim_{n \to \infty} P\{\sup_{0 \leq u \leq s, 1-s \leq u \leq 1} |X_n(u)| > \xi\} = 0$, *for* $\xi > 0$. *Then* $\{X_n, n = 1, 2, \cdots\}$ *is tight.*

We now proceed with the proof of tightness of $\{W^\varepsilon, \varepsilon > 0\}$ using Theorem 3.1. It will suffice to prove two lemmas:

(3.6)   LEMMA. $\{W^\varepsilon(t), \delta \leq t \leq 1\}$, $\varepsilon > 0$, *induces a tight family of measures on* $C[\delta, 1]$ *for all* $\delta > 0$.

PROOF. Let $\tilde{W}^\varepsilon = \pi_{[\delta, 1]} \circ W^\varepsilon$ and $\tilde{W}^+ = \pi_{[\delta, 1]} \circ W^+$. Given $\eta > 0$, there exists a compact subset, K, of $C[\delta, 1]$ such that $P\{\tilde{W}^+ \in K\} > 1 - \eta/2$. Note from equations (1.2) and (2.10) that for $0 < \delta \leq s < t \leq 1$, $p^\varepsilon(s, x, t, y) = p^+(s, x + \varepsilon, t, y + \varepsilon)$. Thus, $P\{\tilde{W}^\varepsilon \in K - \varepsilon \mid \tilde{W}^\varepsilon(\delta) = x - \varepsilon\} = P\{\tilde{W}^+ \in K \mid W^+(\delta) = x\}$ for almost all x. Define $K^* = (\bigcup_{0 < \varepsilon < 1} (K - \varepsilon))^-$; K^* is compact by the Ascoli-Arzela theorem ([2], page 221). For $\varepsilon < 1$

$$P\{\tilde{W}^\varepsilon \in K^*\} \geq P\{\tilde{W}^\varepsilon \in K - \varepsilon\}$$
$$= \int_{-\varepsilon}^{\infty} P\{\tilde{W}^\varepsilon \in K - \varepsilon \mid \tilde{W}^\varepsilon(\delta) = x\} p^\varepsilon(0, 0, \delta, x) \, dx$$

$$= \int_0^\infty P\{\tilde{W}^+ \in K \mid \tilde{W}^+(\delta) = x\} p^\varepsilon(0, 0, \delta, x - \varepsilon) \, dx$$
$$\to \int_0^\infty P\{\tilde{W}^+ \in K \mid \tilde{W}^+(\delta) = x\} p^+(0, 0, \delta, x) \, dx$$
$$= P\{\tilde{W}^+ \in K\} > 1 - \eta/2 \, ,$$

where the convergence is justified by Scheffe's theorem ([2], page 224). Thus, there exists ε_η such that $P\{W^\varepsilon \in K^*\} > 1 - \eta$ for $\varepsilon < \varepsilon_\eta$. As mentioned in Section 1, we are interested in sequences of r.f.'s. For any fixed sequence of ε's tending to 0, the corresponding r.f.'s, W_ε^+, will be tight.

(3.7)   LEMMA. *For* $\eta > 0$, $\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\sup_{0 \leq s \leq \delta} |W^\varepsilon(s)| \leq \eta\} = 1$.

PROOF. First note that it suffices to prove

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{M^\varepsilon(\delta) \leq \eta\} = 1 \, .$$

Using the definition of W^ε, the Markov property of W and generalization of (11.10) of Billingsley (1968) gives

$$P\{\sup_{0 < s < \delta} W^\varepsilon(s) \leq \eta\}$$
$$= P\{M(\delta) \leq \eta, m > -\varepsilon\}/P\{m > -\varepsilon\}$$
$$= \frac{\int_{-\varepsilon}^\eta P\{-\varepsilon \leq m(\delta) < M(\delta) \leq \eta \mid W(\delta) = z\} n_\delta(z) P^z\{m(1 - \delta) > -\varepsilon\} \, dz}{P\{m > -\varepsilon\}}$$
$$= \int_{-\varepsilon}^\eta \sum_{k=-\infty}^\infty [n_\delta(z + 2k(\eta + \varepsilon))$$
$$\quad - n_\delta(z + 2k(\eta + \varepsilon) + 2\varepsilon)] N_{1-\delta}(0, z + \varepsilon) \, dz/N_1(0, \varepsilon) \, .$$

Using dominated convergence, it is possible to take the limit as $\varepsilon \to 0$. Then Fubini's theorem justifies changing order of summation. Then evaluation of the term corresponding to $k = 0$ shows that this term converges to 1 as $\delta \to 0$. The remaining terms ($k \neq 0$) can be bounded by quantities whose sum converges (by monotone convergence) to 0 as $\delta \to 0$. For details see [13].

