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APPLICATIONS OF DUALITY TO A CLASS OF MARKOV PROCESSES[1]

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Let S be a countable set and let ξ_t be a Markov process on the subsets of S. Harris has given criteria for the existence of a dual process $\xi_t{}^*$ on the finite subsets of S. By extending Harris's notion of duality the class of ξ_t which have dual processes is enlarged. The dual processes are then used to study the ergodic behavior of ξ_t. Also treated is a class of ξ_t which have growing dual processes.

1. Introduction. In recent work the study of the ergodic behavior of certain infinite particle systems has been carried out by studying corresponding finite particle systems (e.g., Liggett (1973), (1974), Spitzer (1974), Holley and Liggett (1975), Harris (1976), Holley and Stroock (1976), Griffeath (1976)). The situation can be described as follows. Given countable sets S and Y, let $X = \{0, 1\}^S$ with the product topology, let $C(X)$ be the continuous functions on X with the sup norm and for each $A \in Y$ let F_A be a real valued bounded continuous function defined on X. Then given a Markov process ξ_t on X, does there exist a continuous time Markov process A_t on Y satisfying

$$E_\xi[F_A(\xi_t)] = E_A{}^*[F_{A_t}(\xi)] \tag{1.1}$$

for all $\xi \in X$, $A \in Y$, $t \geqq 0$? Here we use $E(E^*)$ to denote expectation with respect to $\xi_t(A_t)$ and refer to A_t as the dual of ξ_t. If $\{F_A(\cdot)\}_{A \in Y}$ is a determining class of functions for the process ξ_t then A_t can be used to study ξ_t via (1.1).

We always assume that ξ_t is a strong Markov process corresponding to a strongly continuous semigroup of contractions on $C(X)$ and further assume that ξ_t has right continuous paths with left limits. Let $\mathscr{A}$ be the infinitesimal generator of ξ_t and assume that the infinitesimal jump parameters of A_t are $q(A, B)$. Then we can attempt to solve for $q(A, B)$ by taking the derivative with respect to t of both sides of (1.1) to get for each $\xi \in X$ and $A \in Y$

$$\mathscr{A}F_A(\xi) = \sum_{B \in Y} q(A, B)[F_B(\xi) - F_A(\xi)] . \tag{1.2}$$

If the resulting $q(A, B)$ are infinitesimal jump parameters for a nonexplosive Markov chain on Y then $E_A{}^*[F_{A_t}(\xi)]$ and $E_\xi[F_A(\xi_t)]$ are bounded solutions of

$$\frac{du(t, A)}{dt} = \mathscr{A}u(t, A)$$

with initial condition $u(0, A) = F_A(\xi)$. Since the bounded solutions to this

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differential equation are unique up to given initial conditions it follows that (1.1) holds.

In order to further specify the problem we now fix the definitions of Y and F_A. Let $Y = \{A \subset S \mid A \text{ is finite}\}$. Identify each $\xi \in X$ with the subset $\{x \in S \mid \xi(x) = 1\}$ and let $F_A(\xi) = 1$ if $\xi \cap A \neq \emptyset$ and 0 otherwise. With these definitions of Y and F_A, Harris (1976) develops a formula for $q(A, B)$ in the sense of uniquely solving (1.2) subject to the conditions $|S| < \infty$, $\mathscr{A}F_A(\emptyset) = 0$ and $\sum_{B \in Y} q(A, B) = 0$ for all $A \in Y$. The formula is $q(A, B) = \psi\mathscr{A}F_A(B)$ where

$$\psi f(\eta) = \sum_{\xi \subset \eta} (-1)^{|\xi|+1} f(\eta^c \cup \xi) .$$

(1.3) THEOREM (Harris). *Let S be finite. Then ξ_t has a dual process A_t on Y satisfying* (1.1) *if and only if $\psi\mathscr{A}F_A(B) \geqq 0$ for all $A \neq B$ and $\mathscr{A}F_A(\emptyset) = 0$ for all A. The infinitesimal parameters of A_t are given by $q(A, B) = \psi\mathscr{A}F_A(B)$ for $A \neq B$.*

The condition $\mathscr{A}F_A(\emptyset) = 0$ in (1.3) implies that the empty set is an absorbing state for ξ_t. That the empty set must be absorbing if ξ_t has a dual process A_t satisfying (1.1) follows from (1.1) and the definition of $F_A(\xi)$. In order to eliminate the condition $\mathscr{A}F_A(\emptyset) = 0$ from Theorem (1.3) we let $Y_\Delta = Y \cup \{\Delta\}$, define $F_\Delta(\xi) = 1$ for all $\xi \in X$ and ask for a Markov chain A_t on Y_Δ satisfying

$$E_\xi[F_A(\xi_t)] = E_A{}^*[F_{A_t}(\xi)] \qquad \xi \in X,\ A \in Y_\Delta . \tag{1.4}$$

We note that $E_\emptyset[F_A(\xi_t)] = P_A{}^*[A_t = \Delta]$, and hence the empty set is not necessarily absorbing for ξ_t. On the other hand the empty set is absorbing for A_t as is Δ.

