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SLOWING DOWN d-DIMENSIONAL RANDOM WALKS

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If $\{S_n\}$ is a genuinely d-dimensional random walk and $d \geqq 3$, then with probability 1, $n^{-\alpha}|S_n| \to \infty$ as $n \to \infty$ for every $\alpha < \frac{1}{2}$. This follows from a recent result of H. Kesten. In this paper we show that, under certain conditions, there is a constant α_0 depending on the walk, but $\frac{1}{2} - 1/d \leqq \alpha_0 < \frac{1}{2}$, and a deterministic sequence of vectors $\{v_n\}$ such that $\liminf_n n^{-\alpha}|S_n - v_n| = 0$ with probability 1 for every $\alpha \geqq \alpha_0$. In discrete time this phenomenon cannot occur for any $\alpha < \frac{1}{2} - 1/d$; in continuous time it can occur for any $\alpha > 0$.

1. Introduction. Let $\{S_n\}$ be a random walk in R^d, i.e., $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqq 1$, where $X_1, X_2, \cdots$ are i.i.d. random vectors in R^d. Let $\psi : [1, \infty] \to (0, \infty)$ be a function satisfying $t^{-\frac{1}{2}}\psi(t) \downarrow 0$ as $t \to \infty$. If $d \geqq 3$ and $\{S_n\}$ is genuinely d-dimensional, then w.p.1 (with probability one)

$$\psi(n)^{-1}|S_n| \to \infty \qquad \text{as } n \to \infty \tag{1.1}$$

whenever

$$\textstyle\int_1^\infty \psi(t)^{d-2}t^{-d/2}\,dt < \infty\ . \tag{1.2}$$

This remarkable fact was first established by Erdös and Dvoretsky (1951) (see also Itô–McKean (1965) page 164) for the case where $\{S_n\}$ is a simple random walk and, equivalently, for the case where $\{S_n\}$ is a Brownian motion in R^d (in which case n is a continuous time parameter). For these two important special cases, divergence of (1.2) entails

$$\liminf_{n\to\infty} \psi(n)^{-1}|S_n| < \infty \quad \text{w.p. } 1\ .$$

A result of Erickson (1976), Theorem 7 and Corollaries 1 and 2, implies (1.1) under (1.2) when $\{S_n\}$ is in a domain of attraction of a nonsingular stable law in R^d. (Erickson imposes some extra, unnecessary for (1.1), regularity conditions on ψ and the step distribution of the walk.) His result generalizes and extends those of Erdös–Dvoretsky and some analogous results of Takeuchi (1964) and others on rates of escape of stable processes in R^d (see also Fristedt (1974), page 365). The complete result that (1.2) imples (1.1) for *any* genuinely d-dimensional random walk, $d \geqq 3$, was only recently established by H. Kesten (1977). His theorem makes precise the intuitive notion that simple random walk and Brownian motion escape to infinity more slowly than any other random walk. It should be noted that when $\psi(n) \equiv 1$ for all n, then (1.1)

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reduces to the well-known fact that all genuinely d-dimensional random walks are transient if $d \geqq 3$ (see Feller (1971) page 616 for the standard proof of this).

From (1.1)—(1.2) one may easily show that

$$\liminf n^{-\alpha}|S_n| = \infty \quad \text{w.p. } 1 \tag{1.3}$$

for every $\alpha < \frac{1}{2}$, $d \geqq 3$. In light of this it is of some interest to find that under certain circumstances the left-hand side of (1.3) becomes finite for a small interval of values of $\alpha < \frac{1}{2}$ when S_n is replaced by $S_n' = S_n - v_n$ where $\{v_n\}$ is a deterministic sequence of vectors. The purpose of this paper is to clarify this fact.

I wish to thank Professors Kesten and Blumenthal for some useful and stimulating discussions of the problem of rates of escape of random walks and other Markov processes. Professor Kesten had earlier noticed a related slowing down phenomenon in the case of simple random walk.

