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MARTINGALES WITH A COUNTABLE FILTERING INDEX SET

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This paper is concerned with the almost everywhere convergence of martingales indexed by countable filtering sets. It is shown that the convergence is a consequence of the maximal inequality as it is in the classical case. It also contains some results about the law of large numbers when the index belongs to a sector and an optimal condition assuring the almost everywhere convergence of martingales in these sectors.

1. Notation. We first recall some definitions we will use in the following. A set I with a partial order $\leqq$ is filtering to the right if for each α, β in I, there exists γ in I such that $\alpha \leqq \gamma$ and $\beta \leqq \gamma$; γ is called an upper bound of α and β. The notation $\alpha \leqq \beta$ (respectively $\alpha \geqq \beta$) means that α is less than or equal to β (respectively greater than or equal to β). If $\alpha \leqq \beta$ (resp. $\alpha \geqq \beta$) and $\alpha \neq \beta$, then we write $\alpha \ll \beta$ (resp. $\alpha \gg \beta$). Let $(a_\alpha)_{\alpha \in I}$ be a family of real numbers indexed by a set I filtering to the right. The limit superior and inferior of $(\alpha_\alpha)_{\alpha \in I}$ are defined in the following way ($\pm\infty$ included):

$$\lim_{\alpha\to} \sup a_\alpha = \inf_{\alpha \in I} \sup_{\beta \geqq \alpha} a_\beta\,, \qquad \lim_{\alpha\to} \inf a_\alpha = \sup_{\alpha \in I} \inf_{\beta \geqq \alpha} a_\beta\,.$$

If these two numbers are equal and finite, then we say that $(a_\alpha)_{\alpha \in I}$, or more simply, a_α converges. The number

$$\lim_{\alpha\to} a_\alpha = \lim_{\alpha\to} \sup a_\alpha = \lim_{\alpha\to} \inf a_\alpha$$

is called the limit of a_α.

The symbol N designates the set of positive integers, and K_d, d in N, the set of d-tuples of positive integers with the partial order induced by the coordinates. This relation is defined as follows:

$$\alpha = (r_1, r_2, \cdots, r_d) \leqq \beta = (s_1, s_2, \cdots, s_d) \Leftrightarrow r_1 \leqq s_1, r_2 \leqq s_2, \cdots, r_d \leqq s_d\,.$$

The sets K_d are countable and filtering to the right.

An increasing sequence $(\alpha_n)_{n \in N}$ in a partially ordered set I (i.e., $\alpha_n \ll \alpha_{n+1}$ for each n in N) is called a generating sequence (of I) if $I = \bigcup_{n=1}^\infty \{\alpha \in I \mid \alpha \leqq \alpha_n\}$. It is easy to show that a countable partially ordered set is filtering to the right if and only if it contains a generating sequence. But an uncountable set, filtering to the right, does not always contain such a sequence.

In this paper we consider only real random variables (rv) and we always suppose that these are defined on an underlying probability space $(\Omega, \mathscr{F}, P)$.

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The classical martingale theory considers processes indexed by N (or R). Several generalizations of the notion of martingale are possible, when the index set is no longer totally ordered, but filtering to the right ([4], [5]). We now give the definition we will use here. Let I be a countable set, filtering to the right, and $(\mathscr{F}_\alpha)_{\alpha \in I}$ an increasing family of sub-σ-fields of $\mathscr{F}$ (i.e., $\mathscr{F}_{\alpha'} \subset \mathscr{F}_{\alpha''}$ for each $\alpha' \ll \alpha''$). A family $(X_\alpha)_{\alpha \in I}$ of rv is a martingale with respect to $(\mathscr{F}_\alpha)_{\alpha \in I}$ if, for each α in I, X_α is $\mathscr{F}_\alpha$-measurable and integrable, and if for each $\alpha'' \gg \alpha'$, $E(X_{\alpha''} \mid \mathscr{F}_{\alpha'}) = X_{\alpha'}$ a.e. Let $(\mathscr{F}_\alpha)_{\alpha \in I}$ be a decreasing family of sub-σ-fields of $\mathscr{F}$ (i.e., $\mathscr{F}_{\alpha'} \subset \mathscr{F}_{\alpha''}$ for each $\alpha' \gg \alpha''$). A family $(X_\alpha)_{\alpha \in I}$ of rv is a reversed martingale with respect to $(\mathscr{F}_\alpha)_{\alpha \in I}$, if for each α in I, X_α is $\mathscr{F}_\alpha$-measurable and integrable, and if $E(X_{\alpha''} \mid \mathscr{F}_{\alpha'}) = X_{\alpha'}$ a.e. for each $\alpha' \gg \alpha''$.