(1.5) THEOREM. *Let S be finite. Then ξ_t has a dual process A_t on Y_Δ satisfying* (1.4) *if and only if $\psi\mathscr{A}F_A(B) \geqq 0$ for all $A \neq B$, $A, B \in Y$. The infinitesimal parameters of A_t are given by*

$$q(A, B) = \psi\mathscr{A}F_A(B) \qquad A \neq B,\ A, B \in Y ,$$

$$q(A, \Delta) = \mathscr{A}F_A(\emptyset) \qquad A \in Y ,$$

$$q(\Delta, B) = 0 \qquad B \in Y_\Delta .$$

The proof of Theorem (1.5) will be given in Section 2 along with an extension to the case where S is infinite. Applications of these results to the ergodic theory of certain infinite particle processes are also given. In Section 3 we show how to relax the hypothesis in Theorem (1.5) that $\psi\mathscr{A}F_A(B) \geqq 0$ for all $A \neq B$.

In Sections 4 and 5 we illustrate a technique that can be used to prove ergodic theorems for certain processes ξ_t on X which have increasing dual processes A_t on Y, that is, dual process A_t for which $P_A{}^*[\lim_{t\to\infty} \sup |A_t| = \infty] > 0$ where $|A|$ denotes the number of elements in A. Most previous applications of duality treat those processes with duals satisfying $P_A{}^*[|A_t| \leqq |A| \text{ for all } t] = 1$ or $P_A{}^*[A_t = \emptyset \text{ for some } t] = 1$. (An exception is found in Harris (1976), Section 9.)

In order to state our results we need the following definitions. For each probability measure μ on X we let $T(t)\mu$ be the probability measure on X defined on cylinder sets by

$$(T(t)\mu)\{\xi \mid \xi(x_i) = 1, i = 1, \cdots, n\} = \int_X P_\xi[\xi_t(x_i) = 1, i = 1, \cdots, n]\, d\mu(\xi) .$$

Then $T(t)\mu$ represents the state of the process ξ_t at time t when the initial state is μ. A measure ν is *invariant* for ξ_t if $T(t)\nu = \nu$ for all $t \geqq 0$ and we will say that ξ_t is *ergodic* if there exists a measure ν (necessarily invariant) such that $T(t)\mu$ converges weakly to ν as $t \to \infty$ for all initial states μ. Denote by $\delta_\emptyset$ and δ_S the probability measures on X which concentrate on $\xi \equiv 0$ and $\xi \equiv 1$ respectively. Considering $\xi \in X$ as a subset of S we will write $\xi \cup \{x\}$ as $\xi \cup x$ and $\xi \backslash \{x\}$ as $\xi \backslash x$. A Markov process A_t on Y will be called *monotone* if there exists a Markov process (A_t^1, A_t^2) on $\{(A, B) \in Y \times Y \mid A \subset B\}$ such that the marginal processes each have the same finite distributions as A_t, that is, A_t^1 and A_t^2 can be coupled so that $A_t^1 \subset A_t^2$ for all $t \geqq 0$ whenever $A_0^1 \subset A_0^2$ and so that A_t^1 and A_t^2 have the same finite distributions as A_t.

(1.6) THEOREM. *Let ξ_t be a Markov process on X with a dual process A_t on Y satisfying* (1.1). *Suppose that A_t is monotone and that for each $B \in Y$, $A \in Y$, $A \neq \emptyset$*

$$P_A^*[B \subset A_t \text{ for some } t \geqq 0] = 1 , \tag{1.7}$$

then any invariant measure for ξ_t is of the form $\lambda\delta_\emptyset + (1 - \lambda)\delta_S$ for some $0 \leqq \lambda \leqq 1$. *Suppose that for each $B \in Y$, $A \in Y$, $A \neq \emptyset$, $B \neq \emptyset$*

$$\lim_{t\to\infty} P_A^*[A_t \cap B \neq \emptyset] = 1 , \tag{1.8}$$

then $\lim_{t\to\infty} T(t)\mu = \mu(\emptyset)\delta_\emptyset + (1 - \mu(\emptyset))\delta_S$ *for every initial distribution μ.*

Theorem (1.6) is proved in Section 4. Applications of this theorem are found in Section 5. Theorem (1.6) is similar in spirit to Theorem (9.2) in [4] in which Harris gives conditions which imply that any *translation invariant* stationary measure for ξ_t is of the form $\lambda\delta_\emptyset + (1 - \lambda)\delta_S$. Harris does not assume (1.7). Our theorem says that any stationary measure is a convex combination of $\delta_\emptyset$ and δ_S whenever A_t is monotone and (1.7) holds. In particular there is no stationary nontranslation invariant measure for the process (cf. Theorem (3.24) in [6]).

REMARK. It can be shown that A_t is monotone whenever A_t is the dual of ξ_t and $S = \{x, y\}$, a two point space. This result immediately extends to dual processes which are sums of two point generators. In addition we note that branching processes with interference which are duals of certain spin flip processes are also monotone (cf. Holley and Liggett (1975)).