Combining (3.1), (3.6) and (3.7) proves the tightness of $\{W^\varepsilon, \varepsilon > 0\}$.

4. Convergence of conditioned Brownian meander to Brownian excursion.

Our goal in this section is to prove that tied-down Brownian meander is Brownian excursion. We shall prove

(4.1)   THEOREM. $W_\varepsilon^+ \Rightarrow W_0^+$ *as* $\varepsilon \downarrow 0$.

We begin by showing that the finite dimensional distributions converge.

(4.2)   LEMMA. *The finite dimensional distributions of* W_ε^+ *converge to those of* W_0^+ *as* $\varepsilon \downarrow 0$.

PROOF. Because W_ε^+ and W_0^+ are Markov, it suffices to show that the probability transition densities converge. For $0 < t < 1$ and $y > 0$.

$$p_\varepsilon^+(0, 0, t, y) \, dy$$
(4.3)
$$= P\{W^+(t) \in dy \mid W^+(1) \leq \varepsilon\}$$
$$= t^{-\frac{3}{2}} y \exp(-y^2/2t) \, dy \, \frac{N_{1-t}(-y, -y + \varepsilon) - N_{1-t}(-y - \varepsilon, -y)}{1 - \exp(-\varepsilon^2/2)} \, .$$

Using L'Hôpital's rule twice on the ratio above gives

$$\lim_{\varepsilon \downarrow 0} p_\varepsilon^+(0, 0, t, y) = \frac{2y^2 \exp(-y^2/2t(1-t))}{(2\pi t^3(1-t)^3)^{\frac{1}{2}}}$$
$$= p_0^+(0, 0, t, y).$$

For $0 < s < t \leq 1$ and $x, y > 0$

(4.4)
$$p_\varepsilon^+(s, x, t, y)\, dy$$
$$= P\{W^+(t) \in dy \mid W^+(s) = x, W^+(1) \leq \varepsilon\}$$
$$= g(t - s, x, y)\, dy\, \frac{N_{1-t}(-y, -y+\varepsilon) - N_{1-t}(-y-\varepsilon, -y)}{N_{1-s}(-x, -x+\varepsilon) - N_{1-s}(-x-\varepsilon, -x)}.$$

Divide numerator and denominator by $1 - \exp(-\varepsilon^2/2)$ and use the same application of L'Hôpital's rule as in (4.3) to see that

$$\lim_{\varepsilon \downarrow 0} p_\varepsilon^+(s, x, t, y) = p_0^+(s, x, t, y),$$

which completes the proof.

Next we must show that the r.f.'s $\{W_\varepsilon^+, \varepsilon > 0\}$ are tight. We shall use Theorem 3.5.

(4.5)   LEMMA.   *Given $s > 0$, define the random element $\tilde{W}_\varepsilon^+$ of $C[0, 1-s]$ as the projection of W_ε^+ onto $C[0, 1-s]$; that is $\tilde{W}_\varepsilon^+ = \pi_{[0, 1-s]} \circ W_\varepsilon^+$. Then $\{\tilde{W}_\varepsilon^+, \varepsilon > 0\}$ is tight.*

PROOF.   Define $\tilde{W}^+ = \pi_{[0, 1-s]} \circ W^+$. Define the r.f. W^* on $C[0, 1-s]$ as follows; $W^*(0) = 0$, $P\{W^*(1-s) \in dy\} = p_0^+(0, 0, 1-s, y)\, dy$, and $P\{W^* \in A \mid W^*(1-s) = y\} = P\{W^+ \in A \mid W^+(1-s) = y\}$ for all y, for any Borel set A of $C[0, 1-s]$. Given $\eta > 0$, there exists a compact subset, K, of $C[0, 1-s]$ such that $P\{W^* \in K\} > 1 - \eta/2$.