2. Convergence of martingales. The convergence problem of martingales indexed by a countable set, filtering to the right, has been studied by different authors ([4, 7, 13]). One can show ([7]), that an L_1-bounded martingale $(X_\alpha)_{\alpha \in I}$, is always convergent in probability, and converges in L_p, $p > 1$ (resp. in L_1), if and only if it is bounded in L_p (resp. uniformly integrable). We see that for this type of convergence, the behavior of the martingale is not affected by the partial ordering of the index set. The first difficulty appears when we look at the a.e. convergence. It is a well-known fact that the L_1-bounded condition does not imply the a.e. convergence, as it does in the classical case. In the filtering case, one must take the structure of the family of σ-algebras into account. One of the most general conditions is given by Krickeberg in [13] and is called the Vitali condition. The latter, which is automatically verified if $I = N$, can be given in the following way: an increasing family $(\mathscr{F}_\alpha)_{\alpha \in I}$ of sub-σ-fields of $\mathscr{F}$ satisfies the Vitali condition if for each fine covering $(B_\alpha)_{\alpha \in I}$ of an arbitrary measurable set A (i.e., for each α in I, B_α is in $\mathscr{F}_\alpha$ and $A \subset \lim_{\alpha \to} \sup B_\alpha = \bigcap_{\alpha \in I} \bigcup_{\beta \geqq \alpha} B_\beta$ a.e.), and for each $\varepsilon > 0$, there exists a finite set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset I$ and a family of disjoint sets $(L_{\alpha_1}, L_{\alpha_2}, \cdots, L_{\alpha_n})$ such that for each $i = 1, 2, \cdots, n$, L_{α_i} is a set of $\mathscr{F}_{\alpha_i}$, $L_{\alpha_i} \subset B_{\alpha_i}$ and $P\{A \backslash A \cap (\bigcup_{i=1}^n L_{\alpha_i})\} < \varepsilon$. It is proved in [13] that every martingale $(X_\alpha)_{\alpha \in I}$ with respect to a family $(\mathscr{F}_\alpha)_{\alpha \in I}$ satisfying the Vitali condition converges a.e. if it is L-bounded. We present here a different procedure (suggested by S. D. Chatterji) which makes the relation with the classical case clearer. The proof of the a.e. convergence theorem in the case $I = N$ involves the maximal inequality ([7]):

$$\lambda P\{\sup_{n \in N} |X_n| > \lambda\} \leqq \sup_{n \in N} E|X_n| , \qquad \text{for each} \quad \lambda > 0 .$$

It is then of interest to ask if there is such an inequality for generalized martingales. The Vitali condition gives us an indication as to how to proceed.