2. Eliminating the condition $\mathscr{A}F_A(\emptyset) = 0$. In this section Theorem (1.5) is proved and then used to obtain a general ergodic theorem. Examples are given.

PROOF OF THEOREM (1.5). To prove Theorem (1.5) we need only show that for each $A \in Y_\Delta$, the given $q(A, B)$, $q(A, \Delta)$ uniquely solve the set of equations

$$\mathscr{A}F_A(\xi) = \sum_{B\subset S} q(A, B)[F_B(\xi) - F_A(\xi)] + q(A, \Delta)[F_\Delta(\xi) - F_A(\xi)] \qquad \xi \in X$$

subject to $\sum_{B\in Y_\Delta} q(A, B) = 0$. That $q(A, \Delta) = \mathscr{A}F_A(\emptyset)$ and $q(\Delta, B) = 0$ follows immediately. To see that Harris's formula still applies for $q(A, B)$ when $A, B \subset S$ let $\tilde{\xi}_t = \xi_{t\wedge\tau}$ where $\tau = \inf\{t \geqq 0 \mid \xi_t = \emptyset\}$ and let $\tilde{\mathscr{A}}$ be the generator of $\tilde{\xi}_t$. From Harris (1976) we know that $\tilde{q}(A, B) = \psi\tilde{\mathscr{A}}F_A(B)$ uniquely solves the set of equations

$$\tilde{\mathscr{A}}F_A(\xi) = \sum_{B\subset S} \tilde{q}(A, B)[F_B(\xi) - F_A(\xi)] \tag{2.1}$$

subject to $\sum_{B\subset S} \tilde{q}(A, B) = 0$. Let $q(A, B) = \tilde{q}(A, B)$ if $B \neq S$ and let $q(A, S) = \tilde{q}(A, S) - \mathscr{A}F_A(\emptyset)$. Then for $\xi \neq \emptyset$

$$\begin{aligned}\mathscr{A}F_A(\xi) &= \tilde{\mathscr{A}}F_A(\xi)\\ &= \sum_{B\subset S} q(A, B)[F_B(\xi) - F_A(\xi)] + \mathscr{A}F_A(\emptyset)[1 - F_A(\xi)]\,.\end{aligned}$$

Clearly $\sum_{B\in Y_\Delta} q(A, B) = 0$ and so to finish the proof we need to show that $\psi\tilde{\mathscr{A}}F_A(B) = \psi\mathscr{A}F_A(B)$ for $B \subsetneqq S$ and that $\psi\tilde{\mathscr{A}}F_A(S) = \psi\mathscr{A}F_A(S) + \mathscr{A}F_A(\emptyset)$. But this follows from the definition of ψ and the fact that $\tilde{\mathscr{A}}F_A(\xi) = \mathscr{A}F_A(\xi)$ whenever $\xi \neq \emptyset$. The uniqueness of the $q(A, B)$ follows from the relationship between $q(A, B)$ and $\tilde{q}(A, B)$ and the fact that the $\tilde{q}(A, B)$ are the unique solutions of equations (2.1).

As in Harris (1976) we can extend this result to the case of infinite S by considering generators of the form $\mathscr{A} = \sum_n \mathscr{A}_n$ where each $\mathscr{A}_n$ is the infinitesimal generator of a Markov process ξ_t^n on $\{0, 1\}^S$ with the property that there exists a finite set $C_n \subset S$ such that the distribution of $C_n \cap \xi_t^n$ depends only on $C_n \cap \xi_0^n$ and such that $\xi_t^n \cap (S\backslash C_n) = \xi_0^n \cap (S\backslash C_n)$ for all $t \geqq 0$. If each $\mathscr{A}_n$ has a dual generator $\mathscr{A}_n^*$ (i.e., $\mathscr{A}_n^*$ generates a process A_t^n on Y_Δ which is a dual of ξ_t^n), and if $\mathscr{A}^* = \sum_n \mathscr{A}_n^*$ generates a nonexplosive continuous time Markov chain A_t on Y_Δ, then A_t is the dual of ξ_t.

(2.2) THEOREM. *Let ξ_t be a Markov process on X with a dual process A_t on Y_Δ satisfying* (1.4). *Suppose that*

$$P_A^*[A_t \in \{\emptyset, \Delta\} \text{ some } t] = 1 \qquad \text{for all} \quad A \in Y\,. \tag{2.3}$$

Then there exists a unique invariant measure ν for ξ_t and from any initial state μ $\lim_{t\to\infty} T(t)\mu = \nu$. In particular the hypothesis holds if $\inf_{B\in Y; B\neq\emptyset} \mathscr{A}F_B(\emptyset) > 0$.

PROOF. From (1.4)

$$\lim_{t\to\infty} E_\xi[F_A(\xi_t)] = P_A^*[A_t = \Delta \text{ eventually}]\,,$$

which implies the first statement of the theorem. The second statement follows because $\mathscr{A}F_B(\emptyset)$ is the rate at which A_t jumps to the absorbing state Δ. The condition $\inf_{B\in Y; B\neq\emptyset} \mathscr{A}F_B(\emptyset) = \lambda > 0$ will give an exponential rate of convergence to ν since $P_A^*[A_t \notin \{\Delta, \emptyset\}] \leqq e^{-\lambda t}$.

