The Annals of Probability
1977, Vol. 5, No. 2, 210–221

FIRST EXIT TIMES FROM MOVING BOUNDARIES FOR SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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Let $X_1, X_2, \cdots$ be independent random variables such that $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$ and the uniform Lindeberg condition is satisfied. Let $S_n = X_1 + \cdots + X_n$. In this paper, we study the first exit time $N_c = \inf\{n \geqq m: |S_n| \geqq cb(n)\}$ for general lower-class boundaries $b(n)$. Our results extend the theorems of Breiman, Brown, Chow, Robbins and Teicher, Gundy and Siegmund who studied the case $b(n) = n^{\frac{1}{2}}$. We also obtain the limiting moments of N_c in the case $b(n) = n^\alpha$ $(0 < \alpha < \frac{1}{2})$ as analogues of recent results in extended renewal theory.

1. Introduction. In [1], Breiman has proved the following theorem about first exit times from square-root boundaries for sample sums: Let $X_1, X_2, \cdots$ be i.i.d. with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^3 < \infty$. Let $c > 0$, $m \geqq 1$, $S_n = X_1 + \cdots + X_n$, and define $T(c, m) = \inf\{n \geqq m: |S_n| \geqq cn^{\frac{1}{2}}\}$. Then as $n \to \infty$,

$$P[T(c, m) > n] \sim hn^{-\rho(c)}, \tag{1.1}$$

where $\rho(c)$ is an absolute constant depending only on c, and h is a constant depending on c, m and the distribution of X_1 such that $h > 0$ if m is sufficiently large. The key idea in Breiman's proof of (1.1) is to first establish it for $T_c^* = \inf\{t \geqq 1: |W(t)| \geqq ct^{\frac{1}{2}}\}$ by transforming the Wiener process $W(t)$ into the Ornstein–Uhlenbeck process $U(t)$ so that square-root boundaries for $W(t)$ become constant boundaries for $U(t)$. This approach, however, breaks down when one considers more general lower-class boundaries, say $t^{\frac{1}{3}}$ or $t^{\frac{1}{2}}/\log t$, and Breiman mentions the open problem in ([1], pages 15–16) on analogous results for the first exit times from such boundaries.

When $\rho(c)$ is an integer, say $\rho(c) = k$, c turns out to be the smallest positive root c_k of the Hermite polynomial of order $2k$. Since $\rho(c)$ is strictly decreasing in c, (1.1) implies that if m is sufficiently large, then

$$\begin{aligned} ET^k(c, m) &< \infty \quad \text{if} \quad c < c_k, \\ &= \infty \quad \text{if} \quad c \geqq c_k. \end{aligned} \tag{1.2}$$

By making use of suitable martingales, Chow, Robbins and Teicher [3] and later Gundy and Siegmund [7] have proved (1.2) in the case where $k = 1$ and

Received December 8, 1975.

[1] Research partially supported by ONR Grant N00014-67-0108-0018.

AMS 1971 *subject classifications.* Primary 60F05, 60K05.

Key words and phrases. First exit times, lower-class boundaries, Lindeberg condition, uniform invariance principle, delayed sums, extended renewal theory without drift.

$X_1, X_2, \cdots$ are independent random variables such that $EX_n = 0$, $EX_n^2 = 1$ and the Lindeberg condition is satisfied. Their method and result have been extended by Brown [2] for general integers k.

In this paper, we shall study the open problem in Breiman's paper and try to find analogous results for general lower-class boundaries. Instead of restricting ourselves to the i.i.d. case, we shall consider the problem in the setting of independent random variables which satisfy a slightly stronger assumption than the Lindeberg condition. Our main result is Theorem 1 which is stated in Section 2 and proved in Section 4. It covers a general class of boundaries, and applications of the theorem to special boundaries of interest, such as $\log n$, n^α $(0 < \alpha < \frac{1}{2})$ and $n^{\frac{1}{2}}(\log n)^{-\beta}(\log\log n)^\delta$ $(\beta > 0, \delta \leqq 0)$ are given in the corollaries in Section 2.

The main idea in our analysis of first exit times from general moving boundaries $b(n)$ is to divide the family $\{X_1, X_2, \cdots\}$ into suitably chosen increasing blocks $\{X_1, \cdots, X_{\nu(1)}\}$, $\{X_{\nu(1)+1}, \cdots, X_{\nu(2)}\}, \cdots$, so that the block size $\nu(k+1) - \nu(k)$ is of the same order of magnitude as $b^2(\nu(k))$, which is in turn of the same order of magnitude as $b^2(\nu(k+1))$. We can then obtain estimates for $P[T > \nu(k)]$ by using the invariance principle for the successive delayed sums within each block. Since we want the convergence rate in the invariance principle to be uniform over all the blocks, a uniform version of the invariance principle is developed in Section 3.

We note that by (1.2), if $0 \leqq \alpha < \frac{1}{2}$ and $N(c) = \inf\{n \geqq 1 : |S_n| \geqq cn^\alpha\}$, then $EN^p(c) < \infty$ for all $p > 0$ and $c > 0$. It is interesting to consider the limiting behavior of $EN^p(c)$ as $c \to \infty$. In the case where $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = \mu > 0$ and $E(X_1^-)^p < \infty$, then

$$EN^p(c) \sim EN_+^p(c)\,, \tag{1.3}$$

where $N_+(c) = \inf\{n \geqq 1 : S_n \geqq cn^\alpha\}$ is the one-sided first passage time, and Gut [8] has shown that

$$EN_+^p(c) \sim (c/\mu)^{p/(1-\alpha)}\,. \tag{1.4}$$

The result (1.4) is an extension of the classical renewal theorem which deals with the case $\alpha = 0$. In view of (1.3), (1.4) implies that

$$EN^p(c) \sim (c/\mu)^{p/(1-\alpha)}\,. \tag{1.5}$$

In Section 5, we shall find an analogue of (1.5) when $\mu = 0$. Of course, in this driftless case, (1.3) no longer holds and $N(c)$ and $N_+(c)$ have very different asymptotic properties. An analogue of (1.4) for driftless random walks has been considered in [9] for the case $\alpha = 0$.