LEMMA 1. *Let I be a countable set, filtering to the right. If an increasing family $(\mathscr{F}_\alpha)_{\alpha \in I}$ of σ-algebras satisfies the Vitali condition, then for each $\lambda > 0$ and for each martingale $(X_\alpha)_{\alpha \in I}$ with respect to $(\mathscr{F}_\alpha)_{\alpha \in I}$:*

$$\lambda P\{\lim_{\alpha \to} \sup |X_\alpha| > \lambda\} \leqq \sup_{\alpha \in I} E|X_\alpha| .$$

PROOF. With the help of a generating sequence it is easy to see that:

$$\{\lim_{\alpha\to} \sup |X_\alpha| > \lambda\} = \lim_{\alpha\to} \sup \{|X_\alpha| > \lambda\} .$$

Let us put $A = \lim_{\alpha\to} \sup \{|X_\alpha| > \lambda\}$ and for each α in I, $B_\alpha = \{|X_\alpha| > \lambda\}$. The family $(B_\alpha)_{\alpha\in I}$ is then a fine covering of A. The Vitali condition gives us a family of sets $(L_{\alpha_1}, L_{\alpha_2}, \cdots, L_{\alpha_n})$ corresponding to $\varepsilon > 0$, and by writing $L = \bigcup_{i=1}^n L_{\alpha_i}$ we get this expression:

$$\begin{aligned} P(A) &= P(A \cap L) + P(A\backslash A \cap L) \\ &\leqq P(L) + \varepsilon . \end{aligned}$$

Using the martingale condition and an upper bound α of $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$:

$$\begin{aligned} P(L) &\leqq \frac{1}{\lambda} \sum_{i=1}^n \int_{L_{\alpha_i}} |X_{\alpha_i}|\, dP \\ &\leqq \frac{1}{\lambda} \int_L |X_\alpha|\, dP . \end{aligned}$$

The number ε being arbitrary, we conclude that for each $\lambda > 0$

$$\lambda P\{\lim_{\alpha\to} \sup |X_\alpha| > \lambda\} \leqq \sup_{\alpha\in I} E|X_\alpha| .$$

It is not surprising that we get $\lim_{\alpha\to} \sup |X_\alpha|$ instead of $\sup_{\alpha\in I} |X_\alpha|$, because the set $\{\beta \leqq \alpha\}$ can be very complex in a filtering set, whereas it is always finite if $I = N$.

THEOREM 2. *Let I be a countable set, filtering to the right, and $(\mathscr{F}_\alpha)_{\alpha\in I}$ an increasing family of σ-algebras. Let us suppose that for each $\lambda > 0$ and each martingale $(X_\alpha)_{\alpha\in I}$ with respect to $(\mathscr{F}_\alpha)_{\alpha\in I}$, we have*

$$\lambda P\{\lim_{\alpha\to} \sup |X_\alpha| > \lambda\} \leqq \sup_{\alpha\in I} E|X_\alpha| .$$

Then the L_1-bounded martingales with respect to $(\mathscr{F}_\alpha)_{\alpha\in I}$ are a.e. *convergent.*

We see that the structure of the σ-algebras is contained here in the fact that we assume the validity of the inequality for each martingale with respect to $(\mathscr{F}_\alpha)_{\alpha\in I}$. The proof of this theorem can be done in the same way as that given by Baez-Duarte in [1], for the case $I = N$. The key to the proof is the construction of equivalent martingales defined on a compact space and then the use of the Riesz representation theorem.

3. Martingales related to independent random variables. Let us consider a family $(Y_\alpha)_{\alpha\in K_d}$ of independent centered rv. The family of partial sums $S_\alpha = \sum_{\beta\leqq\alpha} Y_\beta$ constitutes a martingale. It can be shown ([9]) that, for each $p \geqq 1$,

$$||\sup_{\alpha\in K_d} |S_\alpha|||_p \leqq A_{d,p} \sup_{\alpha\in K_d} ||S_\alpha||_p ,$$

where the number $A_{d,p}$ depends on d and p only. It is in fact possible to give this inequality for more general filtering sets than the K_d's, the only requirement being their finite dimensionality ([9]). It is clear that this inequality implies the a.e. convergence of S_α under the assumption that $\sup_{\alpha\in K_d} ||S_\alpha||_1 < \infty$.

