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# THE INFINITE SECRETARY PROBLEM AS THE LIMIT OF THE FINITE PROBLEM

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In a recent paper by J. Gianini and S. M. Samuels an "infinite secretary problem" was formulated: an infinite, countable sequence of rankable individuals (rank 1 = best) arrive at times which are independent and uniformly distributed on [0, 1]. As they arrive, only their relative ranks with respect to their predecessors can be observed. Given an increasing cost function $q(\cdot)$, let $v$ be the minimum, among all stopping rules, of the mean of the function $q$ of the actual rank of the individual chosen. Let $v(n)$ be the corresponding minimum for a finite secretary problem with $n$ individuals. Then $\lim v(n) = v$.

**1. Introduction.** We consider an infinite secretary problem as defined by J. Gianini and S. M. Samuels [2]: let $U_i$ $(i = 1, 2, \cdots)$ denote the arrival time of the $i$th best of an infinite, countable sequence of rankable individuals (rank 1 = best); the $U_i$'s are taken to be independent and uniformly distributed on [0, 1]. With the same notation as in [2], for each $s, t \in [0, 1]$ such that $0 \leqq s < t \leqq 1$, we define

$$K_1(s, t) = \min\{j : s < U_j \leqq t\},$$
$$K_{i+1}(s, t) = \min\{j > K_i(s, t) : s < U_j \leqq t\}, \qquad i = 1, 2, \cdots,$$

and

$$Z_i(s, t) = U_{K_i(s,t)}, \qquad i = 1, 2, \cdots.$$

$Z_i(t) = Z_i(0, t)$ is the arrival time of the individual who is $i$th best among those who arrive by time $t$. The sequence $(Z_1(t), Z_2(t), \cdots)$ represents what we can observe up to time $t$, so we define

$$\mathscr{F}_t = \sigma\text{-field generated by } (Z_1(t), Z_2(t), \cdots).$$

$\{\mathscr{F}_t\}$ is an increasing sequence of $\sigma$-fields, and all the stopping rules for the infinite model are adapted to $\{\mathscr{F}_t\}$.

We also use the notation in [2] for the absolute and relative ranks, respectively, of "an individual arriving at time $u$:"

$$X_u = i \quad \text{if} \quad u = U_i$$
$$Y_u = i \quad \text{if} \quad u = Z_i(u)$$
$$X_1 = Y_1 = \infty.$$

Gianini and Samuels have shown in [2] that there exists a stopping rule which

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minimizes the expectation of a specified increasing function $q(\cdot)$ of the absolute rank of the individual selected, and that this optimal stopping rule is of "cutoff-point" type; there is a sequence $0 < t_1 \leqq t_2 \leqq \cdots < 1$ such that the optimal stopping rule $\tau$ selects the first individual arriving after $t_k$ and having relative rank $\leqq k$, if there is such a time and if $\tau$ did not stop before time $t_k$.

$$\tau = \min\{U_i : U_i \geqq t_k \text{ and } Y_{U_i} \leqq k\},$$
$$\tau = 1 \qquad \text{if no such } U_i .$$

The usual, finite secretary problem with $n$ individuals can be formulated in a very similar manner: the best of a sequence of $n$ individuals is equally likely to arrive first, second, $\cdots$, last, and the second best is equally likely to arrive in any of the remaining $n - 1$ positions, and so on. Furthering the analogy with the infinite model, the "arrival times" can be considered to be $\{1/n, 2/n, \cdots, 1\}$, rather than $\{1, 2, \cdots, n\}$. Here also, at each arrival, only the relative ranks of the individuals already seen, including the one under examination, can be observed, and the object is to find a stopping rule to minimize the function $q$ of the absolute rank of the individual selected. It has been shown by Mucci [3, 4] and, for the function $q(i) = i$, by Chow, Moriguti, Robbins and Samuels [1], that an optimal stopping rule exists, and is of "cutoff-point" type.

Let us denote by $v$ and $v(n)$ the minimal expected value for the infinite model and for the finite model with $n$ individuals, respectively. We shall prove here with a probabilistic argument that $\lim v(n) = v$. Chow et al. [1] and Mucci [3, 4] obtained the same result by analytical methods, and without attaching their asymptotic value to any infinite model.

**2. Two intermediate models.** One of the basic features of the infinite model considered here is the "no-recall" property: once an individual has been rejected, he cannot be called back. We shall now introduce two infinite models with a new type of selection procedure; we shall allow, to a certain extent, recalling an individual already seen and dismissed.