$$P\{\tilde{W}_\varepsilon^+ \in K\} = \int_0^\infty P\{\tilde{W}_\varepsilon^+ \in K \mid \tilde{W}_\varepsilon^+(1-s) = y\} p_\varepsilon^+(0, 0, 1-s, y)\, dy$$
$$= \int_0^\infty P\{\tilde{W}^+ \in K \mid \tilde{W}^+(1-s) = y\} p_\varepsilon^+(0, 0, 1-s, y)\, dy$$
$$\to \int_0^\infty P\{\tilde{W}^+ \in K \mid \tilde{W}^+(1-s) = y\} p^+(0, 0, 1-s, y)\, dy$$
$$= P\{W^* \in K\} \geq 1 - \eta/2$$

where convergence follows from Scheffe's theorem ([2], page 224). Given any fixed sequence of ε's tending to 0, the corresponding r.f.'s W_ε^+ will be a tight family.

(4.6)   LEMMA.   *For $\eta > 0$, $\lim_{s \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\sup_{1-s < u < 1} |W_\varepsilon^+(u)| \leq \eta\} = 1$.*

PROOF.   Since $W_\varepsilon^+(t) \geq 0$ it suffices to consider

(4.7)
$$P\{\sup_{1-s \leq u \leq 1} W_\varepsilon^+(u) \leq \eta\} = \int_0^\eta P\{\sup_{1-s \leq u \leq 1} W^+(u) \leq \eta \mid W^+(1-s) = x,$$
$$W^+(1) \leq \varepsilon\} p_\varepsilon^+(0, 0, 1-s, x)\, dx$$
$$= \int_0^\eta h(x, s, \varepsilon) p_\varepsilon^+(0, 0, 1-s, x)\, dx.$$

By Lemma 4.2, $\lim_{\varepsilon \downarrow 0} p_\varepsilon^+ = p_0^+$, thus if $\lim_{\varepsilon \downarrow 0} h(x, s, \varepsilon) = h(x, s)$ and $|h(x, s, \varepsilon)| \leqq 1$ for all (x, s, ε) then by Scheffe's theorem ([2], page 224)

(4.8) $\qquad \lim_{\varepsilon \downarrow 0} \int_0^\eta h(x, s, \varepsilon) p_\varepsilon^+(0, 0, 1 - s, x)\, dx = \int_0^\eta h(x, s) p_0^+(0, 0, 1 - s, x)\, dx .$

In fact, using equations (11.10) and (11.11) of [2],

$$h(x, s, \varepsilon) = P\{M(s) \leqq \eta \mid W(0) = x, m(s) > 0, W(s) \leqq \varepsilon\}$$
$$= \frac{\sum_{k=-\infty}^\infty N_s(2k\eta - x, 2k\eta - x + \varepsilon) - N_s(2k\eta - x - \varepsilon, 2k\eta - x)}{N_s(-x, -x + \varepsilon) - N_s(-x - \varepsilon, -x)}$$
$$\to \sum_{k=-\infty}^\infty n_s'(2k\eta - x)/n_s'(-x) = h(x, s)$$

as $\varepsilon \downarrow 0$ by dominated convergence and L'Hôpital's rule. For details of the domination see [5].

Now consider

$$\int_0^\eta h(x, s) p_0^+(0, 0, 1 - x, x)\, dx = \int_0^\eta h(x, s) \mu_s(dx)$$

by (1.3) $\mu_s \Rightarrow \delta_0$, the measure with atom at $x = 0$, as $s \downarrow 0$. Thus by Theorem 5.5 of Billingsley (1968), it suffices to show that $\lim_{x, s \to 0} h(x, s) = 1$ to prove

(4.9) $\qquad \lim_{s \downarrow 0} \int_0^\eta h(x, s) p_0^+(0, 0, 1 - s, x)\, dx = 1 .$

When (4.7), (4.8) and (4.9) are combined they prove the lemma.