EXAMPLE. Let $\Omega f(\xi) = \sum_{x\in S} \mathscr{B}(x)[f(\xi \cup x) - f(\xi)]$ where $\mathscr{B}(x) \geqq 0$ and $\sup_{x\in S} \mathscr{B}(x) < \infty$. Then Ω generates a process ξ on X with birth rates $\mathscr{B}(x)$ at site $x \in S$. The dual of ξ_t is a process on Y_Δ with infinitesimal parameters $q(A, \Delta) = \sum_{x\in A} \mathscr{B}(x)$ and 0 otherwise. Let $\mathscr{A}$ and $\mathscr{A}^*$ be any dual generators from [4]. Then $\Omega + \mathscr{A}$ will generate a process ξ_t satisfying the hypothesis of (2.2) whenever $\inf_{x\in S} \mathscr{B}(x) > 0$. For an extended discussion of $\Omega + \mathscr{A}$ when $\mathscr{A}$ is the generator for symmetric simple exclusion see [10].

EXAMPLE. *Contact processes with spontaneous birth at one site.* Let $S = Z_1$, the one dimensional integers, $N = \{1, -1\}$, $0 = \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq 2\lambda_1$. Define $\mathscr{A}_0 f(\xi) = \mu\xi(0)[f(\xi\backslash 0) - f(\xi)] + \lambda_{|N\cap\xi|}(1 - \xi(0))[f(\xi \cup 0) - f(\xi)]$ and let $\mathscr{A} = \sum_{x\in S} \mathscr{A}_x$ where $\mathscr{A}_x$ is $\mathscr{A}_0$ translated by x. $\mathscr{A}$ is the generator of the contact process studied in [3], [4] (i.e., $\mathscr{A}$ generates a process ξ_t on the subsets of Z_1 which corresponds to death at site x with intensity μ and birth at site x with intensity λ_k where $k = |\xi_t \cap \{x-1, x+1\}|$). The conditions on λ_1, λ_2 guarantee the existence of a dual process A_t on the finite subsets of Z_1 which satisfies (1.1). The process A_t has the following transition rates. For each $x \in A$

$$\begin{aligned}
&A \to_{\lambda_2-\lambda_1} A \cup \{x+1\}\\
&A \to_{\lambda_2-\lambda_1} A \cup \{x-1\}\\
&A \to_{2\lambda_1-\lambda_2} A \cup \{x+1, x-1\}\\
&A \to_{\mu} A\backslash x
\end{aligned}$$

where $A \to_\lambda B$ means the process jumps from A to B with intensity λ. From symmetry and translation invariance considerations we see that either $A_t = \varnothing$ for some t or A_t spends an infinite amount of time in $\{B \mid B \subset S, 0 \in B\}$. Now consider the process $\hat{\xi}_t$ with generator $\mathscr{A} + \Omega$ where

$$\Omega f(\xi) = \lambda[f(\xi \cup 0) - f(\xi)]$$

for $\lambda > 0$. Hence $\hat{\xi}_t$ is a contact process with spontaneous birth at one site and has a dual process $\hat{A}_t$ on Y_Δ. This dual process will behave like $\hat{A}_t$ with the added transition probability of a jump to Δ at rate λ if $0 \in \hat{A}_t$. From the recurrence property of A_t $P_A[\hat{A}_t = \varnothing$ or $\hat{A}_t = \Delta$ eventually$] = 1$, hence $\hat{\xi}_t$ satisfies the conclusions of Theorem (2.2).

3. The condition $\phi\mathscr{A}F_A(B) \geqq 0$. Holley and Stroock (1976) introduced the following device to handle a situation where the "natural" dual process has a negative transition intensity, a situation that occurs when the solution of equations (1.2) for $q(A, B)$ yields $q(A, B) < 0$ for some $A \neq B$. Assume that S is finite. The extension to infinite S is carried out as in Section 2. Let $Y' = \{(A, k) \mid A \subset S, A \neq \varnothing, k = 0, 1\} \cup \{\Delta\} \cup \{\varnothing\}$ and define $F_{(A,k)}(\eta) = (-1)^k F_A(\eta)$, $F_\Delta(\eta) \equiv 1$, $F_\varnothing(\eta) \equiv 0$. We ask for a Markov chain A_t on Y' satisfying

$$E_\xi[F_{(A,k)}(\xi_t)] = E^*_{(A,k)}[F_{A_t}[\xi]] \tag{3.1}$$

for all $\xi \in X$, $(A, k) \in Y'$, $t \geqq 0$. For each $(A, k) \in Y'$, $\eta \subset S$ we try to solve

$$\begin{aligned} \mathscr{A}F_{(A,k)}(\eta) &= \textstyle\sum_{B \in Y'} q((A, k), B)F_B(\eta) \\ &= \textstyle\sum_{C \subset S} [q((A, k), (C, 0)) - q((A, k), (C, 1))]F_C(\eta) \\ &\quad + q((A, k), \Delta) \end{aligned} \tag{3.2}$$