2. Statement of results. As above we let $S_n = X_1 + \cdots + X_n$, $n \geqq 1$, where $X_1, X_2, \cdots$, is a sequence of independent identically distributed random variables with values in R^d, real Euclidean space of d-dimensions. Let F be the common distribution of the X_i. We say that the walk $\{S_n\}$ is genuinely d-dimensional if F is not concentrated on a $d-1$ dimensional hyperplane. (A hyperplane is a set of the form $\{v \in R^d : \langle v, B\rangle = c\}$ where B is a given nonzero vector, c a given real constant, and $\langle \cdot, \cdot \rangle$ is the usual inner product.) The walk is simple if F assigns mass $(2d)^{-1}$ to each of the $2d$ points $e_1, -e_1, \cdots, e_d, -e_d$, $e_i = i$th unit coordinate vector in R^d. For any vector $v = (v(1), v(2), \cdots, v(d)) \in R^d$ put

$$|v| = \max\{|v(1)|, |v(2)|, \cdots, |v(d)|\}.$$

THEOREM 1. *Assume $d \geqq 3$ and that $\{S_n\}$ is genuinely d-dimensional.*

(i) *For any α satisfying*

$$\alpha < \tfrac{1}{2} - \frac{1}{d} \tag{2.1}$$

we have, for any deterministic sequence $\{u_n\}$ in R^d,

$$\liminf n^{-\alpha}|S_n - u_n| = \infty \quad \text{w.p. } 1. \tag{2.2}$$

(ii) *Suppose that for some $\beta \leqq 1$ and $b < \infty$*

$$\liminf P\{n^{-\beta}|S_n| \leqq b\} > 0, \tag{2.3}$$

then for any α satisfying

$$\beta - \frac{1}{d} \leqq \alpha < \beta \tag{2.4}$$

there exists a deterministic sequence of vectors $\{v_n\}$ such that $|v_n| = O(n^\beta)$ and

$$\liminf n^{-\alpha}|S_n - v_n| = 0 \quad \text{w.p. } 1. \tag{2.5}$$

COROLLARY. *If* $EX_1 = 0$ (= *the* 0 *vector*) *and if the covariance matrix* $(EX_1(i)X_1(j))_{i,j=1,\cdots,d}$ *exists and is nonsingular, then there exists a deterministic sequence* $v_1, v_2, \cdots$ *such that* $|v_n| = O(n^{\frac{1}{2}})$ *and for any* α *satisfying* $\frac{1}{2} - 1/d \leqq \alpha < \frac{1}{2}$

$$(2.6)\qquad \liminf n^{-\alpha}|S_n - v_n| = 0 \quad \text{w.p.1}.$$

REMARKS. 1. Under the assumptions of the corollary, one also gets $\limsup n^{-\alpha}|S_n - v_n| = \infty$ w.p.1 for any $\alpha \leqq \frac{1}{2}$ and any deterministic $v_n = O(n^{\frac{1}{2}})$. To see this note that Law $(n^{-\frac{1}{2}}S_n) \to G$ where G is a nonsingular Gaussian distribution. Hence, putting $C_A = \sup_n (n^{\alpha-\frac{1}{2}}A + n^{-\frac{1}{2}}|v_n|) < \infty$, we get

$$P\{n^{-\alpha}|S_n - v_n| > A \text{ i.o.}\} \geqq \limsup P\{n^{-\alpha}|S_n - v_n| > A\}$$
$$\geqq \lim P\{n^{-\frac{1}{2}}|S_n| > C_A\} = G\{x : |x| > C_A\} > 0 .$$

Consequently $\limsup n^{-\alpha}|S_n - v_n| > A$ w.p.1 for every $A > 0$ by the 0—1 laws.

2. The corollary is a good example of the fact that the means are not always the best centering constants: setting $v_n = 0 = ES_n$ on the left in (2.6) would give ∞ on the right by (1.3).

3. In view of (1.3) it is clear that (2.5) is of interest only when we can take $d \geqq 3$, $\alpha < \frac{1}{2}$. Also a concentration function argument shows that (2.3) is impossible for $\beta < \frac{1}{2}$. Thus β is limited to $\frac{1}{2} \leqq \beta < \frac{5}{6}$.

3. Proof of Theorem 1. Part i) of Theorem 1 is proved in Erickson (1976) (see Remark 7 in Section 5), so we only prove ii).

Assume $d \geqq 3$ and that (2.3) holds for some $\beta \in [\frac{1}{2}, 1]$. If we can find $\{v_n\}$ satisfying (2.5) in the case $\alpha = \beta - 1/d$, then clearly the same $\{v_n\}$ works for all larger α. So throughout this section

$$\alpha = \beta - \frac{1}{d}.$$

Since (2.3) holds we may choose a finite $c_0 > 1$ so large that for some $\delta_0 > 0$ and all $n \geqq 1$

$$(3.1)\qquad P\{n^{-\beta}|S_n| \leqq c_0\} \geqq \delta_0 .$$

We will first show that for all integers ρ sufficiently large there is a deterministic sequence of vectors $\{y_n^\rho\}$, depending on ρ, such that

$$(3.2)\qquad P\{n^{-\alpha}|S_n - y_n^\rho| \leqq \rho^{-1}c_0 2^{d\beta} \text{ i.o.}\} = 1 ,$$