Let us consider now a family $(Y_\alpha)_{\alpha \in K_d}$ of independent and identically distributed (i.i.d.) random variables. The family $(S_\alpha/|\alpha|)_{\alpha \in K_d}$, where $S_\alpha = \sum_{\beta \leqq \alpha} Y_\beta$ and $|\alpha| = \text{Card}\{\beta \in K_d \mid \beta \leqq \alpha\}$, is a reversed martingale. Smythe ([14]) proved that $(S_\alpha/|\alpha|)_{\alpha \in K_d}$ converges a.e. iff $Y_{1,1,\cdots,1} \in L(\log L)^{d-1}$. The convergence problems of multiple Fourier series ([16]) and the derivation of multiple integrals quite naturally raise the question of the a.e. convergence of $S_\alpha/|\alpha|$, when α belongs to a set of the form

$$T_d^\theta = \{(i_1, i_2, \cdots, i_d) \in K_d \mid \theta i_k < i_l \leqq \theta^{-1} i_k;\ l \neq k,\ l, k = 1, 2, \cdots, d\},$$

which we shall call a sector (of K_d), and where $0 < \theta < 1$. A sector T_d^θ with the order of K_d is filtering to the right. We will say that a family $(a_\alpha)_{\alpha \in K_d}$ of real numbers is sectorially convergent if $(a_\alpha)_{\alpha \in T_d^\theta}$ is convergent for each T_d^θ in K_d. We can now formulate the sectorial law of large numbers. (See also [15].)

THEOREM 3. *Let $(Y_\alpha)_{\alpha \in K_d}$ be a family of* i.i.d. rv. *The following two properties are equivalent*:

(a) $(S_\alpha/|\alpha|)_{\alpha \in K_d}$ *is sectorially* a.e. *convergent*;
(b) $Y_{1,1,\cdots,1}$ *is integrable.*

Actually one implication (the "if" part) is contained in the work of Dunford and of Zygmund ([17]). One can also glean from [16] the implication (b) $\Rightarrow$ (a) of Theorem 3, for stationary arrays of rv.

PROOF. We first suppose that the rv Y_α are nonnegative and we prove that (b) $\Rightarrow$ (a). Let T_d^θ be a sector of K_d and $(i_1, i_2, \cdots, i_d)$ be in T_d^θ. If $i_d = n$ is fixed, then $\theta n \leqq i_k < n/\theta$, $k = 1, 2, \cdots, d-1$, and we have

$$\frac{[n/\theta]^{d-1}}{([\theta n] + 1)^{d-1}} \frac{S_{[n/\theta],[n/\theta],\cdots,[n/\theta],n}}{[n/\theta]^{d-1} n} = \frac{S_{[n/\theta],[n/\theta],\cdots,[n/\theta],n}}{([\theta n] + 1)^{d-1} n}$$
$$\geqq \frac{S_{i_1,i_2,\cdots,i_{d-1},n}}{i_1 i_2 \cdots i_{d-1} n},$$

where $[x]$ is the integer part of x, and the last inequality is true for each $(i_1, i_2, \cdots, i_{d-1}, n)$ in T_d^θ. Let us define

$$U_n = \frac{S_{[n/\theta],[n/\theta],\cdots,[n/\theta],n}}{[n/\theta]^{d-1} n}.$$

The family $(U_n)_{n \in N}$ is a reversed martingale and we see that:

$$\sup_{(i_1,i_2,\cdots,i_{d-1}) \in K_{d-1};(i_1,i_2,\cdots,i_{d-1},n) \in T_d^\theta} \frac{S_{i_1,i_2,\cdots,i_{d-1},n}}{i_1 i_2 \cdots i_{d-1} n} \leqq C(\theta) U_n,$$

where $C(\theta) = \sup_{n \in N} ([n/\theta]^{d-1}/([\theta n] + 1)^{d-1})$. Consequently, for each $\lambda > 0$, we have:

$$\lambda P\left\{\sup_{\alpha \in T_d^\theta} \frac{S_\alpha}{|\alpha|} > \lambda\right\} \leqq \lambda P\{\sup_{n \in N} C(\theta) U_n > \lambda\}$$
$$\leqq C(\theta) E(Y_{1,1,\cdots,1}).$$

The inequality for the general case follows from the triangular inequality, and the rv appear with their absolute values. This inequality implies the a.e. sectorial convergence of $S_\alpha/|\alpha|$ under the assumption that $Y_{1,1,\cdots,1}$ is integrable. It is interesting to note that $C(\theta)$ is going to infinity when θ is going to 0 (i.e., when the sector is opening).