so that $q((A, k), B) \geqq 0$ for $(A, k) \neq B$ and $\sum_{B \in Y'} q((A, k), B) = 0$. From Theorem (1.5) we must have $q((A, k), (C, 0)) - q((A, k), (C, 1)) = (-1)^k \psi \mathscr{A}F_A(C)$ whenever $C \neq \varnothing$. Hence for $A \neq C$ and $C \neq \varnothing$ let

$$q((A, 0), (C, 0)) = q((A, 1), (C, 1)) = \psi\mathscr{A}F_A(C) \vee 0$$
$$q((A, 0), (C, 1)) = q((A, 1), (C, 0)) = -[\psi\mathscr{A}F_A(C) \wedge 0]$$

and let $q((A, k), \Delta) = \mathscr{A}F_A(\varnothing)$. Finally let

$$q((A, k), \varnothing) = -\textstyle\sum_{C \subset S; C \neq A, \varnothing} |\psi\mathscr{A}F_A(C)| - \psi\mathscr{A}F_A(A)\,. \tag{3.3}$$

If this last quantity is nonnegative for all $(A, k) \in Y'$ then this set of infinitesimal parameters will satisfy (3.2) subject to the given conditions. The resulting process A_t can be described as follows. The first coordinate of A_t can be viewed as a process on the finite subsets of S with the intensity of a jump from A to C, $C \neq \varnothing$ given by $|\psi\mathscr{A}F_A(C)|$. If $\psi\mathscr{A}F_A(C) < 0$ then a jump from A to C results in a change in the second coordinate. The process A_t will jump from A to $\varnothing$ with intensity (3.3) and from A to Δ with intensity $\mathscr{A}F_A(\varnothing)$. For applications of this result to spin flip models see Holley and Stroock (1976). For formula (3.3) in a more general setting see Griffeath (1976).

EXAMPLE. *Pure jump with births and deaths.* Let $S = \{z, y\}$ and

$$\begin{aligned} \mathscr{A}f(\xi) &= \xi(z)[1 - \xi(y)][f(\{y\}) - f(\{z\})] + \mu[f(\xi\backslash z) - f(\xi)] \\ &\quad + \mathscr{B}[f(\xi \cup z) - f(\xi)]\,. \end{aligned} \tag{3.4}$$

Then $\mathscr{A}$ corresponds to a process in which a particle can jump from z to y with intensity 1 and which has a birth rate $\mathscr{B}$ at z and a death rate μ at z. Then $\mathscr{A}F_A(\varnothing) = \beta$ if $z \in A$, 0 otherwise, and $\psi\mathscr{A}F_A(C)$ has the following form:

		C: $\varnothing$	$\{z\}$	$\{y\}$	$\{z, y\}$
A	$\varnothing$	0	0	0	0
	$\{z\}$	μ	$-\mu$	1	-1
	$\{y\}$	0	1	-1	0
	$\{z, y\}$	0	0	μ	$-\mu$

If $\mu \geqq 2$ then (3.3) holds and consequently a dual process satisfying (3.1) exists. We note that $q((A, k), \varnothing) = \mu - 2$ if $A = \{z\}$ and 0 otherwise and that $q((A, k), \Delta) = \beta$ if $z \in A$ and 0 otherwise.

4. Increasing dual processes. In this section Theorem (1.6) is proved and sufficient criteria are given for the hypotheses of Theorem (1.6) to hold. Here we are assuming that ξ_t has a dual process A_t on Y; if A_t needed to be defined on Y_Δ then the hypothesis (1.7) of Theorem (1.6) could not hold since Δ is absorbing and $P_A{}^*[A_t = \Delta$ for some $t] > 0$ for some $A \subset S$.

PROOF OF THEOREM (1.6). Assume that A_t is monotone and that (1.7) holds. That $\delta_\varnothing$ is invariant follows automatically from (1.1). That δ_S is invariant follows from

$$P_S[\xi_t \cap A \neq \varnothing] = P_A{}^*[A_t \cap S \neq \varnothing] = 1 - P_A{}^*[A_t = \varnothing]$$

and the hypothesis (1.7) which implies that $P_A{}^*[A_t = \varnothing] = 0$ whenever $A \neq \varnothing$. Next let μ be any invariant measure for ξ_t. Integrating (1.1) with respect to μ yields

$$E_A{}^*[\mu\{\eta \mid \eta \cap A_t \neq \varnothing\}] = \mu\{\eta \mid \eta \cap A \neq \varnothing\}$$

so that $\mu\{\eta \mid \eta \cap A_t \neq \varnothing\}$ is a bounded martingale and hence converges a.s. $P_A{}^*$. Let $B \subset S$, $|B| < \infty$ and let $\tau = \inf\{t \geqq 0\colon A_t \supset B\}$. Then

$$\begin{aligned}\mu\{\eta \mid \eta \cap A \neq \varnothing\} &= \lim_{t\to\infty} E_A{}^*[\mu\{\eta \mid \eta \cap A_t \neq \varnothing\}]\\ &= \lim_{t\to\infty} E_A{}^*[E^*_{A_\tau}[\mu\{\eta \mid \eta \cap A_t \neq \varnothing\}]]\\ &\geqq \lim_{t\to\infty} E_A{}^*[E_B{}^*[\mu\{\eta \mid \eta \cap A_t \neq \varnothing\}]]\\ &= \mu\{\eta \mid \eta \cap B \neq \varnothing\},\end{aligned}$$

