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RECURRENCE FOR PRODUCTS OF RENEWAL SEQUENCES[1]

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If $(u_n)_{n=0}^{\infty}$ is a null-recurrent renewal sequence, we prove that there exist two null-recurrent renewal sequences $(v_n)_{n=0}^{\infty}$ and $(w_n)_{n=0}^{\infty}$ such that $(u_n v_n)_{n=0}^{\infty}$ is null-recurrent and $(u_n w_n)_{n=0}^{\infty}$ is transient.

1. Introduction. A renewal sequence $u = (u_n)_{n=0}^{\infty}$ is a sequence of numbers such that there exists a sequence $(f_n(u))_{n=1}^{\infty}$ such that $f_n(u) \geqq 0$ for all $n = 1, 2, \cdots$, $\sum_{n=1}^{\infty} f_n(u) \leqq 1$ and $\sum_{n=0}^{\infty} u_n z^n = (1 - \sum_{n=1}^{\infty} f_n(u) z^n)^{-1}$ if $|z| < 1$. The renewal sequence u will be said *transient* if $\sum_{n=0}^{\infty} u_n < \infty$, *null-recurrent* if $\sum_{n=0}^{\infty} u_n = \infty$ and $\lim_{n\to\infty} u_n = 0$, and *positive-recurrent* if $\limsup_{n\to\infty} u_n > 0$. We refer to Chapter 1 of [2] for the probabilistic background of such sequences and the proofs of various statements that we shall now recall.

It is well known that if $u = (u_n)_{n=0}^{\infty}$ and $v = (v_n)_{n=0}^{\infty}$ are renewal sequences, then $uv = (u_n v_n)_{n=0}^{\infty}$ is a renewal sequence.

If u is transient, since $0 \leqq v_n \leqq 1$ for all n, of course uv is transient. If u is positive-recurrent, it is easy to check that uv has the same type (transience, null-recurrence or positive-recurrence) of v itself. In fact, if we write

$$d(u) = \text{G.C.D.}\,\{n;\, u_n > 0\}\,,$$

the renewal theorem (see [2]) states that

$$\lim_{k\to\infty} u_{kd(u)} = d(u)\,[\sum_{k=1}^{\infty} k f_k(u)]^{-1} > 0$$

when u is positive-recurrent. Hence, if v is positive-recurrent, uv is clearly positive-recurrent. If $d(u) = 1$ and if v is null-recurrent it is also clear that uv is null-recurrent. If $d(u) > 1$ and if v is null-recurrent, it is necessary to use the probabilistic interpretation of renewal sequences to prove that uv is null-recurrent, but this is easy to do.

The situation is completely different if u and v are both null-recurrent. Consider for instance the most famous renewal sequence $w = (w_n)_{n=0}^{\infty}$, with

$$w_n = \binom{2n}{n} 2^{-2n}\,.$$

We have $w_n \sim (\pi n)^{-\frac{1}{2}}$. Hence, if $u = v = w$, then u, v and $uv = w^2$ are null-recurrent, but if $u = w^2$ and $v = w$, then $uv = w^3$ is transient.

We shall prove

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THEOREM 1. *If u is a null-recurrent renewal sequence there exists a null-recurrent renewal sequence v such that uv is null-recurrent.*

THEOREM 2. *If u is a null-recurrent renewal sequence, there exists a null-recurrent renewal sequence w such that uw is transient.*

Let us mention that there exist null-recurrent renewal sequences u such that u^k is another such for all positive integers k. I am indebted to Professor J. F. C. Kingman who has shown me the simple example

$$u_n = [1 + \log(n+1)]^{-1} .$$

The question of infinite products of Markov chains and renewal sequences is developed in [3].

2. Kaluza and moment sequences. To prove our two existence theorems, we need to describe a rich class of renewal sequences, namely *moment sequences.*

Call a *Kaluza sequence* a sequence $u = (u_n)_{n=0}^{\infty}$ of numbers such that $0 \leqq u_n \leqq 1$ and $u_{n+1} \leqq (u_n u_{n+2})^{\frac{1}{2}}$ for $n = 0, 1, 2, \cdots$. It was proved in [1] that $u^t = (u_n{}^t)_{n=0}^{\infty}$ is a renewal sequence for all *real* positive t if and only if u is a Kaluza sequence.

Call a *moment sequence* a sequence $u = (u_n)_{n=0}^{\infty}$ of numbers such that there exists a probability measure ν on $[0, 1]$ with

$$u_n = \int_0^1 x^n \nu(dx) .$$

The Schwarz inequality implies that a moment sequence is a Kaluza sequence, but I do not know any natural probabilistic interpretation of u, as a renewal sequence, in terms of ν.

