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# ON THE INDIVIDUAL ERGODIC THEOREM FOR $K$-AUTOMORPHISMS

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Let $(X, \mathscr{B}(X), P)$ be a probability space and let $T$ be a $K$-automorphism. If $T$ satisfies a Rosenblatt mixing condition of a certain kind, we show that if $\{k_n\}_{n=1}^{\infty}$ is an arbitrary increasing sequence of integers and $g$ belongs to a certain class of functions then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(T^{k_j}x) = E(g) \quad \text{a.s.}$$

**1. Introduction.** Let $(X, \mathscr{B}(X), P)$ be a probability space. Let $T$ be an invertible bimeasurable measure-preserving transformation on $X$. If $\mathscr{S} \subset \mathscr{B}(X)$ then $\mathscr{F}(\mathscr{S})$ will denote the smallest $\sigma$-algebra containing $\mathscr{S}$.

$\mathscr{P} = (A_1, \cdots, A_r)$ is a *partition* of $X$ if each $A_i$ is a measurable set of positive measure, they are disjoint, and their union equals $X$. $\mathscr{P}$ is a *generator for* $T$ if $\mathscr{F}(\{T^iA \mid A \in \mathscr{P}, i = 0, \pm 1, \pm 2, \cdots\}) = \mathscr{B}(X)$.

Let $T^i\mathscr{P}$ denote the partition $\{T^iA_1, \cdots, T^iA_r\}$. Let $\mathscr{A}(\bigcup_{i=l}^{m} T^i\mathscr{P})$ denote the algebra generated by $T^l\mathscr{P}, T^{l+1}\mathscr{P}, \cdots, T^m\mathscr{P}$. Let $\mathscr{A} = \bigcup_{n=1}^{\infty} \mathscr{A}(\bigcup_{i=-n}^{n} T^i\mathscr{P})$; it is clear that $\mathscr{A}$ is an algebra since the $\mathscr{A}(\bigcup_{i=-n}^{n} T^i\mathscr{P})$'s are increasing.

Let $\mathscr{L}(\mathscr{A})$ = closure under the $\| \ \|_\infty$-norm of the linear manifold generated by the class of functions $\{\chi_A\}_{A\in\mathscr{A}}$. And finally define

$$f(n) = \sup_{A,B} |P(A \cap B) - P(A) \cdot P(B)|$$

where

$$A \in \mathscr{F}(\textstyle\bigcup_{i=-\infty}^{0} T^i\mathscr{P}) , \qquad B \in \mathscr{F}(\bigcup_{i=n}^{\infty} T^i\mathscr{P}) \quad \text{for} \quad n \geqq 1 .$$

$T$ satisfies the *Rosenblatt condition* if $\lim_{n\to\infty} f(n) = 0$. $T$ is said to satisfy the *strong Rosenblatt condition* if for every sequence $\{r(j)\}_{j=1}^{\infty}$ so that $r(j) \geqq j$ for $j = 1, 2, \cdots \exists\, \alpha \in (0, 2)$ so that

$$\textstyle\sum_{j=1}^{n} (n - j) f(r(j)) = O(n^\alpha) .$$

*Note* that if $T$ satisfies the Rosenblatt condition then $\sum_{j=1}^{n} (n - j)f(r(j)) = o(n^2)$ for every sequence $\{r(j)\}$ so that $r(j) \geqq j$. Moreover if $f(n) = O(1/n^\beta)$ for $0 < \beta$ then it follows at once that $T$ satisfies the strong Rosenblatt condition.

It is well known that if $\mathscr{P}$ is a generator for $T$ and $T$ satisfies the Rosenblatt condition then $T$ is a $K$-automorphism (see [1]).

**2. The strong Rosenblatt condition and the ergodic theorem.**

(2.1) THEOREM. *If $T$ satisfies the strong Rosenblatt condition then for every*

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*strictly increasing sequence* $\{k_j\}_{j=1}^{\infty}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(T^{k_j}x) = E(g) \quad \text{a.s.}$$

*for every* $g \in \mathscr{L}(\mathscr{A})$.

PROOF. Fix a sequence $\{k_j\}_{j=1}^{\infty}$.