where the inequality follows from the monotone property of A_t and the fact that $A_\tau \supset B$. Therefore for any two nonempty subsets B, A we have $\mu\{\eta \mid \eta \cap B \neq \varnothing\} = \mu\{\eta \mid \eta \cap A \neq \varnothing\}$ which implies that $\mu = \lambda\delta_\varnothing + (1-\lambda)\delta_S$. To prove the last statement of Theorem (1.6) we assume that (1.8) holds. Then we have, for any initial distribution μ,

$$\begin{aligned}(T(t)\mu)\{\xi \mid \xi \cap A \neq \varnothing\} &= \int_X P_\xi[\xi_t \cap A \neq \varnothing]\, d\mu(\xi)\\ &= \int_X P_A{}^*[A_t \cap \xi \neq \varnothing]\, d\mu(\xi)\end{aligned}$$

which goes to $1 - \mu(\varnothing)$ as $t \to \infty$ whenever A is nonempty. □

Before giving an application of Theorem (1.6) we give criteria for the hypothesis (1.7) to hold.

(4.1) PROPOSITION. *Let A_t be a monotone continuous time Markov chain on Y. Suppose that A_t can be coupled with a recurrent Markov chain X_t on S in such a way that $X_t \in A_t$ for all $t \geqq 0$. Then* (1.7) *is equivalent to*

(4.2) $$P^*_{\{x\}}[A_t \supset B \text{ for some } t] > 0$$

for some $x \in S$ and all $|B| < \infty$.

PROOF. Fix $x \in S$, $A \neq \varnothing$. Let $\tau_1 = \inf\{t \geqq 0 \mid x \in A_t\}$ and $\tau_{k+1} = \inf\{t \geqq \tau_k + 1 \mid x \in A_t\}$. Since $X_t \in A_t$ for all $t \geqq 0$ and X_t is recurrent it follows that $\tau_k < \infty$ a.s. for all $k = 1, 2, \cdots$. Let $B \subset S$, $|B| < \infty$. Using the monotone property of A_t and (4.2) we have that

$$P^*_{A_{\tau_k}}[A_t \supset B \text{ for some } t \leqq 1] \geqq P^*_{\{x\}}[A_t \supset B \text{ for some } t \leqq 1] = 1 - \rho > 0 \quad \text{a.s.} \quad P_A{}^* .$$

Hence $P_A{}^*[A_t \supset B$ for some $t \leqq \tau_k] \geqq 1 - \rho^k$ which goes to 1 as k goes to ∞, proving the proposition. □

5. Applications of Theorem (1.6).

a. *Recurrent simple exclusion with variable birth rates.* Let $p(x, y)$ be a symmetric irreducible probability transition function for a recurrent Markov chain on S. Let $r(x, y) \geqq 0$ and $\sum_{y \in S} r(x, y) = 1$. Finally let the infinitesimal generator of ξ_t be given by

$$\mathscr{A}f(\xi) = \sum_{x,y \in S} \xi(x)[1 - \xi(y)]p(x, y)[f(\xi_{xy}) - f(\xi)] + \sum_{x,y \in S} \xi(y)[1 - \xi(x)]r(x, y)[f(\xi_x) - f(\xi)] \tag{5.1}$$

where $\xi_x(u) = \xi(u)$ if $u \neq x$ and $\xi_x(x) = 1 - \xi(x)$ and where $\xi_{xy}(u) = \xi(u)$ if $u \neq x, y$, $\xi_{xy}(x) = \xi(y)$ and $\xi_{xy}(y) = (x)$. Then ξ_t is a simple exclusion process with one particle motion determined by $p(x, y)$ modified so that a particle at x is created at a rate $\sum_{y:\xi(y)=1} r(x, y)$. By expressing $F_A(\xi)$ as $1 - \prod_{x \in A}(1 - \xi(x))$ and substituting into (5.1) we can solve equations (1.2) for the parameters $q(A, B)$. The parameters are given by

$$q(A, (A\backslash x) \cup y) = p(x, y) \qquad \text{if} \quad x \in A\,, \quad y \in A$$
$$q(A, A \cup y) = \sum_{x \in A} r(x, y)\,.$$

The resulting dual process A_t has the following description. Think of A_t as a finite collection of particles on S. The particles move as a simple exclusion process using $p(x, y)$ but modified so that at each empty site y a particle is created at the rate $\sum_{x \in A_t} r(x, y)$. If $p(x, y)$ or $r(x, y)$ is recurrent then A_t satisfies the hypotheses of Theorem (1.6) (via Proposition (4.1)) and hence the only invariant measure for ξ_t is of the form $\lambda\delta_\varnothing + (1 - \lambda)\delta_S$ for some $0 \leqq \lambda \leqq 1$. Further results in the case where $p(x, y)$ is transient and $\sum_{y:y \neq x}(x, y) = 0$ for all but finitely many x are found in the author's Ph.D. thesis (UCLA, 1975).