We divide the interval [0, 1] into $n$ equal subintervals; at the end of each subinterval, we must decide whether to stop and select the best individual who has arrived in that subinterval, or to continue observing. That decision has to be based on either the relative ranks of all the individuals who have arrived before the end of the subinterval in question (full memory), or only on the relative ranks of the best arrivals in each of the preceding subintervals (finite memory).

First, we need to introduce some new variables, to represent the arrival times and ranks of the $n$ individuals who are best among those arriving in one of the $n$ equal subintervals. $T_k$ will represent the arrival time of the best among all individuals arriving in $((k-1)/n, k/n]$, and $Q_i$ will represent the absolute rank, among all individuals, of the $i$th best among the individuals arriving at times $T_1, T_2, \cdots, T_n$.

Formally,

$$T_k = Z_1\left(\frac{k-1}{n}, \frac{k}{n}\right), \qquad k = 1, 2, \cdots, n \tag{2.1}$$

$$Q_0 \equiv 0$$

$$Q_{i+1} = \inf\{j > Q_i : U_j \in \{T_1, T_2, \cdots, T_n\}\}, \qquad i = 0, 1, \cdots, n-1. \tag{2.2}$$

We note that, since the best among the individuals arriving at times $T_1, T_2, \cdots, T_n$ has to be the best among all individuals, $Q_1 \equiv 1$.

We also define modified absolute and relative ranks; for each $k = 1, 2, \cdots, n$, $X_1(k)$ will represent the absolute rank of the individual arriving at $T_k$, among the $n$ individuals arriving at $T_1, T_2, \cdots, T_n$, while $Y_1(k)$ will represent the relative rank of the individual arriving at $T_k$, among the individuals arriving at $T_1, \cdots, T_k$.

So, for $k = 1, 2, \cdots, n$,

$$X_1(k) = i \quad \text{if} \quad Z_1\left(\frac{k-1}{n}, \frac{k}{n}\right) = U_{Q_i}, \qquad i = 1, 2, \cdots, n \tag{2.3}$$

$$Y_1(k) = j \quad \begin{array}{l}\text{if for some } i \in \{j, \cdots, n\},\ T_k = U_{Q_i} \text{ and } j-1 \\ \text{exactly of } U_{Q_1}, \cdots, U_{Q_{i-1}} \text{ are in } (0, k-1)/n], \\ j = 1, 2, \cdots, k.\end{array} \tag{2.4}$$

$X_1(k)$ and $Y_1(k)$ are ranks defined on the finite sequence of individuals arriving at $T_1, \cdots, T_n$. Now, we must define the absolute rank, $X_2(k)$, of the individual arriving at $T_k$, among all individuals, and the relative rank, $Y_2(k)$, of the individual arriving at $T_k$, among all individuals who arrive in $(0, k/n]$.

For $k = 1, 2, \cdots, n$,

$$X_2(k) = i \quad \text{if} \quad Z_1\left(\frac{k-1}{n}, \frac{k}{n}\right) = U_i, \qquad i = 1, 2, \cdots \tag{2.5}$$

$$Y_2(k) = j \quad \text{if} \quad Z_1\left(\frac{k-1}{n}, \frac{k}{n}\right) = Z_j\left(\frac{k}{n}\right), \qquad j = 1, 2, \cdots. \tag{2.6}$$

We note that, by definitions (2.2) and (2.3) of $Q_i$ and $X_1(k)$,

$$X_2(k) = Q_{X_1(k)}. \tag{2.7}$$

We shall now introduce the two models which will serve as intermediates between the finite and infinite models.

(i) *Finite memory model.* In this model, the decision whether to select the individual arriving at $T_k$, or to continue observing, is based on the values of $Y_1(1), \cdots, Y_1(k)$ alone; let

$$\mathscr{C}_1 = \{\tau : \{\tau = k\} \in \mathscr{B}(Y_1(1), \cdots, Y_1(k)), k = 1, 2, \cdots, n\}, \tag{2.8}$$

where, by "$\tau = k$," we mean "select the individual arriving at $T_k$." Then we define

$$v_1(n) = \inf_{\tau \in \mathscr{C}_1} \{E[q(X_2(\tau))]\} . \tag{2.9}$$

$v_1(n)$ is the value for the "infinite model with finite memory and partial recall."