(1) We will first prove the theorem for $g(x) = \chi_A(x)$ where $A \in \mathscr{A}$. Now if $A \in \mathscr{A}$ then $A \in \mathscr{A}(\bigcup_{i=-l_0}^{l_0} T^i \mathscr{P})$ for some $l_0 \geqq 1$.

(2) It therefore follows that

$$T^{-l_0}A \in \mathscr{A}(\bigcup_{i=-2l_0}^{0} T^i\mathscr{P}) \subset \mathscr{F}(\bigcup_{i=-\infty}^{0} T^i\mathscr{P}) .$$

(3) For $\varepsilon > 0$ let

$$B_{n,\varepsilon} = \left\{x \,\middle|\, \left|\frac{1}{n} \sum_{j=i}^{n} \chi_A(T^{k_j}x) - P(A)\right| \leqq \varepsilon\right\} .$$

(4) Using Chebyshev's inequality we obtain

$$\begin{aligned}
P(B_{n,\varepsilon}^c) &\leqq \frac{1}{\varepsilon^2} \int_X \left|\frac{1}{n} \sum_{j=1}^{n} \chi_A(T^{k_j}x) - P(A)\right|^2 P(dx) \\
&= \frac{1}{\varepsilon^2} \cdot \frac{1}{n^2} \sum_{j,l=1}^{n} \int_X [\chi_A(T^{k_j}x)\chi_A(T^{k_l}x) - P(A)^2]P(dx) \\
&= \frac{1}{\varepsilon^2}\left[\frac{1}{n^2} \sum_{j,l=1}^{n} P(T^{k_j}A \cap T^{k_l}A) - P(A)^2\right] \\
&\leqq \frac{1}{\varepsilon^2} \cdot \frac{1}{n^2} \cdot \sum_{j,l=1}^{n} |P(T^{k_j}A \cap T^{k_l}A) - P(A)^2| \\
&= \frac{1}{\varepsilon^2} \cdot \frac{1}{n^2} [n \cdot |P(A) - P(A)^2| + 2 \sum_{1\leqq j<l\leqq n} |P(T^{k_l-k_j}A \cap A) - P(A)^2] .
\end{aligned}$$

(5) For each integer $s = 1, 2, \cdots$ there exists an integer $j(s) \geqq 1$ so that

$$|P(T^{k_{j(s)}} + s^{-k_{j(s)}}A \cap A) - P(A)^2 \geqq |P(T^{k_{j+s}-k_j}A \cap A) - P(A)^2|$$

for $j = 1, 2, \cdots$.

This follows from the fact that if $k_{j+s} - k_j$ is bounded for all $j$, there is nothing to prove; if $k_{j+s} - k_j$ is unbounded then it follows from the fact that $T$ is strongly mixing (by the Rosenblatt condition).

Now we define $r(s) = k_{j(s)+s} - k_{j(s)}$ for $s = 1, 2, \cdots$. Notice that $r(s) \geqq s$ for $s = 1, 2, \cdots$ since $\{k_j\}_{j=1}^{\infty}$ is a strictly increasing sequence.

(6) Observe now that

$$T^{-l_0}A \in \mathscr{F}(\bigcup_{i=-\infty}^{0} T^i\mathscr{P}) \qquad \text{and} \qquad T^{r(s)-l_0}A \in \mathscr{F}(\bigcup_{i=r(s)-2l_0}^{\infty} T^i\mathscr{P}) .$$

Therefore we obtain the following inequality for s $\geqq 2l_0 + 1$:

$$|P(T^{r(s)}A \cap A) - P(A)^2| = |P(T^{r(s)-l_0}A \cap T^{-l_0}A) - P(A)^2| \leqq f(r(s) - 2l_0) .$$