("i.o." = "infinitely often"), and

$$(3.3)\qquad y_n^\rho = O(n^\beta) \qquad \text{uniformly in } \rho .$$

Write $D = 2^d$ and partition the closed hypercube $[-c_0 D^{\beta(k+1)}, c_0 D^{\beta(k+1)}]^d = \{x \in R^d : |x| \leqq c_0 D^{\beta(k+1)}\}$ into

$$r_k \equiv \rho^d D^k$$

subcubes $C_{1,k}, \cdots, C_{r_k,k}$ with centers $U_{1,k}, \cdots, U_{r_k,k}$. Each subcube has side length L_k where

$$L_k = 2c_0 D^{\beta(k+1)} r_k^{-1/d} = 2\rho^{-1} c_0 D^{\alpha k+\beta} .$$

Thus $C_{ik} = \{x \in R^d : |x - U_{ik}| \leqq \frac{1}{2}L_k\}$. Bring in a sequence $\{Y_n\}$, $n = 1, 2, \cdots$, of totally independent random variables in R^d independent of the walk $\{S_n\}$ (but defined on the same sample space) and which satisfy

$$P\{Y_n = U_{i,k}\} = 1/r_k$$

for $k = 1, 2, \cdots$, $i = 1, 2, \cdots, r_k$, $n \in (D^k, D^{k+1}]$. (When appropriate, $(a, b]$ denotes only the integers in $(a, b]$.) Let

$$\varepsilon_0 = \rho^{-1} c_0 D^\beta = \rho^{-1} c_0 2^{d\beta} .$$

We will now prove that for all $k \geqq 1$

$$\begin{aligned} P\{|S_n - Y_n| \leqq \varepsilon_0 n^\alpha \text{ for some } n \in (D^k, D^{k+1}]\} \\ \geqq (\delta_0 - \tfrac{1}{2}D^3\rho^{-d})(D-1)\rho^{-d} . \end{aligned} \tag{3.4}$$

To see this we note first that $\varepsilon_0 n^\alpha > \varepsilon_0 D^{\alpha k} = \frac{1}{2}L_k$ for $n \in (D^k, D^{k+1}]$, so if we write

$$B_n = \{|S_n - Y_n| \leqq \tfrac{1}{2}L_k\} ,$$

we have

$$\begin{aligned} P\{|S_n - Y_n| &\leqq \varepsilon_0 n^\alpha \text{ for some } n \in (D^k, D^{k+1}]\} \\ &\geqq P\{\textstyle\bigcup_{D^k < n \leqq D^{k+1}} B_n\} \\ &\geqq \textstyle\sum_{D^k < n \leqq D^{k+1}} P(B_n) - \sum_{D^k < n_1 < n_2 \leqq D^{k+1}} P(B_{n_1} B_{n_2}) . \end{aligned} \tag{3.5}$$

By definition of Y_n and by (3.1) we have

$$\begin{aligned} P(B_n) &= (1/r_k) \textstyle\sum_{i=1}^{r_k} P\{|S_n - U_{i,k}| \leqq \frac{1}{2}L_k\} \\ &\geqq (1/r_k) P(\textstyle\bigcup_{i=1}^{r_k} \{|S_n - U_{i,k}| \leqq \frac{1}{2}L_k\}) \\ &= (1/r_k) P\{|S_n| \leqq c_0 D^{\beta(k+1)}\} \\ &\geqq (1/r_k) P\{n^{-\beta}|S_n| \leqq c_0\} \\ &> \delta_0 / r_k , \end{aligned}$$

whenever $n \in (D^k, D^{k+1}]$. Hence

$$\textstyle\sum_{D^k < n \leqq D^{k+1}} P(B_n) \geqq \delta_0 (D-1)(D^k/r_k) = \delta_0(D-1)\rho^{-d} . \tag{3.6}$$

Now for any $S_n \in R^d$ at most $D = 2^d$ of the r_k inequalities

$$|S_n - U_{i,k}| \leqq \tfrac{1}{2}L_k , \qquad i = 1, 2, \cdots, r_k$$

can hold simultaneously, so, if we let N_n denote the number which occur, we obtain

$$\begin{aligned} P(B_{n_1} B_{n_2}) &= (1/r_k)^2 \textstyle\sum_{i=1}^{r_k} \sum_{j=1}^{r_k} P\{|S_{n_1} - U_{i,k}| \leqq \frac{1}{2}L_k, |S_{n_2} - U_{j,k}| \leqq \frac{1}{2}L_k\} \\ &= (1/r_k)^2 E(N_{n_1} N_{n_2}) \leqq D^2/r_k^2 \end{aligned}$$

and hence

$$\sum_{D^k<n_1<n_2\leqq D^{k+1}} P(B_{n_1}B_{n_2}) \leqq \tfrac{1}{2}D^3(D-1)\rho^{-2d} . \tag{3.7}$$

Clearly (3.5)—(3.7) give us (3.4).