By (2.7), we can also write

$$v_1(n) = \inf_{\tau \in \mathscr{C}_1} \{E[q(Q_{X_1(\tau)})]\}$$

and thus, this model reduces to a finite model where the absolute ranks, instead of the integers $\{1, 2, \cdots, n\}$, are $n$ random variables $\{Q_1, Q_2, \cdots, Q_n\}$. These random variables have the following properties:

$$\begin{aligned} &\text{(a)} \quad Q_i \geqq i \quad \text{with probability } 1; \\ &\text{(b)} \quad Q_i \quad \text{is independent of} \quad (X_1(1), \cdots, X_1(n), Y_1(1), \cdots, Y_1(n)); \\ &\text{(c)} \quad P[Q_{i+1} - Q_i = k] = (1 - i/n)(i/n)^{k-1} . \end{aligned} \tag{2.10}$$

(a) follows immediately from definition (2.2).

(b) is proved by induction on $i$: we first note that $Q_1 = 1$ with probability 1, because the best individual of all will necessarily be best in the interval in which he arrives, so (b) holds for $i = 1$. Let us assume (b) holds for some $i \geqq 1$; then, on $\{Q_i = j\}$,

$$\begin{aligned} I\{Q_{i+1} = k\} = I\{&U_{j+1}, \cdots, \text{ and } U_{k-1} \text{ are in one of the } i \text{ subintervals} \\ &\text{already occupied by } U_1, \cdots, \text{ or } U_j, \text{ while } U_k \\ &\text{is not in one of those subintervals}\} . \end{aligned} \tag{2.11}$$

This does not depend on the order of arrival of the individuals concerned;

$$\begin{aligned} &P[Q_{i+1} = k \mid X_1(1), \cdots, X_1(n)] \\ &\quad = \textstyle\sum_{j=i}^{k-1} P[Q_{i+1} = k \mid I\{Q_i = j\}, X_1(1), \cdots, X_1(n)] P[Q_i = j \mid X_1(1), \cdots, X_1(n)] \\ &\quad = \textstyle\sum_{j=i}^{k-1} P[Q_{i+1} = k \mid Q_i = j] P[Q_i = j] = P[Q_{i+1} = k] ; \end{aligned}$$

thus, $Q_{i+1}$ is independent of $X_1(1), \cdots, X_1(n)$, and since $Y_1(1), \cdots, Y_1(n)$ are functions of $(X_1(1), \cdots, X_1(n))$, $Q_{i+1}$ is also independent of $Y_1(1), \cdots, Y_1(n)$.

(c)

$$P[Q_{i+1} - Q_i = k] = \textstyle\sum_{j=i}^{\infty} P[Q_{i+1} - Q_i = k \mid Q_i = j] P[Q_i = j] .$$

Now, by (2.11),

$$P[Q_{i+1} - Q_i = k \mid Q_i = j] = P[Q_{i+1} = j + k \mid Q_i = j] = (1 - i/n)(i/n)^{k-1} ;$$

and finally,

$$P[Q_{i+1} - Q_i = k] = \textstyle\sum_{j=1}^{\infty} (1 - i/n)(i/n)^{k-1} P[Q_i = j] = (1 - i/n)(i/n)^{k-1} . \qquad \square$$

Now, we are ready to prove

LEMMA 1. *$v_1(n)$, the value for the "random ranks" model, is equal to the value for a finite model with $n$ individuals, and with cost function $q_1(i) = E[q(Q_i)]$.*

Indeed, if the same stopping rule is used for both models, it gives the same expected cost.

PROOF. Recalling (2.10b),

$$\begin{aligned} E[q(Q_{X_1(\tau)})] &= E[E(q(Q_{X_1(\tau)} \mid Q_1, Q_2, \cdots, Q_n)] \\ &= E[\textstyle\sum_{j=1}^n q(Q_j)P[X_1(\tau) = j \mid Q_1, Q_2, \cdots, Q_n]] \\ &= \textstyle\sum_{j=1}^n E[q(Q_j)]P[X_1(\tau) = j] \\ &= \textstyle\sum_{j=1}^n q_1(j)P[X_1(\tau) = j] \\ &= E[q_1(X_1(\tau))] . \end{aligned} \qquad \square$$

It follows in particular that for the infinite model with finite memory and partial recall, there is an optimal stopping rule $\tau_1$, and that $\tau_1$ is of "cutoff-point" type.