(7) From the fact that $T$ satisfies the strong Rosenblatt condition we see that $\exists\, \alpha \in (0, 2)$ so that

$$\sum_{s=2l_0+1}^{n-1} (n - s)f(r(s) - 2l_0) = O(n^{\alpha}) .$$

(8) Combining (4), (5), (6) and (7) we get the estimate

$$P(B^c_{n,\varepsilon}) \leqq \frac{P(A)-P(A)^2}{\varepsilon^2 n} + \frac{2}{\varepsilon^2 n^2}\sum_{1\leqq j<l\leqq n}|P(T^{k_l-k_j}A\cap A)-P(A)^2|$$
$$= \frac{P(A)-P(A)^2}{\varepsilon^2 n} + \frac{2}{\varepsilon^2 n^2}\sum_{s=1}^{n-1}\sum_{j=1}^{n-s}|P(T^{k_{j+s}-k_j}A\cap A)-P(A)^2|$$
$$\leqq \frac{P(A)-P(A)^2}{\varepsilon^2 n} + \frac{2}{\varepsilon^2 n^2}\sum_{s=1}^{n-1}(n-s)|P(T^{r(s)}A\cap A)-P(A)^2|$$
$$\leqq \frac{2}{\varepsilon^2 n} + \frac{2}{\varepsilon^2 n^2}\sum_{s=1}^{2l_0}(n-s)\cdot 2+\frac{2}{\varepsilon^2 n^2}\sum_{2=2l_0+1}^{n-1}(n-s)|P(T^{r(s)}A\cap A-P(A)^2|$$
$$\leqq \frac{2}{\varepsilon^2 n} + \frac{8l_0 n}{\varepsilon^2 n^2} + \frac{2}{\varepsilon^2 n^2}\sum_{s=2l_0+1}^{n-1}(n-s)f(r(s)-2l_0)$$
$$= O\left(\frac{1}{n}\right) + O\left(\frac{1}{n^{2-\alpha}}\right).$$

(9) Now remember that $2-\alpha>0$. This means we can choose a positive integer $\bar{j}$ so that $\bar{j}\geqq 2$ and $\bar{j}(2-\alpha)>1$; this together with (8) gives us

$$\sum_{n=1}^{\infty} P(B^c_{n^{\bar{j}},\varepsilon}) < \infty .$$

Use the Borel–Cantelli lemma now to conclude that the set

$$C_\varepsilon = \{x\,|\,x\in B_{n^{\bar{j}},\varepsilon} \text{ for all except finitely many } n\text{'s}\}$$

has $P(C_\varepsilon)=1$. Let $C=\bigcap_{j=1}^{\infty} C_{(1/j)}$. It is clear that $P(C)=1$.

(10) If $x\in C$ then.

$$\lim_{n\to\infty}\left|\frac{1}{n^{\bar{j}}}\sum_{j=1}^{n^{\bar{j}}}\chi_A(T^{k_j}x)-P(A)\right| = 0 .$$

(11) Now suppose $n^{\bar{j}}\leqq m\leqq (n+1)^{\bar{j}}$. Then

$$\left|\frac{1}{m}\sum_{j=1}^{m}\chi_A(T^{k_j}x)-P(A)\right|$$
$$\leqq \left|\frac{n^{\bar{j}}}{m}\cdot\frac{1}{n^{\bar{j}}}\sum_{j=1}^{n^{\bar{j}}}\chi_A(T^{k_j}x)-P(A)\right| + \frac{1}{m}\sum_{j=n^{\bar{j}}+1}^{m}\chi_A(T^{k_j}x)$$
$$\leqq \frac{n^{\bar{j}}}{m}\left|\frac{1}{n^{\bar{j}}}\sum_{j=1}^{n^{\bar{j}}}\chi_A(T^{k_j}x)-P(A)\right| + \left|\frac{n^{\bar{j}}}{m}P(\mathrm{A})-P(A)\right| + \frac{((n+1)^{\bar{j}}-n^{\bar{j}})}{n^{\bar{j}}}$$
$$\leqq \left|\frac{1}{n^{\bar{j}}}\sum_{j=1}^{n^{\bar{j}}}\chi_A(T^{k_j}x)-P(A)\right| + (P(A)+1)\frac{((n+1)^{\bar{j}}-n^{\bar{j}})}{n^{\bar{j}}}$$
$$= \left|\frac{1}{n^{\bar{j}}}\sum_{j=1}^{n^{\bar{j}}}\chi_A(T^{k_j}x)-P(A)\right| + O\left(\frac{1}{n}\right).$$