2. First exit times from general lower-class boundaries.

THEOREM 1. *Suppose $X_1, X_2, \cdots$ are independent random variables such that $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, and*

$$n^{-1}\textstyle\sum_{i=k+1}^{k+n} EX_i^2 I_{[|X_i| > \varepsilon n^{\frac{1}{2}}]} \to 0 \quad \text{uniformly in } k \text{ as } n \to \infty \text{ for every } \varepsilon > 0\,. \tag{2.1}$$

Let $S_n = X_1 + \cdots + X_n$, *and let* $(b(n))_{n \geqq m}$ *be an ultimately nondecreasing sequence of positive numbers such that*

$$\limsup_{n\to\infty} n^{-\frac{1}{2}} b(n) < \infty \,, \tag{2.2a}$$

$$\limsup_{n\to\infty} b(an)/b(n) < \infty \qquad \textit{for every integer} \quad a > 1 \,. \tag{2.2b}$$

For $c > 0$, *define*

$$N_c = \inf\{n \geqq m : |S_n| \geqq cb(n)\} \,. \tag{2.3}$$

Let $m = \nu(1) < \nu(2) < \cdots$ *be an increasing sequence of positive integers satisfying the following conditions*:

$$\limsup_{k\to\infty} (\nu(k+1) - \nu(k))/b^2(\nu(k)) < \infty \,, \tag{2.4a}$$

$$\liminf_{k\to\infty} (\nu(k+1) - \nu(k))/b^2(\nu(k)) > 0 \,. \tag{2.4b}$$

By defining $\nu(t)$ *to be linear on* $[i, i+1]$, $i = 1, 2, \cdots$, *we can regard* ν *as an increasing continuous function on* $[1, \infty)$. *Let* $\Psi : [m, \infty) \to [1, \infty)$ *be the inverse function of* ν. *Then for every sufficiently large* c, *there exists a finite positive number* θ_c *such that*

$$\begin{aligned} E\exp(\theta\Psi(N_c)) &< \infty \qquad \textit{if} \quad \theta < \theta_c \,, \\ &= \infty \qquad \textit{if} \quad \theta > \theta_c \,. \end{aligned} \tag{2.5}$$

REMARKS. (i) Condition (2.1) is obviously satisfied when $X_1, X_2, \cdots$ are i.i.d. with $EX_1^2 < \infty$. If $X_1, X_2, \cdots$ are independent and $\sup_{n\geqq 1} EX_n^2\varphi(|X_n|) < \infty$ for some nondecreasing function φ on $[0, \infty)$ such that $\lim_{x\to\infty} \varphi(x) = \infty$, then it is easy to see that (2.1) also holds. We note that (2.1) is a slightly strengthened form of the Lindeberg condition. It guarantees that the sequence of partial sums S_n not only obeys the central limit theorem but also the invariance principle. In fact, it implies an even stronger result. If instead of S_n, we look at the sequence of delayed sums $S_{k,n} = X_{k+1} + \cdots + X_{k+n}$ $(n = 1, 2, \cdots)$, then for every fixed k, the sequence $S_{k,n}$ obeys the invariance principle and the convergence to Brownian motion is in some sense uniform in k. This result, which will be proved in Section 3, is what we need in the proof of Theorem 1.

(ii) Given any sequence $(b(n))_{n\geqq m}$ of positive numbers such that $\inf_{n\geqq m} b(n) > 0$, there always exists an increasing sequence $(\nu(k))_{k\geqq 1}$ of positive integers satisfying (2.4a) and (2.4b). For example, set $\nu(1) = m$ and choose a positive number α such that $\inf_{j\geqq m} \alpha b^2(j) \geqq 2$. For $k = 1, 2, \cdots$, define $\nu(k+1) = \nu(k) + [\alpha b^2(\nu(k))]$.

The proof of Theorem 1 will be given in Section 4. In the rest of this section, we shall consider some applications of Theorem 1 to different types of boundaries $b(n)$. In all the following corollaries, we assume that $X_1, X_2, \cdots$ are independent with $EX_n = 0$, $EX_n^2 = 1$ $(n = 1, 2, \cdots)$ such that (2.1) is satisfied and define N_c as in Theorem 1.

COROLLARY 1. *Let* $0 \leqq \alpha < \frac{1}{2}$. *Let* $b(n) = n^\alpha g(n)$, $n \geqq m$, *where* $g(t)$ *is a positive and ultimately nondecreasing function on* $[m, \infty)$. *If* $\lim_{t\to\infty} g(t) = \infty$, *assume* *further that the following conditions hold*:

(2.6a) $$g(t+t^\rho)/g(t) = 1 + O(t^{\rho-1}) \qquad \textit{for every} \quad 0<\rho<1\,;$$

(2.6b) $$\limsup_{t\to\infty} g(tg(t))/g(t) < \infty\,.$$

Then

(2.7) $$\lim_{t\to\infty} t^{-\rho}g(t) = 0 \qquad \textit{for every} \quad \rho > 0\,.$$

Furthermore, given any sufficiently large c, there exists a finite positive constant $\theta(c)$ *such that*

(2.8) $$\begin{aligned} E\exp(\theta N_c^{1-2\alpha}/g^2(N_c)) &< \infty \qquad \textit{if} \quad \theta < \theta(c)\,,\\ &= \infty \qquad \textit{if} \quad \theta > \theta(c)\,. \end{aligned}$$

PROOF. To prove (2.7), we shall assume that $\lim_{t\to\infty} g(t) = \infty$. Let $\beta > \limsup_{t\to\infty} g(tg(t))/g(t)$. Given any $\delta > 1/\beta$, choose $\alpha \geqq m$ such that $g(\alpha) > \delta\beta$, $g(t)$ is nondecreasing for $t \geqq \alpha$ and $\beta > g(tg(t))/g(t)$ if $t \geqq \alpha$. Therefore

$$g(\alpha) > \beta^{-1}g(\alpha g(\alpha)) > \beta^{-2}g(\alpha g^2(\alpha)) > \cdots > \beta^{-k}g(\alpha g^k(\alpha)) > \cdots\,.$$

Hence for $k \geqq 1$, $g(\alpha\delta^k\beta^k) \leqq g(\alpha g^k(\alpha)) < \beta^k g(\alpha)$. A simple change of variable then implies that $g(t) = O(t^{1-(\log\delta)/(\log\delta\beta)})$ as $t\to\infty$. Since δ can be arbitrarily large, (2.7) holds.