REMARKS. 1. If $q$ is bounded, so is $q_1$. Let $L = \sup_i q_1(i)$. Let $\tau$ be defined by

$$\begin{aligned} \tau &= k \qquad \text{if} \quad k = \inf\{j > [n/2], Y_1(j) = 1\} \\ \tau &= n \qquad \text{if no such} \quad j \quad \text{exists} . \end{aligned}$$

It is easy to see that, if $n$ is even,

$$v_1(n) \leqq E[q_1(X_1(\tau))] \leqq q(1)/4 + 3L/4 < L .$$

2. We know by (2.10a) that $Q_i \geqq i$; it follows that $q(i) \leqq q_1(i)$ for $i = 1, 2, \cdots$ and that

$$v(n) \leqq v_1(n) . \tag{2.12}$$

(ii) *Full memory model.* This model is also a finite model, since we must only make $n$ decisions, but the decision concerning the individual arriving at $T_k$ is based on the relative ranks of all the individuals who have arrived before time $k/n$: let

$$\mathscr{C}_2 = \{\tau : \{\tau = k\} \in \mathscr{F}_{k/n}\} , \tag{2.13}$$

where by "$\tau = k$" we mean "select the individual arriving at $T_k$."

Now, we define:

$$v_2(n, i-1) = \inf_{\tau \in \mathscr{C}_2, \tau \geqq i} \{E[q(X_2(\tau))]\} ,$$

and

$$v_2(n) = v_2(n, 0) = \inf_{\tau \in \mathscr{C}_2} \{E[q(X_2(\tau))]\} . \tag{2.14}$$

By Proposition 3.1 in Gianini and Samuels [2],

$$E[q(X_2(k)) \mid \mathscr{F}_{k/n}] = R_j(k/n) \qquad \text{on} \quad \{Y_2(k) = j\} , \tag{2.15}$$

where $R_j(\cdot)$ is defined in (3.1), in [2] as:

$$R_j(t) = \textstyle\sum_{k=i}^{\infty} \binom{k-1}{i-1} q(k) t^i (1-t)^{k-i} , \quad t \in (0, 1] ,$$

and by Proposition 2.1 in Gianini and Samuels [2], $Y_2(k)$ is independent of

$\mathscr{F}_{(k-1)/n}$, and hence, the $Y_2(k)$ $(k = 1, 2, \cdots, n)$ are independent. Moreover,

$$P[Y_2(k) = j \mid \mathscr{F}_{(k-1)/n}] = P[Y_2(k) = j] = \frac{1}{k}\left(\frac{k-1}{k}\right)^{j-1}. \tag{2.16}$$

From (2.15), it is easy to see that the optimal rules are based only on the $Y_2(k)$'s, that is, there is an optimal rule $\tau$ such that $\{\tau \leqq k/n\} \in \mathscr{B}\{Y_2(i) : i \leqq k\}$. Hence,

$$v_2(n, k) = \operatorname{ess\,inf}_{\tau > k/n} E[q(X_2(\tau)) \mid \mathscr{F}_{k/n}] = \text{constant},$$

and we have the recursive equations

$$v_2(n, n-1) = E[q(X_2(n))] = \textstyle\sum_{j=1}^{\infty} q(j) \dfrac{1}{n}\left(\dfrac{n-1}{n}\right)^{j-1},$$

and, by (2.15) and (2.16),

$$v_2(n, k-1) = \textstyle\sum_{j=1}^{\infty} \dfrac{1}{k}\left(\dfrac{k-1}{k}\right)^{j-1} \min\left(R_j\left(\dfrac{k}{n}\right), v_2(n, k)\right).$$

This shows immediately that there is an optimal policy $\tau_2$, which selects the individual arriving at $T_k$ if his relative rank $j$ is such that $R_j(k/n) \leqq v_2(n, k)$, and no selection has been made previously. This optimal stopping rule is of "cutoff-point" type:

$$\begin{aligned} &\tau_2 \cong (k_1, k_2, \cdots, k_n), \qquad \text{with} \quad k_i = \min\{k : R_i(k/n) \leqq v_2(n, k)\} \\ &\tau_2 = r \qquad \text{if} \quad r = \inf\{j : j \geqq k_i \text{ and } Y_2(j) \leqq i\}. \end{aligned} \tag{2.17}$$

We also note that

$$v_2(n) = R_1(k_1/n). \tag{2.18}$$

REMARKS. 1. If the individual arriving at $T_k$ has relative rank $j \geqq M$ and if the function $q$ is truncated at $q(M)$, then the expected cost for selecting that individual is $R_j(k/n) \geqq q(j) = q(M)$. An optimal policy for this model will not select such an individual.