To prove (a) $\Rightarrow$ (b), we first remark, by the lemma of Borel–Cantelli, that

$$\sum_{\alpha \in T_d^\theta} P\{|Y_{1,1,\cdots,1}| > |\alpha|\} < \infty$$

is a consequence of the a.e. sectorial convergence of $S_\alpha/|\alpha|$. Let us define $\pi_n = \{(i_1, i_2, \cdots, i_d) \in K_d \mid i_d = n\}$. There exist two real numbers θ_1 and θ_2, depending on θ, such that

$$\theta_1 n^{d-1} \leqq \operatorname{Card}(\pi_n \cap T_d^\theta) \leqq \theta_2 n^{d-1}.$$

Therefore

$$\theta_1 \sum_{n=1}^\infty n^{d-1} P\{\theta^{d-1}|Y_{1,1,\cdots,1}| > n^d\} \leqq \sum_{\alpha \in T_d^\theta} P\{|Y_{1,1,\cdots,1}| > |\alpha|\}$$
$$\leqq \theta_2 \sum_{n=1}^\infty n^{d-1} P\left\{\frac{|Y_{1,1,\cdots,1}|}{\theta^{d-1}} > n^d\right\}.$$

By changing the variable n, we can see that the convergence of $\sum_{\alpha \in T_d^\theta} P\{|Y_{1,1,\cdots,1}| > |\alpha|\}$ is equivalent to $\sum_{n=1}^\infty P\{|Y_{1,1,\cdots,1}| > n\} < \infty$. The last inequality is precisely the condition required for the integrability of $Y_{1,1,\cdots,1}$.

4. Integrability of the supremum. Let $(Y_n)_{n \in N}$ be a family of i.i.d. random variables. Burkholder in [2] proved the equivalence of $E(\sup_{n \in N}(|S_n|/n)) < \infty$, $E(\sup_{n \in N}(|Y_n|/n)) < \infty$ and $E(|Y_1| \log_+ |Y_1|) < \infty$. We give here the generalization of this theorem for the rv indexed by K_d, and for this purpose, we need a different proof of this classical theorem. The only nonobvious step is that the integrability of $\sup_{n \in N}(|Y_n|/n)$ implies $E(|Y_1| \log_+ |Y_1|) < \infty$ (or equivalently $Y_1 \in L \log L$). Let us write

$$\left\{\sup_{n \in N} \frac{|Y_n|}{n} > m\right\} = \bigcup_{n=1}^\infty \left\{\frac{|Y_1|}{1} \leqq m, \frac{|Y_2|}{2} \leqq m, \cdots, \frac{|Y_{n-1}|}{n-1} \leqq m, \frac{|Y_n|}{n} > m\right\}.$$

The sets in the right member are disjoint and therefore

$$\infty > \sum_{m=1}^\infty P\left\{\sup_{n \in N} \frac{|Y_n|}{n} > m\right\}$$
$$= \sum_{m=1}^\infty \sum_{n=1}^\infty \left(P\{|Y_1| > mn\} \prod_{j=1}^{n-1} \left(1 - P\left\{\frac{|Y_1|}{m} > j\right\}\right)\right).$$