We now prove (2.8). By (2.7), (2.2a) halds. Also (2.2b) holds in view of (2.6b). Let us first assume that $\lim_{t\to\infty} g(t) = \infty$. Set $\gamma = 1/(1-2\alpha)$ and let $\nu(k) = [k^\gamma g^{2\gamma}(k^\gamma)]$ for all large k. Then for all large k,

$$\begin{aligned} &\{(k+1)^\gamma - k^\gamma\}g^{2\gamma}(k^\gamma) - 1\\ (2.9)\qquad &\leqq \nu(k+1) - \nu(k)\\ &\leqq 1 + \{(k+1)^\gamma - k^\gamma\}g^{2\gamma}((k+1)^\gamma) + k^\gamma\{g^{2\gamma}((k+1)^\gamma) - g^{2\gamma}(k^\gamma)\}\,; \end{aligned}$$

(2.10) $$b^2(\nu(k)) \sim k^{2\alpha\gamma}g^{4\alpha\gamma}(k^\gamma)g^2(\nu(k))\,.$$

The ultimate monotonicity of $g(t)$ and (2.6b) imply that for each $d > 0$,

(2.11) $$\limsup_{t\to\infty} g(tg^d(t))/g(t) < \infty\,.$$

Furthermore it follows from (2.6a) that

(2.12) $$g^{2\gamma}((k+1)^\gamma) - g^{2\gamma}(k^\gamma) = O(g^{2\gamma}(k^\gamma)/k)\,.$$

Using (2.9), (2.10), (2.11), (2.12), it is easy to see that (2.4a) and (2.4b) hold. Hence Theorem 1 is applicable.

We note that for all large k, if $t = [k^\gamma g^{2\gamma}(k^\gamma)]$, then $\Psi(t) = k \geqq t^{1/\gamma}/g^2(t+1)$. By (2.11), we also have $\Psi(t) = O(t^{1/\gamma}/g^2(t))$. Therefore (2.8) follows from (2.5).

In the case where $\lim_{t\to\infty} g(t) < \infty$, we define $\nu(k) = [k^\gamma]$ for all large k. Then a similar argument as before establishes (2.8).

SOME SPECIAL CASES. (i) Suppose $(b(n))_{n\geqq m}$ is an ultimately nondecreasing sequence of positive numbers such that $\lim_{n\to\infty} b(n) = A < \infty$. Then $\alpha = 0$ in

Corollary 1 and so given c sufficiently large, there exists a positive constant $\theta(c, A)$ such that $E\exp(\theta N_c) < \infty$ if $\theta < \theta(c, A)$, but $= \infty$ if $\theta > \theta(c, A)$. Hence N_c has a finite moment generating function $m(\theta)$ in some neighborhood of the origin. This is the well-known theorem of Stein [12] when $X_1, X_2, \cdots$ are i.i.d. Our result says further that $m(\theta)$ cannot be finite for all θ if c is sufficiently large.

(ii) Suppose $b(n) = \log n$ for $n \geqq m$. Then the conditions in Corollary 1 are satisfied and so given c sufficiently large, there exists a positive constant $\theta(c)$ such that

$$(2.13)\qquad \begin{aligned} E\exp(\theta N_c/(\log N_c)^2) &< \infty \quad \text{if} \quad \theta < \theta(c)\,, \\ &= \infty \quad \text{if} \quad \theta > \theta(c)\,. \end{aligned}$$

(iii) Suppose $b(n) = n^\alpha(\log n)^\beta$ for $n \geqq m$, where $0 < \alpha < \frac{1}{2}$ and $\beta \geqq 0$. Then by Corollary 1, for every sufficiently large c, there exists $\theta(c) > 0$ such that

$$(2.14)\qquad \begin{aligned} E\exp(\theta N_c^{1-2\alpha}/(\log N_c)^{2\beta}) &< \infty \quad \text{if} \quad \theta < \theta(c)\,, \\ &= \infty \quad \text{if} \quad \theta > \theta(c)\,. \end{aligned}$$

COROLLARY 2. *Let $0 < \alpha < \frac{1}{2}$. For $n \geqq m$, let $b(n) = n^\alpha/g(n)$, where $g(t)$ is a positive and ultimately nondecreasing function on $[m, \infty)$ such that $b(n)$ is ultimately nondecreasing. If $\lim_{t\to\infty} g(t) = \infty$, assume further that* (2.6a) *holds and*

$$(2.15)\qquad \liminf_{t\to\infty} g(t/g(t))/g(t) > 0\,.$$

Then (2.7) *holds, and for every sufficiently large c, there exists a finite positive constant $\theta(c)$ such that*

$$(2.16)\qquad \begin{aligned} E\exp(\theta N_c^{1-2\alpha}g^2(N_c)) &< \infty \quad \textit{if} \quad \theta < \theta(c)\,, \\ &= \infty \quad \textit{if} \quad \theta > \theta(c)\,. \end{aligned}$$