Thus consider

$$h(x, s) = \sum_{k=-\infty}^\infty \frac{n_s'(2k\eta - x)}{n_s'(-x)}$$
$$= 1 + \sum_{k=1}^\infty \frac{(2k\eta + x)n_s(2k\eta + x) - (2k\eta - x)n_s(2k\eta - x)}{xn_s(-x)} .$$

Let $a_k(s, x)$ be the kth term in the above summation. Then

$$a_k(s, x) = \exp\left[-\frac{(2k\eta)^2 + 4k\eta x}{2s}\right] + \exp\left[-\frac{(2k\eta)^2 - 4k\eta x}{2s}\right]$$
$$+ 2k\eta \frac{\exp\left[-\dfrac{(2k\eta)^2 + 4k\eta x}{2s}\right] - \exp\left[-\dfrac{(2k\eta)^2 - 4k\eta x}{2s}\right]}{x} .$$

For $x \leqq \eta/2$, the ratio in the above expression is less than $8k\eta|f'(c)|$ where $f(y) = \exp(-y/2s)$ and $c = (2k\eta)^2 - 4k\eta x$; furthermore, the second term dominates the first so (for $x < \eta/2$)

$$|a_K(s, x)| \leqq 2 \exp\left[-\frac{(2k\eta)^2 - 4k\eta x}{2s}\right] + \frac{16k^2\eta^2}{2s} \exp\left[-\frac{(2k\eta)^2 - 4k\eta x}{2s}\right]$$
$$\leqq (2 + 8k^2\eta^2 s^{-1}) e^{-k^2\eta^2/2s} .$$

Since for $c > 0$,

(4.10) $\qquad \sum_{k=1}^\infty e^{-k^2/2c} < \left(\frac{\pi c}{2}\right)^{\frac{1}{2}} \qquad$ and $\qquad \sum_{k=1}^\infty k^2 e^{-k^2/2c} < c\left(\frac{\pi c}{2}\right)^{\frac{1}{2}}$

it follows that (for $0 < x < \eta/2$)

$$\sum_{k=1}^{\infty} |a_k(s, x)| \leqq 5\eta^{-1}(2\pi s)^{\frac{1}{2}} \to 0$$

as $s \downarrow 0$, completing the proof of Lemma 4.6.

When combined with Theorem 3.4, Lemmas 4.5 and 4.6 imply tightness of $\{W_\varepsilon^+, \varepsilon > 0\}$ which together with Lemma 4.2 implies Theorem 4.1.

5. Convergence of conditioned Brownian bridge to Brownian excursion. This section is devoted to proving

(5.1)   THEOREM. $W_0^\varepsilon \Rightarrow W_0^+$ as $\varepsilon \downarrow 0$.

The proof will follow the same pattern as the method of Sections 3 and 4. First we consider finite dimensional distributions, then the assumptions needed to apply Theorem 3.5.

(5.2)   LEMMA.   *The finite dimensional distributions of W_0^ε converge to those of W_0^+ as $\varepsilon \downarrow 0$.*

PROOF.   Because W_0^ε and W_0^+ are Markov it suffices to show that the transition probability densities converge. Using arguments similar to those previously used the transition probabilities are derived.

$$p_0^\varepsilon(0, 0, t, y) = \frac{[1 - \exp(-2\varepsilon(y + \varepsilon)/t)][1 - \exp(-2\varepsilon(y + \varepsilon)/(1 - t))]}{1 - \exp(-2\varepsilon^2)} n_{t(1-t)}(y)$$

$$p_0^\varepsilon(s, x, t, y) = \frac{[1 - \exp(-2\varepsilon(y + \varepsilon)/(1 - t))][1 - \exp(-2(y + \varepsilon)(x + \varepsilon)/(t - s))]}{1 - \exp(-2\varepsilon(x + \varepsilon)/(1 - s))}$$

$$\times n_u\left(y - \frac{1 - t}{1 - s} x\right)$$

for $x, y > -\varepsilon$ and $0 < s < t < 1$; where $u = (t - s)(1 - t)/(1 - s)$. Use of L'Hôpital's rule gives $\lim_{\varepsilon \downarrow 0} p_0^\varepsilon = p_0^+$, completing the proof.