It is well known that for a family of nonnegative real numbers $(a_n)_{n \in N}$, the convergence of $\prod_{n=1}^\infty (1 - a_n)$ is equivalent to $\sum_{n=1}^\infty a_n < \infty$. By hypothesis Y_1 is integrable and therefore $\prod_{j=1}^\infty (1 - P\{|Y_1| > j\})$ is convergent. The relation $L_\infty \subset L \log L$ permits us to suppose that Y_1 is not bounded. For this reason, there exists $m_0 \in N$ such that $0 < P\{|Y_1|/m > 1\} < 1$, for each $m \geqq m_0$. Consequently, the product $\prod_{j=1}^{\infty} (1 - P\{|Y_1|/m > j\})$ converges to a nonzero limit $a(m)$, if $m \geqq m_0$, and we have:

$$\prod_{j=1}^{n-1} \left(1 - P\left\{\frac{|Y_1|}{m} > j\right\}\right) \geqq \prod_{j=1}^{\infty} \left(1 - P\left\{\frac{|Y_1|}{m} > j\right\}\right) = a(m) \geqq a(m_0) ,$$

for each $m \geqq m_0$. Finally

$$\infty > \sum_{m=1}^{\infty} P\left\{\sup_{n \in N} \frac{|Y_n|}{n} > m\right\}$$
$$\geqq \sum_{m=m_0}^{\infty} \sum_{n=1}^{\infty} \left(P\{|Y_1| > mn\} \prod_{j=1}^{n-1}\left(1 - P\left\{\frac{|Y_1|}{m} > j\right\}\right)\right)$$
$$\geqq a(m_0) \sum_{m=m_0}^{\infty} \sum_{n=1}^{\infty} P\{|Y_1| > mn\} .$$

The proof is complete because $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P\{|Y_1| > mn\} < \infty$ is equivalent to $E(|Y_1| \log_+ |Y_1|) < \infty$ ([14]). An easy consequence of this proof is the following lemma:

LEMMA 4. *Let $(Y_n)_{n \in N}$ be a family of* i.i.d. *random variables. If* $\sup_{n \in N} (|Y_n|/n)$ *is in $L(\log L)^{d-1}$, d in N, then Y is in $L(\log L)^d$.*

PROOF. It is proven in [14] that the hypothesis is equivalent to $\sum_{\alpha \in K_d} P\{\sup_{n \in N} (|Y_n|/n) > |\alpha|\} < \infty$. We now transform this expression, using the method described previously:

$$\sum_{\alpha \in K_d} P\{\sup_{n \in N} (|Y_n|/n) > |\alpha|\}$$
$$= \sum_{\alpha \in K_d} \sum_{n=1}^{\infty} (P\{|Y_1| > n|\alpha|\} \prod_{j=1}^{n-1} (1 - P\{|Y_1|/|\alpha| > j\})) .$$

Therefore we have $\sum_{\alpha \in K_{d+1}} P\{|Y_1| > |\alpha|\} < \infty$ and the proof is completed.

THEOREM 5. *Let $(Y_\alpha)_{\alpha \in K_d}$ be a family of* i.i.d. *random variables. The following properties are equivalent*:

(a) $\sup_{\alpha \in K_d} (|S_\alpha|/|\alpha|)$ *is integrable*;
(b) $\sup_{\alpha \in K_d} (|Y_\alpha|/|\alpha|)$ *is integrable*;
(c) $Y_{1,1,\cdots,1}$ *is in $L(\log L)^d$*.

PROOF. It suffices to prove (b) $\Rightarrow$ (c), the other implications being trivial consequences of the martingale theory. We know that for $d = 1$ the proposition is true and we now suppose the same for d. Let us define

$$Z_{i_1,i_2,\cdots,i_d} = \sup_{i_{d+1} \in N} \left|\frac{Y_{i_1,i_2,\cdots,i_d,i_{d+1}}}{i_{d+1}}\right| .$$

The rv $(Z_\alpha)_{\alpha \in K_d}$ are still i.i.d. Our hypothesis implies that

$$\sup_{i_{d+1} \in N} \left|\frac{Y_{1,1,\cdots,1,i_{d+1}}}{i_{d+1}}\right|$$

is in $L(\log L)^d$, and the result follows from the preceding lemma.

We will now give the analogous result for the sectors of K_d.