PROOF. The ultimate monotonicity of $g(t)$ and (2.15) imply that for every $d > 0$, $\liminf_{t\to\infty} g(t/g^d(t))/g(t) > 0$. From this, it is easy to see that (2.7) holds. To prove (2.16), set $\gamma = 1/(1-2\alpha)$. If $\lim_{t\to\infty} g(t) < \infty$, then let $\nu(k) = [k^\gamma]$ and proceed as in the proof of Corollary 1. Now assume that $\lim_{t\to\infty} g(t) = \infty$. Define $\nu(k) = [k^\gamma/g^{2\gamma}(k^\gamma)]$. In view of (2.7), $\lim_{k\to\infty}\nu(k) = \infty = \lim_{n\to\infty} b(n)$. Proceeding as in the proof of Corollary 1, it can be shown that (2.4a) and (2.4b) hold. Therefore $\lim_{k\to\infty}(\nu(k+1) - \nu(k)) = \infty$ and $\nu(k)$ is strictly inceasing for all large k, say $k \geqq k_0$. We can also have $m = \nu(1) < \nu(2) < \cdots < \nu(k_0 - 1) < \nu(k_0)$ by redefining these terms if necessary. Hence (2.16) follows from Theorem 1.

COROLLARY 3. *Let $\alpha > 0$. Let $b(n) = n^{\frac{1}{2}}(\log n)^{-\alpha}g(n)$ for $n \geqq m$, where $g(t)$ is a positive and ultimately nondecreasing function on $[m, \infty)$. If $\lim_{t\to\infty} g(t) = \infty$, assume further that the following conditions hold*:

$$(2.17\text{a})\qquad g((1+\delta t^{-\rho})e^t)/g(e^t) = 1 + O(t^{-1-\rho}) \quad \textit{for all} \quad \delta > 0 \quad \textit{and} \quad \rho > 0\,;$$

$$(2.17\text{b})\qquad \limsup_{t\to\infty} g(e^{tg(t)})/g(e^t) < \infty\,.$$

Then

$$(2.18)\qquad \lim_{t\to\infty} (\log t)^{-\rho} g(t) = 0 \qquad \textit{for every} \quad \rho > 0 .$$

Furthermore, given any sufficiently large c, there exists a finite positive constant $\theta(c)$ *such that*

$$(2.19)\qquad \begin{aligned} E \exp(\theta(\log N_c)^{1+2\alpha}/g^2(N_c)) &< \infty \qquad \textit{if} \quad \theta < \theta(c) , \\ &= \infty \qquad \textit{if} \quad \theta > \theta(c) . \end{aligned}$$

REMARK. Write $\log_1 t = \log t$, $\log_2 t = \log\log t$, and in general set $\log_k t = \log(\log_{k-1} t)$. Let $g(t) = \log_k t$. If $k \geqq 1$, then $g(t)$ satisfies conditions (2.6a), (2.6b) and (2.15). If $k \geqq 2$, then conditions (2.17a) and (2.17b) also hold. Therefore if $b(n) = n^{\frac{1}{2}}(\log n)^{-\alpha}(\log_k n)^{\beta}$, where $\alpha > 0$, $\beta \geqq 0$ and $k \geqq 2$, then (2.19) reduces to

$$(2.20)\qquad \begin{aligned} E \exp(\theta(\log N_c)^{1+2\alpha}/(\log_k N_c)^{2\beta}) &< \infty \qquad \text{if} \quad \theta < \theta(c) , \\ &= \infty \qquad \text{if} \quad \theta > \theta(c) . \end{aligned}$$

PROOF OF COROLLARY 3. To prove (2.18), we may assume that $\lim_{t\to\infty} g(t) = \infty$ and proceed as the proof of (2.7). Therefore $b(n)$ satisfies (2.2a) and (2.2b). We now prove (2.20). First assume that $\lim_{t\to\infty} g(t) = \infty$. Set $\gamma = 1/(1 + 2\alpha)$ and let $\nu(k) = [\exp(k^\gamma g^{2\gamma}(\exp(k^\gamma)))]$ for all large k. The ultimate monotonicity of $g(t)$ and (2.17b) imply that for each $d > 0$,

$$(2.21)\qquad \limsup_{t\to\infty} g(\exp(tg^d(t)))/g(e^t) < \infty .$$

Making use of (2.17a) and (2.21), it can be shown as in the proof of Corollary 1 that (2.4a) and (2.4b) hold, and the desired conclusion (2.20) follows from Theorem 1. In the case where $\lim_{t\to\infty} g(t) < \infty$, we simply define $\nu(k) = [\exp(k^\gamma)]$ instead and proceed as before.

COROLLARY 4. *Let* $\alpha \geqq 0$. *Let* $b(n) = n^{\frac{1}{2}}(\log n)^{-\alpha}/g(n)$ *for* $n \geqq m$, *where* $g(t)$ *is a positive and ultimately nondecreasing function on* $[m, \infty)$ *such that* $b(n)$ *is ultimately nondecreasing. If* $\lim_{t\to\infty} g(t) = \infty$, *assume further that* (2.17a) *and* (2.18) *hold and*

$$(2.22)\qquad \liminf_{t\to\infty} g(e^{t/g^d(t)})/g(e^t) > 0 \qquad \textit{for every} \quad d > 0 .$$

Then for c sufficiently large, there exists a finite positive constant $\theta(c)$ *such that*

$$(2.23)\qquad \begin{aligned} E \exp(\theta(\log N_c)^{1+2\alpha} g^2(N_c)) &< \infty \qquad \textit{if} \quad \theta < \theta(c) , \\ &= \infty \qquad \textit{if} \quad \theta > \theta(c) . \end{aligned}$$

We note that if $\alpha = 0$, then (2.23) can be written as

$$(2.24)\qquad \begin{aligned} EN_c^{\theta g^2(N_c)} &< \infty \qquad \text{if} \quad \theta < \theta(c) , \\ &= \infty \qquad \text{if} \quad \theta > \theta(c) . \end{aligned}$$

In the particular case where $\lim_{t\to\infty} g^2(t) = A < \infty$, (2.24) says that N_c has finite moments of order lower than $A\theta(c)$, but infinite moments of order higher than

$A\theta(c)$. This corresponds to the results of Breiman and Brown referred to in Section 1.