A great many classical renewal sequences are moment sequences; for instance, if $\alpha > 0$, $s(\alpha) = ((n+1)^{-\alpha})_{n=0}^{\infty}$ is a moment sequence:

$$(n+1)^{-\alpha} = \int_0^1 x^n \left[\log \frac{1}{x}\right]^{\alpha-1} \frac{dx}{x\Gamma(\alpha)} .$$

Another example is

$$\binom{2n}{n} \frac{1}{2^{2n}} = \int_0^1 (\cos \pi t)^{2n}\, dt = \int_0^1 x^n \nu(dx) ,$$

where $\nu(dx)$ is the measure on $[0, 1]$ carried from Lebesgue measure on $[0, 1]$ by the map $t \mapsto \cos^2 \pi t$. (This answers a question raised on page 19 of [2].)

The last example of a moment sequence is $[1 + \log(n+1)]^{-1}$; it is enough to check that for $s > 0$,

$$[1 + \log(s+1)]^{-1} = \int_0^{+\infty} e^{-sy} \mu(dy) ,$$

where μ is the distribution of $X(T)$, where $(X(t))_{t\geqq 0}$ is the Lévy process with distribution $1_{(0,\infty)}(x) e^{-x}(x^{t-1}/\Gamma(t))\, dx$ and T a random variable independent of $(X(t))_{t\geqq 0}$, such that $P[T > t] = e^{-t}$ for $t \geqq 0$.

3. Proof of Theorem 1. Denote l^1 the Banach space of sequences of real numbers $x = (x_n)_{n=0}^{\infty}$ such that $||x||_1 = \sum_{n=1}^{\infty} |x_n| < \infty$, and c_0 the Banach space of sequences of real numbers $y = (y_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} y_n = 0$, with norm $||y||_\infty = \sup_n |y_n|$. The linear subspace F of c_0 generated by the null-recurrent renewal sequences $s(\alpha) = ((n+1)^{-\alpha})_{n=0}^{\infty}$ where $0 < \alpha \leqq 1$ is dense in c_0, because if it were not true, there would exist x in l^1, different from 0, such that the analytic function in α defined by the sum of the Dirichlet series

$$\sum_{n=0}^{\infty} \frac{x_n}{(n+1)^\alpha}$$

would be equal to zero for all α in $(0, 1]$ which implies $x_n = 0$ for all $n = 0, 1, 2, \cdots$.

Now, if for all null-recurrent renewal sequences $v = (v_n)_{n=0}^{\infty}$, we have $\sum_{n=0}^{\infty} u_n v_n < \infty$, this implies that $\sum_{n=0}^{\infty} |u_n y_n| < \infty$ for all y in F. Since F is dense in c_0, by the Banach-Steinhaus theorem, this implies that u belongs to l^1 and u is transient, a contradiction.

REMARK. Instead of using the Baire category theorem when we apply Banach–Steinhaus, we could give a constructive proof of the existence of v; v would be a linear combination with positive coefficients of some $s(\alpha)$ and hence a moment sequence.

4. Proof of Theorem 2. Write $U(x) = \sum_{n=0}^{\infty} u_n x^n$, where $0 \leqq x < 1$. Since u is a null-recurrent renewal sequence $\lim_{x\to 1^-} U(x) = \infty$ and $\lim_{x\to 1^-} (1-x)U(x) = 0$. We introduce the increasing function $G(x)$ on $[0, 1)$

$$G(x) = [\sup_{x \leqq t < 1} (1-t)U(t)]^{-\frac{1}{2}},$$

and consider the positive unbounded measure $dG(x)$ on $[0, 1)$. We have $\int_0^1 dG(x) = \infty$ and

$$\int_0^1 (1-x)U(x)\, dG(x) \leqq \int_0^1 \frac{dG(x)}{G^2(x)} = 1 .$$

Since $\lim_{x\to 1^-} U(x) = \infty$, this implies that

$$A = \int_0^1 (1-x)\, dG(x) < \infty .$$

We define now the probability measure ν on $[0, 1]$ by

$$\nu(dx) = \frac{1}{A}(1-x)\, dG(x),$$

and the moment sequence w by

$$w_n = \int_0^1 x^n \nu(dx) .$$

Then, since ν has no atom at $\{1\}$, $\lim_{n\to\infty} w_n = 0$. Furthermore

$$\sum_{n=0}^{\infty} w_n = \frac{1}{A} \int_0^1 (1-x)^{-1}(1-x)\, dG(x) = \infty .$$

Hence w is a null-recurrent renewal sequence. Now:

$$\sum_{n=0}^{\infty} u_n w_n = \frac{1}{A} \int_0^1 U(x)(1 - x)\, dG(x) < \infty ,$$

and uw is transient.

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