2. If $\tau$ is a policy for the infinite model, then

$$\tau_n' = \textstyle\sum_{k=1}^{n} kI\{(k-1)/n \leqq \tau < k/n\}$$

is a policy for the full memory model with partial recall. By the definition (2.1) of $T_k$ it is clear that $X_2(\tau_n') \leqq X_\tau$ and this implies that

$$v_2(n) \leqq v. \tag{2.19}$$

3. Since $\mathscr{C}_1 \subset \mathscr{C}_2$ (see (2.8) and (2.13)), by definitions (2.9) and (2.14) of $v_1(n)$ and $v_2(n)$, it follows that

$$v_2(n) \leqq v_1(n). \tag{2.20}$$

**3. Asymptotic results.** We have proved (see (2.12), (2.19) and (2.20)) some inequalities relating $v(n)$, $v_1(n)$, $v_2(n)$ and $v$. Now, we shall prove:

LEMMA 2. $v(n) \leqq v$.

PROOF. Let us consider an infinite model, and let us define

$$\xi(t) = \#\{i : 1 \leqq i \leqq n, U_i \leqq t\} .$$

We consider the $\sigma$-algebra

$$\mathscr{H}_t = \mathscr{F}_t \vee \mathscr{B}(\xi(t)) ,$$

and we let $v'(n)$ be the value of the infinite model if we allow strategies adapted to $\mathscr{H}_t$: $\mathscr{H}_t$ contains $\mathscr{F}_t$ and hence, $v'(n) \leqq v$. But the arrival times $U_i$ ($i = 1, 2, \cdots, n$) of the $n$ best individuals, are measurable with respect to the corresponding $\mathscr{H}_{U_i}$'s, so the optimal rule here is simply the optimal rule for the finite model with $n$ individuals. Thus,

$$v'(n) = v(n) \leqq v . \qquad \square$$

We shall now restrict ourselves to functions $q$ which are truncated at $q(M)$: $q(k) = q(M)$, $k \geqq M$. For any such function, we shall show that the value $v(n)$ for the finite model approaches, as $n$ tends to infinity, the value $v$ for the infinite model. To do this, we shall use the intermediate values $v_1(n)$ and $v_2(n)$. Then, we shall extend the result to any increasing function $q$.

LEMMA 3. *For any truncated function $q$,*

$$v_2(n) \to v , \quad n \to \infty .$$

PROOF. Let us consider the optimal policy $\tau_2$, defined in (2.17), for the full memory model with partial recall. We can adapt $\tau_2$ to the infinite model. Let

$$\theta(t) = i \qquad \text{if} \quad k_i \leqq tn < k_{i+1}$$
$$\tau = \inf\{t : Y_t \leqq \theta(t)\} .$$

It is clear that $\tau_2 = r$ iff $\tau \in ((r-1)/n, r/n]$. We want to show that for any $\varepsilon > 0$ and $n$ sufficiently large,

$$E[q(X_\tau)] \leqq E[q(X_2(\tau_2))] + \varepsilon . \tag{3.1}$$

Then we shall have

$$v \leqq E[q(X_\tau)] \leqq E[q(X_2(\tau_2))] + \varepsilon = v_2(n) + \varepsilon .$$

Since by (2.19), $v_2(n) \leqq v$, this implies that $v_2(n) \to v$, $n \to \infty$ and proves the lemma.

Let us prove (3.1). Since $\tau$ and $\tau_2$ stop in the same subinterval $((r-1)/n, r/n]$, and since $k_1$ is such that $P[\tau_2 \geqq k_1] = 1$,

$$\begin{aligned} E[q(X_\tau)] - E[q(X_2(\tau_2))] &\leqq q(M)P[\tau \neq \tau_2] \\ &= q(M) \textstyle\sum_{r=k_1+1}^{n} P[\tau \neq \tau_2, \tau_2 \in ((r-1)/n, r/n]] \\ &\leqq q(M) \textstyle\sum_{r=k_1+1}^{n} P(A_r) , \end{aligned}$$

where $A_r =$ "at least two of the $M$ best arrivals in $(0, r/n]$ occur in $((r-1)/n, r/n]$;"