An example of a function $g(t)$ such that $\lim_{t\to\infty} g(t) = \infty$ and (2.22) holds is $\log_k t$ with $k \geqq 2$. The proof of Corollary 4 is similar to that of Corollaries 2 and 3, and the details are omitted here.

3. A uniform version of the invariance principle for sums of independent random variables.

THEOREM 2. *Suppose* $X_1, X_2, \cdots$ *are independent random variables such that* $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, *and* (2.1) *holds. For* $k = 0, 1, \cdots$, *define*

$$(3.1) \qquad \zeta_{k,n}(0) = 0\,, \qquad \zeta_{k,n}(i/n) = n^{-\frac{1}{2}}(X_{k+1} + \cdots + X_{k+i}) \quad \text{and}$$
$$\zeta_{k,n}(t) \text{ is linear on } [(i-1)/n, i/n]\,, \quad i = 1, \cdots, n\,.$$

Then for each k, *there exists a standard Wiener process* $W(t)$, $0 \leqq t \leqq 1$, *such that for every* $\delta > 0$, *we have* (*by redefining the random variables on a new probability space if necessary*) *as* $n \to \infty$,

$$(3.2) \qquad P[\max_{0\leqq t\leqq 1} |W(t) - \zeta_{k,n}(t)| \geqq \delta] \to 0 \quad \text{uniformly in } k \text{ as } n \to \infty\,.$$

PROOF. Given any $\eta > 0$, we can choose $\rho > 0$ such that

$$(3.3) \qquad P[\max_{|t-s|\leqq 2\rho, 0\leqq s,t\leqq 1} |W(t) - W(s)| \geqq \delta/4] \leqq \eta\,.$$

Take $\varepsilon > 0$ such that

$$(3.4) \qquad \eta > 128\varepsilon^2\rho^{-2}\,.$$

With this choice of ε, we now truncate the X_j's. Define

$$X_j^{(n)} = X_j I_{[|X_j|\leqq \varepsilon n^{\frac{1}{2}}]}\,, \qquad \tilde{X}_j^{(n)} = X_j - X_j^{(n)}\,,$$
$$Y_j^{(n)} = n^{-\frac{1}{2}}(X_j^{(n)} - EX_j^{(n)})\,.$$

Let $\xi_{k,n}(i) = Y_{k+1}^{(n)} + \cdots + Y_{k+i}^{(n)}$ for $i = 1, \cdots, n$. By the Skorohod embedding theorem, we have (on some probability space) the representation

$$(3.5) \qquad \xi_{k,n}(i) = W(T_{k,n}(i))\,, \qquad i = 1, \cdots, n\,,$$

where $T_{k,n}(i) = \sum_{j=1}^{i} \tau_{k,n}(j)$ and $\tau_{k,n}(1), \cdots, \tau_{k,n}(n)$ are independent random variables such that for $i = 1, \cdots, n$,

$$E\tau_{k,n}(i) = E(Y_{k+i}^{(n)})^2\,, \qquad E(\tau_{k,n}(i))^2 \leqq 32E(Y_{k+i}^{(n)})^4$$

(cf. [11], pages 166–169). Noting that $|Y_j^{(n)}| \leqq 2\varepsilon$, we obtain that

$$(3.6) \qquad E(Y_j^{(n)})^4 \leqq 4\varepsilon^2 E(Y_j^{(n)})^2 \leqq 4\varepsilon^2 n^{-1} E(X_j^{(n)})^2 \leqq 4\varepsilon^2 n^{-1}\,.$$

In view of (3.6), it follows from the Kolmogorov inequality that

$$(3.7) \qquad P[\max_{1\leqq i\leqq n} |T_{k,n}(i) - ET_{k,n}(i)| \geqq \rho]$$
$$\leqq \rho^{-2} \sum_{i=1}^{n} E(\tau_{k,n}(i))^2 \leqq 32\rho^{-2} \sum_{i=1}^{n} E(Y_{k+i}^{(n)})^4 \leqq 128\varepsilon^2\rho^{-2} < \eta\,.$$

The last inequality in (3.7) follows from (3.4).

By (2.1), we can choose n_0 such that

$$(3.8)\qquad v(k, n) \leqq \min(\rho/2, \varepsilon\delta/8, \varepsilon^2\eta) \quad \text{for all} \quad n \geqq n_0 \quad \text{and all} \quad k\,,$$
$$\text{where} \quad v(k, n) = n^{-1} \textstyle\sum_{i=k+1}^{k+n} EX_i^2 I_{[|X_i| > \varepsilon n^{\frac{1}{2}}]}\,.$$

Making use of (3.8), we obtain the following inequalities:

$$(3.9\text{a})\qquad P[X_i \neq X_i^{(n)} \text{ for some } i = k+1, \cdots, k+n] \leqq \varepsilon^{-2} v(k, n) \leqq \eta\,;$$

$$(3.9\text{b})\qquad \max_{1\leq i\leq n} n^{-\frac{1}{2}} |\textstyle\sum_{j=1}^{i} EX_{k+j}^{(n)}| \leqq \varepsilon^{-1} v(k, n) \leqq \delta/8\,;$$

$$(3.9\text{c})\qquad \max_{1\leq i\leq n} |ET_{k,n}(i) - (i/n)| \leqq \textstyle\sum_{j=1}^{n} |E(Y_{k+j}^{(n)})^2 - 1/n|$$
$$\leqq n^{-1} \textstyle\sum_{j=1}^{n} \{|E(X_{k+j}^{(n)})^2 - 1| + (E\tilde{X}_{k+j}^{(n)})^2\}$$
$$\leqq 2v(k, n) \leqq \rho\,.$$

From the inequalities (3.3), (3.5), (3.7) and (3.9), the desired conclusion (3.2) follows.

The key step in the above proof lies in the truncation, and the rest of the argument has been straightforward. In fact if the X_j's are uniformly bounded, then the desired conclusion (3.2) follows immediately from a corresponding theorem of Freedman for martingales (cf. [6], pages 90–93).