b. *Biased voter model.* Let $S = Z_d$, the d dimensional integers, let $\beta \geqq 0$ and for $x \in Z_d$ let $N_x = \{u \in Z_d \mid |u - x| = 1\}$. Let

$$\mathscr{A}f(\eta) = (1 + \beta) \sum_{x \in Z_d} (1 - \eta(x))(\sum_{u \in N_x} \eta(u))[f(\eta \cup x) - f(\eta)] + \sum_{x \in Z_d} \eta(x)(\sum_{u \in N_x}(1 - \eta(u))[f(\eta\backslash x) - f(\eta)]\,.$$

$\mathscr{A}$ generates a process ξ_t with the following interpretation. Consider the sites of Z_d as occupied by two opposing factions, the ones and zeros. Suppose at any time t the ones occupy the set ξ_t and the zeros the set $\xi_t{}^c$. A one will replace a neighboring zero with a one with an intensity $1 + \beta$ while a zero will replace a neighboring one with a zero with an intensity 1. If $\beta = 0$ this is the voter model for which Holley and Liggett (1975) characterized the invariant measures and their domains of attraction by studying the dual process of ξ_t. It is also the fair invasion process considered by Clifford and Sudbury (1973). If $\beta = 0$ the dual process A_t has the property that $P_A{}^*[|A_t| \leqq |A|$ for all $t \geqq 0] = 1$. If $\beta > 0$ the dual process has probability one of growing infinitely large, hence we treat this case using Theorem (1.6).

The infinitesimal jump parameters of A_t are derived in Section 7.c (Harris (1976)) and are given by $q(A, C) = 1$ if $x \in A$, $y \in N_x$ and $C = (A \backslash x) \cup y$ and by $q(A, C) = \beta$ if $x \in A$, $y \in N_x$ and $C = A \cup y$. That is, A_t is a particle process in which each particle jumps to a neighboring site with intensity one (independent for each particle and site). If the site is occupied the two particles coalesce, otherwise the particle occupies the new site. In addition, each particle in A_t produces a particle in an unoccupied neighboring site with intensity β.

(5.2) THEOREM. *Suppose $\beta > 0$. Then the set of extreme invariant measures of ξ_t is $\{\delta_\varnothing, \delta_{Z_d}\}$. If μ is a translation invariant measure then* $\lim_{t\to\infty} T(t)\mu = \mu(\varnothing)\delta_\varnothing + (1 - \mu(\varnothing))\delta_{Z_d}$.

PROOF. It is clear that A_t is monotone and that $P_A{}^*[A_t \supset B] > 0$ for any $|A| > 0$ and $0 \leqq |B| < \infty$. We now show that the hypotheses of Proposition (4.1) hold. Let $||x||$ be the Euclidean distance between x and the origin. Define a partial ordering on Z_d by $x \prec y$ if $||x|| < ||y||$. For $x, y \in Z_d$ let $q_{xy} = 1$ if $||x - y|| = 1$ and $x \prec y$ and let $q_{xy} = 1 + \beta$ if $||x - y|| = 1$ and $x \succ y$. Then the set q_{xy} are the infinitesimal parameters of X_t, a recurrent Markov chain on Z_d. To couple X_t with A_t let X_t make the same transitions as a tagged particle in A_t until this tagged particle produces another particle in a neighboring site. If the neighboring site is a step towards the origin, X_t jumps to this neighboring site and follows the newly created particle. If the neighboring site is a step away from the origin, X_t does not jump and continues to follow the original tagged particle. The motion of X_t continues in this way. Hence Proposition (4.1) and Theorem (1.6) imply that the extreme invariant measures are $\{\delta_\varnothing, \delta_{Z_d}\}$. To prove the second statement let μ_t be the distribution of ξ_t when the process is started in the translation invariant state μ. Then μ_t is translation invariant, hence

$$\begin{aligned} \frac{d}{dt}\mu_t\{\xi \mid \xi(x) = 1\} &= -\textstyle\sum_{y\in N_x} \mu_t\{\xi \mid \xi(x) = 1, \xi(y) = 0\} \\ &\quad + (1 + \beta) \textstyle\sum_{y\in N_x} \mu_t\{\xi \mid \xi(x) = 0, \xi(y) = 1\} \\ &= \beta \textstyle\sum_{y\in N_x} \mu_t\{\xi \mid \xi(x) = 0, \xi(y) = 1\} \geqq 0 \,. \end{aligned} \tag{5.3}$$

Therefore $\mu_t\{\xi \mid \xi(x) = 1\}$ increases to a number $1 - \lambda$. Let ν be any weak limit of $\{\mu_t\}_{t\geqq 0}$. We now show that $\nu\{\xi \mid \xi(x) = 0, \xi(y) = 1\} = 0$ for all $x \in Z_d$, $y \in N_x$ and, hence, that $\nu = \lambda\delta_\varnothing + (1 - \lambda)\delta_S$. Let $f(t) = \mu_t\{\xi \mid \xi(x) = 1\}$. By (5.3) $f'(t) \geqq 0$ and a direct calculation will yield $\sup_{t\geqq 0} |f''(t)| < \infty$. Since $\int_0^\infty f'(s)\, ds < \infty$, an application of the mean value theorem gives $\lim_{s\to\infty} f'(s) = 0$ and, hence, (5.3) yields $\lim_{s\to\infty} \mu_s\{\xi \mid \xi(x) = 0, \xi(y) = 1\} = \nu\{\xi \mid \xi(x) = 0, \xi(y) = 1\} = 0$.