From (3.3) we see immediately that if

$$\rho > (D^3/2\delta_0)^{1/d} \equiv \rho_0$$

then

$$\begin{aligned} P\{n^{-\alpha}|S_n - Y_n| \leqq \varepsilon_0 \ \text{i.o.}\} \\ &\geqq \liminf_k P\{|S_n - Y_n| \leqq \varepsilon_0 n^\alpha \ \text{for some} \ n \in (D^k, D^{k+1}]\} \\ &> 0 . \end{aligned}$$

Consequently by Kolmogorov's 0—1 law for tail events (Feller (1971), page 124)

$$P\{n^{-\alpha}|S_n - Y_n| \leqq \varepsilon_0 \ \text{i.o.}\} = 1 . \tag{3.8}$$

For any event A on the process $\{S_n, Y_n\}_{n=1}^\infty$,

$$P(A) = \int_{\{y\}\in\hat\Omega} P(A \mid \{Y_n\} = \{y_n\}) \, d\hat P(\{y_n\})$$

where $\hat\Omega \subset R^d \times R^d \times \cdots$ is the range of the process $\{Y_n\}$ and $\hat P$ is the probability induced on $\hat\Omega$ by the Law $(\{Y_n\})$. It follows from this and (3.8) that for $\hat P$-almost all sequences $\{y_n\} \in \hat\Omega$ we have

$$P\{n^{-\alpha}|S_n - Y_n| \leqq \varepsilon_0 \ \text{i.o.} \mid Y_1 = y_1, Y_2 = y_2, \cdots\} = 1 . \tag{3.9}$$

But the two processes $\{S_n\}$ and $\{Y_n\}$ are independent, so we also have for $\hat P$-almost all $\{y_n\}$

$$\text{Law}(\{S_n - Y_n\} \mid Y_1 = y_1, Y_2 = y_2, \cdots) = \text{Law}(\{S_n - y_n\}) . \tag{3.10}$$

Now let $\{y_n\}$ be a fixed sequence of constants in R^d for which both (3.9) and (3.10) hold. Then

$$P\{n^{-\alpha}|S_n - y_n| \leqq \varepsilon_0 \ \text{i.o.}\} = 1 ,$$

which is exactly (3.2). For this sequence $\{y_n\}$, note that when $n \in [D^k, D^{k+1}]$ the vector y_n, one of the r_k possible values of Y_n, is a point in the hypercube $[-c_0 D^{\beta(k+1)}, c_0 D^{\beta(k+1)}]^d$, so

$$n^{-\beta}|y_n| \leqq n^{-\beta} c_0 D^{\beta(k+1)} \leqq c_0 D^\beta = c_0 2^{d\beta} ,$$

and thus (3.3) also holds.

It is now a simple matter to complete the proof of Theorem 1. Assume that (nonrandom) y_n^ρ have been chosen to satisfy (3.2)—(3.3). There is no loss in generality if we suppose (3.2) holds for each $\rho \geqq 1$. Now (3.2) is equivalent to $\lim_{r\to\infty} P\{n^{-\alpha}|S_n - y_n^\rho| \leqq c_1/\rho$ for some $n \in (t, r]\} = 1$ for every $t \geqq 1$ where $c_1 = 2^{d\beta}c_0$. Choose inductively a sequence of integers $t_1 < t_2 < \cdots$ as follows. Choose $t_1 = 1$ and, after the integers $t_1, t_2, \cdots, t_{\rho-1}$ have been chosen, choose

$t_\rho \geqq 1 + t_{\rho-1}$ to satisfy

$$P\left\{n^{-\alpha}|S_n - y_n^\rho| \leqq \frac{c_1}{\rho} \text{ for some } n \in (t_{\rho-1}, t_\rho]\right\} \geqq 1 - \frac{1}{\rho}.$$

We now define a sequence $\{v_n\}$, which will satisfy (2.5), by

$$v_n = \sum_{k=1}^\infty y_n^k I_{(t_{k-1}, t_k]}(n) \equiv y_n^\rho \quad \text{for} \quad t_{\rho-1} < n \leqq t_\rho, \qquad \rho = 1, 2, \cdots.$$