(5.3)   LEMMA.   *Given $s > 0$, define the random element $\tilde{W}_0^\varepsilon$ of $C[s, 1 - s]$ as the projection of W_0^ε onto $C[s, 1 - s]$. Then $\{\tilde{W}_0^\varepsilon, \varepsilon > 0\}$ is tight.*

PROOF.   The proof is similar to (3.6) and (4.5). Define the random element $\tilde{W}^*$ of $C[s, 1 - s]$ as follows: The joint distribution of $(\tilde{W}^*(s), \tilde{W}^*(1 - s))$ has density $p_0^+(0, 0, s, x)p_0^+(s, x, t, y)$. $P\{\tilde{W}^* \in A \mid \tilde{W}^*(s) = x, \tilde{W}^*(1 - s) = y\} = P\{\tilde{W}^+ \in A \mid \tilde{W}^+(s) = x, \tilde{W}^+(1 - s) = y\}$ for all $x, y > 0$ and for all Borel sets A of $C[s, 1 - s]$. Note that, as in (3.6), $P\{\tilde{W}^+ \in A \mid \tilde{W}^+(s) = x, \tilde{W}^+(1 - s) = y\} = P\{\tilde{W}^\varepsilon \in A - \varepsilon \mid \tilde{W}^\varepsilon(s) = x - \varepsilon, \tilde{W}^\varepsilon(1 - \varepsilon) = y - \varepsilon\}$ for almost all y and all Borel A. As before, given $\eta > 0$, there exists a compact set K such that $P\{\tilde{W}^* \in K\} \geqq 1 - \eta/2$. Define $K^* = (\bigcup_{0 < \varepsilon < 1} (K - \varepsilon))^-$ as before and show that $\lim_{\varepsilon \downarrow 0} P\{\tilde{W}_0^\varepsilon \in K^*\} \geqq P\{\tilde{W}^* \in K\} \geqq 1 - \eta/2$, following the proof of Lemma 3.6.

(5.4)   LEMMA.   *For $\eta > 0$,*

$$\lim_{s \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\sup_{0 \leqq t \leqq s} |W_0^\varepsilon(t)| \leqq \eta\} = 1 \qquad and$$

$$\lim_{s \downarrow 0} \lim_{\varepsilon \downarrow 0} P\{\sup_{1 - s < t < 1} |W_0^\varepsilon(t)| \leqq \eta\} = 1 \, .$$

PROOF. It suffices to consider

$$P\{\sup_{0 \leq t \leq s} W_0^\varepsilon(t) \leq \eta\} = P\{M_0^\varepsilon(s) \leq \eta\}$$
$$= P\{M_0(s) \leq \eta, m_0 > -\varepsilon\}/P\{m_0 > -\varepsilon\}.$$

By (11.40) of Billingsley (1968), $P\{m_0 > -\varepsilon\} = 1 - \exp(-2\varepsilon^2)$. By the Markov property, (11.10), (11.40) of Billingsley (1968) and the relationship between W and W_0

$$P\{M_0(s) \leq \eta, m_0 > -\varepsilon\}$$
$$= \int_\varepsilon^\eta P\{M(s) \leq \eta, m(s) > -\varepsilon \,|\, W(s) = z\}$$
$$\times P\{m(1 - s) > -\varepsilon \,|\, W(1 - s) = z\} n_{s(1-s)}(z)\, dz$$
$$= \int_{-\varepsilon}^\eta \left[\sum_{k=-\infty}^\infty [n_s(z + 2k(\eta + \varepsilon)) - n_s(z + 2k(\eta + \varepsilon) + 2\varepsilon)]\right.$$
$$\left. \times [1 - \exp(-2\varepsilon(z + \varepsilon)/(1 - s))]n_{s(1-s)}(z)\, dz\right]/n_s(z).$$

Using L'Hôpital's rule twice and the dominated convergence theorem gives

$$\lim_{\varepsilon \downarrow 0} P\{M_0^\varepsilon(s) \leq \eta\}$$
$$= \int_0^\varepsilon \sum_{k=-\infty}^\infty 2s^{-1}(z + 2k\eta) \exp(-2k\eta(z + k\eta)/s)z(1 - s)^{-1}n_{s(1-s)}(z)\, dz.$$

The term for $k = 0$ approaches 1 as $s \downarrow 0$. The remaining terms are bounded by

$$\leq \int_0^\eta \sum_{k=-\infty;k\neq 0}^\infty 2(s(1 - s))^{-1}(z + 2k\eta)zn_{s(1-s)}(z)\, dz$$
$$\leq (s(1 - s))^{-1} \int_{|z|>\eta} z^2 n_{s(1-s)}(z)\, dz \to 0$$

as $s \downarrow 0$ by dominated convergence. The second statement of the lemma follows from the first because of symmetry of W_0 on $[0, 1]$.

Theorem 5.1 follows from Lemmas 5.4, 5.3, and Theorem 3.5 plus Lemma 5.2.