As an immediate corollary of Theorem 2, we obtain the following theorem which will be used in the sequel.

THEOREM 3. *Suppose* $X_1, X_2, \cdots$ *are independent random variables such that* $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, *and* (2.1) *holds. Let* $S_{k,n} = X_{k+1} + \cdots + X_{k+n}$.

(i) *Let* $\Phi(x)$ *denote the distribution function of the standard normal distribution. Then as* $n \to \infty$,

$$(3.10)\qquad P[S_{k,n} < n^{\frac{1}{2}}x] \to \Phi(x) \qquad \textit{uniformly in} \quad k = 0, 1, \cdots \quad \textit{and} \quad -\infty < x < \infty\,.$$

(ii) *Let* $W(t)$ *be the standard Wiener process. Then as* $n \to \infty$,

$$(3.11)\qquad P[\max_{1\leq j\leq n} |S_{k,j}| < n^{\frac{1}{2}}\lambda,\ n^{\frac{1}{2}}\alpha < S_{k,n} < n^{\frac{1}{2}}\beta]$$
$$\to P[\max_{0\leq t\leq 1} |W(t)| < \lambda,\ \alpha < |W(1)| < \beta]$$
$$\textit{uniformly in} \quad \lambda > 0,\ -\lambda \leqq \alpha < \beta \leqq \lambda \quad \textit{and} \quad k = 0, 1, \cdots\,.$$

4. Proof of Theorem 1. We shall show that

$$(4.1)\qquad P[\Psi(N_c) > k] = O(p^k) \qquad \text{for some} \quad 0 < p = p(c) < 1\,;$$

$$(4.2)\qquad \liminf_{k\to\infty} q^{-k} P[\Psi(N_c) > k] > 0$$
$$\text{for some} \quad 0 < q = q(c) < 1 \quad \text{if} \quad c \geqq c_0\,.$$

From (4.1), $\Psi(N_c)$ has a finite moment generating function in some neighborhood of the origin, while (4.2) implies that if $c \geqq c_0$, then $E\exp(\theta\Psi(N_c)) = \infty$ for $\theta \geqq |\log q|$. Therefore defining $\theta_c = \sup\{\theta : E\exp(\theta\Psi(N_c)) < \infty\}$, we obtain (2.5).

Assume first that $\lim_{n\to\infty} b(n) = \infty$. Then by (2.4b),

$$\lim_{k\to\infty} (\nu(k+1) - \nu(k)) = \infty . \tag{4.3}$$

To prove (4.1), we first show that

$$\limsup_{k\to\infty} b^2(\nu(k+1))/(\nu(k+1) - \nu(k)) < \infty . \tag{4.4}$$

In view of (2.4b), it suffices to show that

$$\limsup_{k\to\infty} b(\nu(k+1))/b(\nu(k)) < \infty . \tag{4.5}$$

By (2.4a), $\nu(k+1)-\nu(k) = O(b^2(\nu(k)))$. Therefore by (2.2a), $\nu(k+1) = O(\nu(k))$. Hence (4.5) follows from (2.2b).

Let $S_{k,n} = X_{k+1} + \cdots + X_{k+n}$. Suppose $b(n)$ is nondecreasing for $n \geqq \nu(k_0)$. By Theorem 3(i), (4.3) and (4.4), given $c > 0$, there exist a positive number p with $p < 1$ and an integer $k_1 \geqq k_0$ such that for all $i \geqq k_1$,

$$P[|S_{\nu(i),\nu(i+1)-\nu(i)}| < 2cb(\nu(i+1))] \leqq p .$$

Therefore for $k > k_1$,

$$P[\Psi(N_c) > k] = P[N_c > \nu(k)] \leqq \prod_{i=k_1}^{k-1} P[|S_{\nu(i),\nu(i+1)-\nu(i)}| < 2cb(\nu(i+1))] \leqq p^{k-k_1} .$$

We now prove (4.2). By (2.4a), there exist $\alpha > 0$ and $k_2 \geqq k_0$ such that $b^2(\nu(k)) > \alpha^2(\nu(k+1) - \nu(k))$ if $k \geqq k_2$. For $c > 0$ and $k \geqq 1$, let

$$A_{k,c} = [N_c > \nu(k), |S_{\nu(k)}| < \tfrac{1}{2}cb(\nu(k))] ;$$
$$B_{k,c} = [\max_{1\leqq j\leqq \nu(k+1)-\nu(k)}|S_{\nu(k),j}| < \tfrac{1}{2}c\alpha(\nu(k+1) - \nu(k))^{\frac{1}{2}} \text{ and } 0 > S_{\nu(k),\nu(k+1)-\nu(k)} > -\tfrac{1}{2}c\alpha(\nu(k+1) - \nu(k))^{\frac{1}{2}}] ;$$
$$D_{k,c} = [\max_{1\leqq j\leqq \nu(k+1)-\nu(k)}|S_{\nu(k),j}| < \tfrac{1}{2}c\alpha(\nu(k+1) - \nu(k))^{\frac{1}{2}} \text{ and } 0 < S_{\nu(k),\nu(k+1)-\nu(k)} < \tfrac{1}{2}c\alpha(\nu(k+1) - \nu(k))^{\frac{1}{2}}] .$$

We note that for $k \geqq k_2$, the event

$$[\max_{1\leqq j\leqq \nu(k+1)-\nu(k)}|x + S_{\nu(k),j}| < cb(\nu(k)) \text{ and } |x + S_{\nu(k),\nu(k+1)-\nu(k)}| < \tfrac{1}{2}cb(\nu(k+1))]$$

contains $B_{k,c}$ if $0 \leqq x < \frac{1}{2}cb(\nu(k))$, and contains $D_{k,c}$ if $0 \geqq x > -\frac{1}{2}cb(\nu(k))$. Therefore for $k \geqq k_2$,

$$P(A_{k+1,c} \mid A_{k,c}) \geqq \min \{P(B_{k,c}), P(D_{k,c})\} . \tag{4.6}$$

By Theorem 3 (ii) and (4.3), we obtain that as $k \to \infty$,

$$P(B_{k,c}) \to P[\max_{0\leqq t\leqq 1} |W(t)| < \tfrac{1}{2}c\alpha \text{ and } 0 > W(1) > -\tfrac{1}{2}c\alpha] , \tag{4.7}$$

$$P(D_{k,c}) \to P[\max_{0\leqq t\leqq 1} |W(t)| < \tfrac{1}{2}c\alpha \text{ and } 0 < W(1) < \tfrac{1}{2}c\alpha] , \tag{4.8}$$

and the convergence in (4.7) and (4.8) is uniform in $c > 0$.