Clearly $v_n = O(n^\beta)$ by (3.3). That (2.5) holds can be seen as follows. Let $\varepsilon > 0$ be fixed but arbitrary. For any $\rho > c_1/\varepsilon$ we have $c_1/(\rho + 1) < \varepsilon$, so

$$\begin{aligned} &P\{n^{-\alpha}|S_n - v_n| \leqq \varepsilon \text{ for some } n > t_\rho\} \\ &\geqq P\left\{n^{-\alpha}|S_n - y_n^{\rho+1}| \leqq \frac{c_1}{\rho+1} \text{ for some } n \in (t_\rho, t_{\rho+1}]\right\} \\ &\geqq 1 - \frac{1}{\rho+1}. \end{aligned}$$

Consequently

$$\begin{aligned} &P\{\liminf n^{-\alpha}|S_n - v_n| \leqq \varepsilon\} \\ &\geqq \lim_{\rho\to\infty} P\{n^{-\alpha}|S_n - v_n| \leqq \varepsilon \text{ for some } n > t_\rho\} \\ &= 1. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (2.5) follows and the proof of Theorem 1 is done.

4. Slowing down Brownian motion. Let $\{Z_t\}_{t\in[0,\infty)}$ be d-dimensional Brownian motion. By a method very similar to that used in the proof of Theorem 1 (with $\beta = \frac{1}{2}$), one may prove that for every $\alpha \in (0, \frac{1}{2})$

$$\liminf_{t\to\infty} t^{-\alpha}|Z_t - v_t| = 0 \quad \text{w.p. } 1 \tag{4.1}$$

for some deterministic function $v_t \in R^d$ which depends on α and is continuous in t. There is no need to restrict to $\alpha > \frac{1}{2} - 1/d$. The proof is sketched blow.

Since $\{Z_t\}$ with t restricted to the multiples of a given $\delta > 0$ is a random walk, it would seem that (4.1) contradicts (2.2). However, the only conclusion (2.2) allows is that for each $\delta > 0$ and $\alpha < \frac{1}{2} - 1/d$,

$$\lim_n n^{-\alpha}|Z_{n\delta} - U_n| = \infty \quad \text{w.p. } 1. \tag{4.2}$$

There are uncountably many null events, one for each $\delta > 0$, on which (4.2) fails. These null events may, indeed do, add up to a nonnull event. So there is no contradiction.

Here, briefly, is how to prove (4.1). In what follows $\delta_0, \delta_1, \varepsilon_0, c_0, c_1, c_2, \cdots$ are appropriate finite positive constants (independent of k and t); their appropriate values are to be determined by the reader. Let $P\{|Z_t| \leqq c_0 t^{\frac{1}{2}}\} \geqq \delta_0$ for all t. Let $\gamma \in (0, \frac{1}{2})$ be fixed and pick $D > 1$ so that $D^{\frac{1}{2}-\gamma}$ is an integer. Put $\Delta = D^{1-d(\frac{1}{2}-\gamma)}$ and let T_k be the set of time points $D^k + j\Delta^k$, $j = 1, 2, \cdots, q_k$, where $q_k = (D^{k+1} - D^k)/(D-1)\Delta^k = D^{d(\frac{1}{2}-\gamma)k}$. Put $T = \bigcup_{k=1}^\infty T_k$. Note that T is asymptotically dense in $[0, \infty)$ if and only if $\gamma < \frac{1}{2} - 1/d$. Partition $[-c_1 D^{k/2}$,

$c_1 D^{k/2}]^d$ into r_k subcubes with centers $u_{1,k}, \cdots, u_{r_k,k}$, and side length L_k, where $r_k = c_2 D^{d(\frac{1}{2}-\gamma)k}$ and $L_k = c_3 D^{\gamma k}$. Bring in independent vectors $\{Y_t\}_{t \in T}$, as in the proof of Theorem 1, so that $P\{Y_t = u_{i,k}\} = 1/r_k$, $i = 1, \cdots, r_k$, $t \in T_k$ $k = 1, 2, \cdots$. Then $P\{t^{-\gamma}|Z_t - Y_t| \leqq \varepsilon_0$ for some $t \in T_k\} \geqq \delta_1 > 0$, δ_1 independent of k, provided all the constants are chosen correctly. This leads, as in the proof of Theorem 1, to a deterministic sequence $\{v_t\}_{t \in T}$ so that $\liminf_{t \to \infty, t \in T} t^{-\gamma}|Z_t - v_t| < \infty$ w.p. 1 and then $\liminf_{t \to \infty, t \in T} t^{-\alpha}|Z_t - v_t| = 0$ w.p. 1 for any $\alpha > \gamma$. To get a continuous v simply interpolate v_t linearly between consecutive points of T.

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