To see other types of results possible if duality is used we give the following propositions. We continue to assume that ξ_t is the biased voter model with $\beta > 0$.

(5.4) PROPOSITION. Let $H = \{z \in Z_d \mid z = (z_1, \cdots, z_d),\ z_1 \geqq 0\}$. *Let* $\xi \in X$ *and suppose that* $\xi(u) = 1$ *for all* $u \in H$. *Then* $\lim_{t\to\infty} P_\xi[\xi_t(x) = 1] = 1$ *for all* x.

PROOF. From duality we need only show that $P^*_{\{x\}}[A_t \cap H \neq \emptyset]$ goes to 1 as t goes to ∞. But this is a consequence of coupling A_t with a random walk W_t on Z_d so that $W_t \in A_t$ for all t and so that $\Pr[W_t \in H$ for all sufficiently large $t] = 1$. The jump parameters of W_t are given by

$$\begin{aligned} s(x, y) &= 1 && \text{if } \|x - y\| = 1 \text{ and } x_1 \geqq y_1 \\ &= 1 + \beta && \text{if } \|x - y\| = 1 \text{ and } x_1 < y_1 \end{aligned}$$

where $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$. □

In the statement of Proposition (5.5) X_t is a simple random walk on Z_d if X_t is a Markov process on Z_d with jump parameters $p(x, y) = 1$ if $\|x - y\| = 1$ and 0 elsewhere.

(5.5) PROPOSITION. *Let* $d \geqq 3$ *and let* X_t *be a simple random walk on* Z_d. *Let* $\xi \subset Z_d$ *satisfy*

$$\inf_{x \in Z_d} \liminf_{t\to\infty} \Pr[X_t \in \xi \mid X_0 = x] = \rho > 0\,.$$

Then $\lim_{t\to\infty} P_\xi[\xi_t(x) = 1] = 1$ *for all* x.

PROOF. This proposition is an immediate consequence of the duality relationship between ξ_t and A_t and the next lemma.

(5.6) LEMMA. *Suppose that* $d \geqq 3$ *and that* X_t *and* ξ *satisfy the hypotheses of Proposition* (5.5). *Then* $\lim_{t\to\infty} P_x{}^*[A_t \cap \xi = \emptyset] = 0$ *for all* x.

PROOF. Fix $n \geqq 2$ and $\varepsilon > 0$. Let X_t^i, $i = 1, \cdots, n$ be n independent copies of X_t. Let

$$\mathscr{B} = \{X_t^i \neq X_t^j \text{ for all } t \geqq 0, \text{ for all } i \neq j\}\,.$$

From the transience of the simple random walk on $Z_d (d \geqq 3)$ we can find a subset $A = \{x_1, \cdots, x_n\} \subset Z_d$ such that $\hat{P}_A[\mathscr{B}] > 1 - \varepsilon$ where $\hat{P}_A$ refers to the joint probability distribution of $(X_t^1, \cdots, X_t^n)$ started in $(x_1, \cdots, x_n)$ (cf. Liggett (1974)). Next start both processes A_t and $(X_t^1, \cdots, X_t^n)$ on the set A and couple them so that $A_t \supset \{X_t^1, \cdots, X_t^n\}$ for as long as possible, that is, until two initial particles of A_t coalesce. Let

$$\tau = \inf\{t \geqq 0 \mid A_t \text{ does not contain } \{X_t^1, \cdots, X_t^n\}\}\,.$$

From the choice of A and the coupling of A_t and $\{X_t^1, \cdots, X_t^n\}$ we see that $P_A{}^*[\tau < \infty] \leqq \varepsilon$. Then

$$\begin{aligned} P_A{}^*[A_t \cap \xi = \emptyset] &\leqq P_A{}^*[A_t \cap \xi = \emptyset, \tau > t] + P_A{}^*[\tau \leqq t] \\ &\leqq \hat{P}_A[X_t^i \notin \xi;\ i = 1, \cdots, n] + P_A{}^*[\tau \leqq t]\,. \end{aligned}$$

Taking limits as $t \to \infty$ we obtain

$$\lim_{t\to\infty} \sup P_A^*[A_t \cap \xi = \emptyset] \leqq (1-\rho)^n + \varepsilon .$$

Since $P_x^*[A_t \supset A$ for some $t] = 1$ and since A_t is monotone we have that $\lim_{t\to\infty} \sup P_x^*[A_t \cap \xi = \emptyset] \leqq (1-\rho)^n + \varepsilon$. We complete the lemma by letting $\varepsilon \to 0$ and $n \to \infty$. □

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