Therefore from (10) and the last inequality we obtain: if $x\in C$ then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_A(T^{k_j}x) = P(A) .$$

It is clear now that we can extend a.s. convergence to simple functions whose characteristic function components come from sets in $\mathscr{A}$, and of course then we can extend a.s. convergence to functions which can be uniformly approximated by these simple functions. □

Krengel proves in [3] that we can find a strictly increasing sequence $\{k_j\}_{j=1}^{\infty}$ and a set $A \in \mathscr{B}(X)$ so that

$$\limsup \frac{1}{n} \sum_{j=1}^{n} \chi_A(T^{k_j}x) = 1 \quad \text{a.s.}$$

and

$$\liminf \frac{1}{n} \sum_{j=1}^{n} \chi_A(T^{k_j}x) = 0 \quad \text{a.s.}$$

In view of this fact we cannot have pointwise convergence for all strictly increasing sequences and all $f \in L^p(X, P)$; this means that the best we can hope for is that $\exists$ a dense set $\mathscr{D} \subset L^p(X, P)$ so that the individual ergodic theorem holds for every strictly increasing sequence $\{k_j\}$ and all $g \in \mathscr{D}$.

(2.2) COROLLARY. *If $\mathscr{P}$ is a generator for $T$ and $T$ satisfies the strong Rosenblatt condition then $\exists$ a closed linear subspace $\mathscr{D} \subset L^\infty(X, P)$ so that $\mathscr{D}$ is dense in $L^p(X, P)$ for $1 \leqq p < \infty$ and for every strictly increasing sequence $\{k_j\}_{j=1}^{\infty}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(T^{k_j}x) = E(g) \quad \text{a.s.}$$

*for every $g \in \mathscr{D}$.*

PROOF. Observe that if $\mathscr{P}$ is a generator for $T$ then $\mathscr{F}(\mathscr{A}) = \mathscr{B}(X)$. From this it follows that $\mathscr{D} = \mathscr{L}(\mathscr{A})$ has the desired properties. □

In [2], J.-P. Conze proves:

(2.3) THEOREM. *If $\{k_j\}_{j=1}^{\infty}$ is a sequence of integers with positive lower density then*

$$\left\{ f \in L^1(X, P) \,\middle|\, \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(T^{k_j}x) = E(f) \ \text{a.s.} \right\}$$

*is a closed subset of $L^1(X, P)$.*

This result allows us to prove:

(2.4) THEOREM. *If $\{k_j\}_{j=1}^{\infty}$ is a sequence of integers with positive lower density, $\mathscr{P}$ is a generator for $T$, and $T$ satisfies the strong Rosenblatt condition, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(T^{k_j}x) = E(f) \quad \text{a.s.} \qquad \textit{for} \quad \forall f \in L^1(X, P)\,.$$

PROOF. Combine (2.2) and (2.3).

**3. Topological spaces.** Let $X$ in addition be a topological space with topology $\mathscr{T}$ (i.e., $\mathscr{T}$ is the class of open sets).

DEFINITION. We say that $(\mathscr{P}, T)$ is a generator for $\mathscr{T}$ if $\exists$ a class of sets

$\mathscr{B} \subset \mathscr{A}$ so that

(i) if $B \in \mathscr{B}$ then interior $(B) \neq \emptyset$;

(ii) if $x \in X$ and $x \in 0_x \in \mathscr{T}$ then $\exists\, B_x \in \mathscr{B}$ so that $B_x \subset 0_x$ and $x \in$ interior $(B_x)$.

Let

$C(X) =$ (I) the bounded continuous functions if $X$ is compact.

$=$ (II) the bounded continuous functions which vanish at infinity if $X$ is not compact.

(3.1) LEMMA. *If $(\mathscr{P}, T)$ is a generator for $\mathscr{T}$ then $C(X) \subset \mathscr{L}(\mathscr{A})$.*

PROOF. Let $h \in C(X)$ be real valued and fix $\varepsilon > 0$.

(1) Choose a positive integer $m$ so large that $(2/m)\|h\|_\infty < \varepsilon$.