In view of (4.6), (4.7) and (4.8), we can choose $i_0 \geqq k_2$ such that

$$P(A_{i+1,c} \mid A_{i,c}) \geqq q(c) > 0 \quad \text{for all } i \geqq i_0 \text{ and } c > 0 . \tag{4.9}$$

Choose c_0 large enough so that

$$(4.10)\qquad P[|S_n| < c_0 b(n) \text{ for all } m \leqq n \leqq \nu(i_0) \text{ and } |S_{\nu(i_0)}| \leqq \tfrac{1}{2}c_0 b(\nu(i_0))] = \lambda > 0 .$$

From (4.9) and (4.10), it follows that if $c \geqq c_0$ and $k > i_0$,

$$P[\Psi(N_c) > k] = P[N_c > \nu(k)] \geqq P(A_{k,c}) \geqq P(A_{k,c} \mid A_{k-1,c})P(A_{k-1,c}) \geqq \cdots \geqq P(A_{i_0,c}) \prod_{i=i_0}^{k-1} P(A_{i+1,c} \mid A_{i,c}) \geqq \lambda(q(c))^{k-i_0} .$$

Hence (4.2) holds.

We now assume that $\lim_{n\to\infty} b(n) < \infty$. Then by (2.4a), $\limsup_{k\to\infty} (\nu(k+1) - \nu(k)) < \beta < \infty$. It then follows that $t/\beta \leqq \Psi(t) \leqq t$ for all large t. Let $0 < \beta_1 \leqq b(n) \leqq \beta_2$ for all $n \geqq m$. By Theorem 3(i), given $c > 0$, there exist a positive number p with $p < 1$ and a positive integer j such that $P[|S_{h,j}| < 2c\beta_2] \leqq p$ for all $h = 0, 1, \cdots$. Therefore for $k = 1, 2, \cdots$

$$P[N_c > kj] \leqq \prod_{i=1}^{k} P[|S_{(i-1)j,j}| < 2c\beta_2] \leqq p^k .$$

Hence (4.1) holds.

To prove (4.2), define $A_{k,c} = [N_c > k[c^2], |S_{k[c^2]}| < \tfrac{1}{2}c\beta_1]$. Then using Theorem 3(ii) and a similar argument as that leading to (4.9), we can choose $c_0 > 0$ and $0 < q < 1$ such that for all $c \geqq c_0$,

$$(4.11)\qquad P(A_{1,c}) \geqq q \quad \text{and} \quad P(A_{k+1,c} \mid A_{k,c}) \geqq q , \qquad k = 1, 2, \cdots .$$

From (4.11), it follows as before that $P[N_c > k[c^2]] \geqq q^k$ for $k = 1, 2, \cdots$. Hence (4.2) holds.

REMARK. An examination of the above proof shows that to obtain the upper bound (4.1), we need the condition (4.4). This condition explains why Theorem 1 covers only boundaries $b(n)$ which do not grow faster than the square-root boundary, i.e., so that (2.2a) holds. If $\lim_{n\to\infty} b^2(n)/n = \infty$, it is obvious that condition (4.4) cannot be satisfied for any choice of the subsequence $(\nu(i))$. In such cases, however, by combining the ideas in the derivation of (4.1) with those commonly used in the proof of the law of the iterated logarithm, we can obtain another type of upper bound. As an example, let $b(n) = \{2(1 - \varepsilon)n \log \log n\}^{\frac{1}{2}}$ $(0 < \varepsilon < 1)$. Let $N = \inf\{n \geqq 3 : |S_n| \geqq b(n)\}$, where $S_n = X_1 + \cdots + X_n$ and $X_1, X_2, \cdots$ are independent with $EX_n = 0$, $EX_n{}^2 = 1$, $n = 1, 2, \cdots$. For simplicity, assume

$$(4.12)\qquad \sup_{n\geqq 1} E|X_n|^{2+\delta} < \infty \qquad \text{for some} \quad 0 < \delta < 1 .$$

Given $0 < \varepsilon_1 < \varepsilon$, choose an integer $\alpha \geqq 3$ such that

$$(1 - \varepsilon)^{\frac{1}{2}}(\alpha^{\frac{1}{2}} + 1) < (1 - \varepsilon_1)^{\frac{1}{2}}(\alpha - 1)^{\frac{1}{2}} .$$

Letting $\nu(i) = \alpha^i$, we can therefore choose j_1 such that for all $j \geqq j_1$,

$$b(\nu(j + 1)) + b(\nu(j)) < \{2(1 - \varepsilon_1)\alpha^j(\alpha - 1) \log j\}^{\frac{1}{2}} .$$

Then for $k > j_1$,

$$\begin{aligned} &P[N > \nu(k)] \\ (4.13)\qquad &\leqq \prod_{j=j_1}^{k-1} P[|S_{\nu(j),\nu(j+1)-\nu(j)}| < b(\nu(j+1)) + b(\nu(j))] \\ &\leqq \prod_{j=j_1}^{k-1} P[|X_{\alpha^j+1} + \cdots + X_{\alpha^{j+1}}| < \{2(1-\varepsilon_1)\alpha^j(\alpha-1)\log j\}^{\frac{1}{2}}] \\ &\leqq \prod_{j=j_1}^{k-1} \{1 - 2\Phi(-\{2(1-\varepsilon_1)\log j\}^{\frac{1}{2}}) + O(\alpha^{-j\delta/2})\} . \end{aligned}$$