but as $P(A_r) \leqq \frac{1}{2}M(M-1)r^{-2}$,

$$E[q(X_\tau)] - E[q(X_2(\tau_2))] \leqq q(M)\frac{M(M-1)}{2}\sum_{r=k_1+1}^{n}\frac{1}{r^2},$$

and, to prove (3.1), we only need to prove that $k_1 = k_1(n) \to \infty$, $n \to \infty$. This is easy, because by (2.18) and (2.20),

$$R_1(k_1/n) = v_2(n) \leqq v_1(n)$$

but we have seen that $v_1(n) < q(M)$ while, for any fixed $k$,

$$R_1(k/n) = \sum_{j=1}^{\infty} q(j)(j-1)(k/n)(1-k/n)^{j-1} \to q(M), \quad n \to \infty,$$

and $k_1$ bounded would yield a contradiction. Thus (3.1) holds, and the lemma is proved. □

LEMMA 4. *For any truncated function* $q$,

$$v(n) - v_1(n) \to 0, \quad n \to \infty.$$

PROOF. We have seen in (2.12) that $v(n) \leqq v_1(n)$. Let $\tau(n)$ be an optimal stopping rule for the finite model; $\tau(n)$ is also a stopping rule for the infinite model with finite memory and partial recall; we have

$$q(X_1(\tau(n))) = q(X_2(\tau(n))) = q(Q_{X_1(\tau(n))})$$

if either of the following occur:

(i) $X_1(\tau(n)) \geqq M$, because then $X_2(\tau(n)) \geqq M$, and

$$q(X_1(\tau(n))) = q(X_2(\tau(n))) = q(M);$$

(ii) $X_1(\tau(n)) < M$ and $Q_{X_1(\tau(n))} = X_1(\tau(n))$.

This means that $q(X_1(\tau(n))) \neq q(X_2(\tau(n)))$ implies that $Q_i > i$ for some $i \in \{1, 2, \cdots, M-1\}$. Hence, by (2.10)c,

$$\begin{aligned} E[q(X_2(\tau(n)))] - E[q(X_1(\tau(n)))] &\leqq q(M)P[Q_i > i \text{ for some } i \in \{1, 2, \cdots, M-1\}] \\ &= q(M)(1 - \textstyle\prod_{j=1}^{M-1}(1-j/n)); \end{aligned}$$

and, since $E[q(X_1(\tau(n)))] = v(n)$ and $E[q(X_2(\tau(n)))] \geqq v_1(n)$, this implies

$$v_1(n) - v(n) \leqq q(M)(1 - \textstyle\prod_{j=1}^{M-1}(1-j/n));$$

the right-hand side tends to 0 as $n$ tends to $\infty$, and this, together with (2.12), proves the lemma. □

THEOREM. *For any function* $q$,

$$v(n) \to v, \quad n \to \infty.$$

PROOF. (a) If $q$ is truncated at $q(M)$, by Lemma 2, we have $v(n) \leqq v$; moreover, from (2.20), it follows that

$$v - v(n) = v - v_2(n) + v_2(n) - v(n) \leqq (v - v_2(n)) + (v_1(n) - v(n)),$$

and the right-hand side tends to 0 as $n$ tends to $\infty$, by Lemmas 3 and 4. Thus the theorem is proved if $q$ is truncated.

(b) If $q$ is not truncated, then by (2.20) we have

$$\limsup v(n) \leqq v\,. \tag{3.2}$$

Let $q_M$ be the function $q$, truncated at $q(M)$;

$$\begin{aligned} q_M(k) &= q(k) \qquad \text{if} \quad k < M \\ &= q(M) \qquad \text{if} \quad k \geqq M\,. \end{aligned}$$

Let $v_M(n)$ and $v_M$ be the minimal expected costs corresponding to the function $q_M$ in the finite and infinite models, respectively. Then, as $q_M(k) \leqq q(k)$ for all $k$, $M$,

$$v_M(n) \leqq v(n) \qquad \text{for all} \quad n, M\,.$$

The theorem holds for the truncated functions $q_M$:

$$v_M = \lim_n v_M(n) \leqq \liminf v(n)\,,$$

and since, by Proposition (5.5) in Gianini and Samuels [2], $v_M \to v$, $M \to \infty$, we have

$$v = \lim_M v_M \leqq \liminf v(n)\,.$$

This, and (3.2), prove the theorem. □

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