(2) Let $A_k = \{x \mid ((k-1)/m)\|h\|_\infty < h(x) < ((k+1)/m)\|h\|_\infty\}$ for $k = -m, -m+1, \cdots, m$. Then $A_{-m}, \cdots, A_m \in \mathscr{T}$ and $\bigcup_{k=-m}^{m} A_k = X$.

(3) Choose a compact set $K$ so that if $x \in K^c$ then $|h(x)| < \varepsilon$. That this is possible follows from the definition of $C(X)$.

(4) For $k = -m, \cdots, m$ and $x \in A_k$ choose $B_x \in \mathscr{B}$ so that $x \in$ interior $(B_x)$ $B_x \subset A_k$. Then $\bigcup_{x \in X}$ interior $(B_x) = X$ and therefore we can find $x_1, \cdots, x_l$ so that $\bigcup_{j=1}^{l} B_{x_j} \supset K$.

(5) Define

$$\begin{aligned} C_{x_1} &= B_{x_1} \\ C_{x_2} &= B_{x_2} \backslash B_{x_1} \\ &\vdots \\ C_{x_l} &= B_{x_l} \backslash \textstyle\bigcup_{j=1}^{l-1} B_{x_j}\,. \end{aligned}$$

Then: (i) $C_{x_1}, \cdots, C_{x_l} \in \mathscr{A}$ and they are disjoint.

(ii) If $x \in (\bigcup_{j=1}^{l} C_{x_j})^c$ then $|h(x)| < \varepsilon$. This follows from (3) and (4).

(6) From (4) and (5) it follows that each $C_{x_j} \subset A_{k_j}$ for some $k_j$.

Define $c_j = (k_j/m)\|h\|_\infty$ for $j = 1, 2, \cdots, l$.

(7) Let $g(x) = \sum_{j=1}^{l} c_j \chi_{C_{x_j}}(x)$. Then $g(x) \in \mathscr{L}(\mathscr{A})$ and

$$\|g(x) - h(x)\|_\infty \leqq \frac{2}{m}\|h\|_\infty < \varepsilon\,. \qquad \square$$

(3.2) COROLLARY. *If $(\mathscr{P}, T)$ is a generator for $\mathscr{T}$ and $T$ satisfies the strong Rosenblatt condition then for every strictly increasing sequence $\{k_j\}_{j=1}^{\infty}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(T^{k_j}x) = E(g) \quad \text{a.s.}$$

*for all $g \in C(X)$.*

PROOF. Immediate from Lemma 3.1 and Theorem 2.1.

(3.3) REMARK. If $\mathscr{B}(X) = \mathscr{F}(\mathscr{T}) =$ Borel sets in the topological space $(X, \mathscr{T})$, $\mathscr{P}$ is a generator for $T$, and $(\mathscr{P}, T)$ is a generator for $\mathscr{T}$, then Corollary 2.2 holds with $\mathscr{D} = C(X)$. This follows simply from the fact that in this case Corollary 3.2 is in force and $C(X)$ is dense in $L^p(X, P)$ for $1 \leqq p < \infty$.

**4. Concluding remarks.** J.-P. Conze proves in [2] some individual ergodic theorems for subsequences if $T$ has Lebesgue spectrum; however, his techniques are not related to ours. In [4] N. F. G. Martin proves that if $f(n)$ decays at an exponential rate and $\mathscr{P}$ is a generator for $T$ then $T$ is a Bernoulli shift; however our strong Rosenblatt condition for $f(n)$ is considerably weaker than exponential decay.

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## REFERENCES

[1] Blum, J. R. and Hanson, D. L. (1963). Some asymptotic convergence theorems. In *Ergodic Theory*, 55–70. Academic Press, New York.

[2] Conze, J.-P. (1973). Convergence des moyennes pour des sous-suites. *Bull. Soc. Math. France, Memoires* **35** 7–15.

[3] Krengel, U. (1971). On the individual ergodic theorem for subsequences. *Ann. Math. Statist.* **42** 1091–1095.

[4] Martin, N. F. G. (1974). The Rosenblatt mixing condition and Bernoulli shifts. *Ann. Probability* **2** 333–338.

[5] Rosenblatt, M. (1956). A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci.* **42** 43–47.

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