For the last relation above, we use (4.12) and a theorem of Esseen ([5], page 43). It then follows from (4.13) that

$$P[N > \alpha^k] \leqq \exp\{-k^{\varepsilon_1}(\pi(1-\varepsilon_1)\log k)^{-\frac{1}{2}}(\varepsilon_1^{-1} + o(1))\} .$$

Since ε_1 can be any number $< \varepsilon$, we have proved that

$$(4.14)\qquad E\exp(\theta(\log N)^\eta) < \infty \qquad \text{for all} \quad \theta \quad \text{if} \quad \eta < \varepsilon .$$

5. Asymptotic moments of first exit times related to extended renewal theory without drift. In this section, we shall find an analogue of (1.5) in the case $\mu = 0$. We shall drop the assumption that the X_i's are identically distributed and use the setting of Theorem 1.

THEOREM 4. *Suppose $X_1, X_2, \cdots$ are independent random variables such that $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, and (2.1) holds. Let $S_n = X_1 + \cdots + X_n$, $0 \leqq \alpha < \frac{1}{2}$, and let $W(t)$, $t \geqq 0$, be the standard Wiener process. Define*

$$N(c) = \inf\{n \geqq 1 : |S_n| \geqq cn^\alpha\} , \qquad \tau = \inf\{t > 0 : |W(t)| \geqq t^\alpha\} .$$

Let $a_p = E\tau^p$ for $p > 0$. Then as $c \to \infty$, $c^{-2/(1-2\alpha)}N(c)$ converges to τ in distribution, and

$$(5.1)\qquad EN^p(c) \sim a_p c^{2p/(1-2\alpha)} \qquad \text{for all} \quad p > 0 .$$

LEMMA. *Suppose $X_1, X_2, \cdots$ are independent with $EX_n = 0$ and $EX_n^2 = 1$ for all n. Then for $0 < \alpha < \frac{1}{2}$,*

$$\begin{aligned} (5.2)\qquad &\lim_{\delta\downarrow 0} \sup_{m\geqq 1} P[|S_n| \geqq m^{\frac{1}{2}}(n/m)^\alpha \ \textit{for some} \ 1 \leqq n \leqq \delta m] \\ &= 0 = \lim_{\delta\downarrow 0} P[|W(t)| \geqq t^\alpha \ \textit{for some} \ 0 < t \leqq \delta] . \end{aligned}$$

PROOF. By the Hájek–Rényi–Chow inequality (cf. [4], page 25),

$$\begin{aligned} &P[m^{\alpha-\frac{1}{2}}n^{-\alpha}|S_n| \geqq 1 \ \text{for some} \ 1 \leqq n \leqq \delta m] \\ (5.3)\qquad &\leqq m^{2\alpha-1} \textstyle\sum_{1\leqq n\leqq \delta m} n^{-2\alpha}EX_n^2 \\ &\leqq (1-2\alpha)^{-1}\delta^{1-2\alpha} \to 0 \qquad \text{as} \quad \delta \downarrow 0 . \end{aligned}$$

We remark that when $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = 0$ and $EX_1^2 = 1$, (5.2) is a special case of a theorem of Robbins and Siegmund for more general boundaries of the form $m^{\frac{1}{2}}g(n/m)$ (cf. Theorem 2(ii) of [10]). In the above lemma where we deal with $g(t) = t^\alpha$ $(0 < \alpha < \frac{1}{2})$, the X_i's need not be identically distributed and we do not even assume the Lindeberg condition.

PROOF OF THEOREM 4. Let $\gamma = 1/(1-2\alpha)$. We note that for all $x > 0$,

$$(5.4)\qquad P[N(c) > c^{2\gamma}x] = P[|S_n| < c^\gamma(n/c^{2\gamma})^\alpha \ \text{for all} \ 1 \leqq n \leqq c^{2\gamma}x] .$$

It follows easily from Theorem 2 that for every $0 < \delta < x$,

(5.5) $$P[|S_{[c^{2\gamma}t]}| < c^{\gamma}([c^{2\gamma}t]/c^{2\gamma})^{\alpha} \text{ for all } \delta \leqq t \leqq x] \to P[|W(t)| < t^{\alpha} \text{ for all } \delta \leqq t \leqq x] \quad \text{as} \quad c \to \infty .$$

Using the preceding lemma, we obtain from (5.4) and (5.5) that as $c \to \infty$,

$$P[N(c) > c^{2\gamma}x] \to P[|W(t)| < t^{\alpha} \text{ for all } 0 < t \leqq x] = P[\tau > x] .$$

To prove (5.1), we need only show that $(c^{-2\gamma}N(c))^{p}$ is uniformly integrable. We shall show that in fact there exist positive constants $\lambda < 1$ and c_0 such that

(5.6) $$P[N(c) > c^{2\gamma}k^{\gamma}] \leqq \lambda^{k-1} \qquad \text{for all} \quad c \geqq c_0 \quad \text{and} \quad k = 1, 2, \cdots .$$

Let $\nu(k) = [c^{2\gamma}k^{\gamma}]$. Then by Theorem 3(i), there exist positive constants $\lambda < 1$ and c_0 such that for all $c \geqq c_0$ and $k = 1, 2, \cdots$

$$P[|S_{\nu(k),\nu(k+1)-\nu(k)}| \leqq 2c(\nu(k+1))^{\alpha}] \leqq \lambda .$$

Therefore if $c \geqq c_0$ and $k = 1, 2, \cdots$, then

$$P[N(c) > c^{2\gamma}k^{\gamma}] \leqq \prod_{i=1}^{k-1} P[|S_{\nu(i),\nu(i+1)-\nu(i)}| \leqq 2c(\nu(i+1))^{\alpha}] \leqq \lambda^{k-1} .$$

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