6. The suprema of Brownian meander and Brownian excursion.

In this section the continuity theorem (Theorem 5.1, Billingsley (1968)) is used in conjunction with Theorems 2.1 and 4.1 to derive the distribution of M^+ and M_0^+, respectively. Chung (1975, 1976) derives the distribution of M_0^+ using a different approach. Durrett and Iglehart (1977) derive the distribution of M^+ as a corollary of a more general theorem.

(6.1)   THEOREM. $P\{M^+ \leq x\} = 1 + 2 \sum_{k=1}^\infty (-1)^k \exp(-k^2x^2/2)$, for $x > 0$.

PROOF. It suffices to evaluate $\lim_{\varepsilon \downarrow 0} P\{M^\varepsilon < x\}$. By definition of W^ε and (10.17) and (11.12) of Billingsley (1968) we have

$$P\{M_\varepsilon^+ \leq x\} = P\{-\varepsilon < m < M \leq x\}/P\{-\varepsilon < m\}$$
$$= \sum_{k=-\infty}^\infty (-1)^k N(k(x + \varepsilon) - \varepsilon, k(x + \varepsilon) + \varepsilon)/N(-\varepsilon, \varepsilon).$$

By L'Hôpital's rule and dominated convergence this quantity has the desired limit.

(6.2)   THEOREM. $P\{M_0^+ \leq x\} = 1 + 2 \sum_{k=1}^\infty (1 - 4k^2x^2) \exp(-2k^2x^2)$, for $x > 0$.

PROOF. As above it suffices to evaluate $\lim_{\varepsilon \downarrow 0} P\{M_0^\varepsilon \leq x\}$. By definition of W_0^ε, and (11.38) and (11.40) of Billingsley (1968), we have

$$
\begin{aligned}
P\{M_0^\varepsilon < x\} &= P\{-\varepsilon < m_0 < M_0 < x\}/P\{-\varepsilon < m_0\} \\
&= \sum_{k=-\infty}^{\infty} [\exp(-2k^2(x + \varepsilon)^2) \\
&\quad - \exp(-2(x + k(x + \varepsilon))^2)]/(1 - \exp(-2\varepsilon^2)) \\
&= 1 + \sum_{k=1}^{\infty} [2 \exp(-2(k(x + \varepsilon))^2) - \exp(-2(k(x + \varepsilon) - \varepsilon)^2) \\
&\quad - \exp(-2(k(x + \varepsilon) + \varepsilon)^2)]/(1 - \exp(-2\varepsilon^2)) .
\end{aligned}
$$

Multiplying numerator and denominator by ε^{-2} and using L'Hôpital's rule twice on each shows that the kth term in the above expression converges to $2(1 - 4k^2x^2)\exp(-2k^2x^2)$. Therefore, the sum converges to the desired limit by dominated convergence; the numerator of the kth term in the above sum equals

$$
\exp(-2k^2(x + \varepsilon)^2)[2 - \exp(-2\varepsilon^2)[\exp(4k\varepsilon(x + \varepsilon)) + \exp(-4k\varepsilon(x + \varepsilon))]]
$$

which is less in absolute value than

$$
\begin{aligned}
\exp(-2k^2x^2) &\max (|2 - \exp(-2\varepsilon^2)[2]|, |2 - \exp(-2\varepsilon^2)[2 - (4k\varepsilon(x + \varepsilon))^2]|) \\
&\leq \exp(-2k^2x^2)[2 - 2\exp(-2\varepsilon^2) + \exp(-2\varepsilon^2)(4k\varepsilon(x + \varepsilon))^2] \\
&\leq \exp(-2k^2x^2)(4\varepsilon^2 + (4k\varepsilon(x + \varepsilon))^2) .
\end{aligned}
$$

Thus since $1 - \exp(-2\varepsilon^2) \geq 2\varepsilon^2 - 2\varepsilon^4$ it follows that the kth term of the sum in question is less in absolute value than $2\exp(-2k^2x^2)(1 + (2k(x + \varepsilon))^2)/(1 - \varepsilon^2)$ which in turn is less than $2(1 + 16k^2x^2)\exp(-2k^2x^2)$ for $\varepsilon \leq x$. By (4.10) the above serves as a dominating series in application of the dominated convergence theorem.

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