THEOREM 6. *Let* $(Y_\alpha)_{\alpha\in K_d}$ *be a family of nonnegative* i.i.d. *random variables and* T_d^θ *be a sector in* K_d. *The two following properties are equivalent*:

(a) $\sup_{\alpha\in T_d^\theta} S_\alpha/|\alpha|$ *is integrable*;
(b) $Y_{1,1,\cdots,1}$ *is in* $L\log L$.

PROOF. Let $(Y_n)_{n\in N}$ be a family of independent rv distributed like $Y_{1,1,\cdots,1}$. It is easy to prove that (a) is equivalent to $E(\sup_{n\in N}(S_{n^d}/n^d)) < \infty$. If $(n_k)_{k\in N}$ is a subsequence such that $\sup_{k\in N}(n_{k+1}/n_k) < \infty$, then according to [8], [10], $E(\sup_{k\in N}(S_{n_k}/n_k)) < \infty$ is equivalent to $Y_1 \in L\log L$. Theorem 6 follows from the fact that $n_k = k^d$ verifies the inequality $n_{k+1}/n_k \leqq 2^d$.

5. Sectorial convergence of martingales. The a.e. convergence of martingales indexed by K_d has been studied by Cairoli in [4]. We recall the essential results for the case $d = 2$.

Let $\mathscr{M}_2$ be the class of martingales $(X_{mn}, \mathscr{F}_m \otimes \mathscr{F}_n)_{(m,n)\in K_2}$ defined on $[0, 1) \otimes [0, 1)$ equipped with the Lebesgue measure. In [4] it is proved that the optimal condition which assures the a.e. convergence of the martingales in $\mathscr{M}_2$ is:

$$\sup_{(m,n)\in K_2} E(|X_{mn}| \log_+ |X_{mn}|) < \infty .$$

What is the optimal condition that leads to the a.e. sectorial convergence of these martingales? Looking at the last section of this paper, one could conjecture that, in a sector of K_2, the $L\log L$ integrability condition should be replaced by simple integrability. The following theorem is a negative answer to this conjecture.

THEOREM 7. *Given an arbitrary sector* T_2^θ *and a function* $\phi(t)$, $t \geqq 0$, *increasing, convex, such that* $\phi(t) = o(t\log t)$ *for* $t \to \infty$, *there exists a uniformly integrable martingale* $(E(X \mid \mathscr{F}_m \otimes \mathscr{F}_n))_{(m,n)\in K_2}$ *belonging to the class* $\mathscr{M}_2$, *with* $E(\phi(|X|)) < \infty$ *and*

$$P\{\limsup_{(m,n)\to;(m,n)\in T_2^\theta} |E(X \mid \mathscr{F}_m \otimes \mathscr{F}_n)| > \liminf_{(m,n)\to;(m,n)\in T_2^\theta} |E(X \mid \mathscr{F}_m \otimes \mathscr{F}_n)|\} > 0 .$$

The result also holds for reversed martingales.

PROOF. We first treat the case of martingales. The idea of the proof is to modify Cairoli's counterexample given in [4]. We refer the reader to this paper to avoid giving all the details.

Let us consider the set $[0, 1) \otimes [0, 1)$ equipped with Lebesgue measure, and let $\mathscr{F}_n^*$ be the σ-algebra defined in $[0, 1)$ and generated by the family $\{[(i-1)/2^n, i/2^n); 1 \leqq i \leqq 2^n\}$. For a given $0 < \theta < 1$ and a given $m \in N$, we introduce the numbers

$$I(m, \theta) = \max\{i \in N \mid (i, j) \in T_2^\theta, j = -i + m\} ,$$
$$J(m, \theta) = \min\{i \in N \mid (i, j) \in T_2^\theta, j = -i + m\} ,$$

where T_2^θ is the sector of K_2 corresponding to θ. Lemma 1 ([4], page 11) is replaced by

LEMMA 8. *Let A_m be the set of points in R^2 between the t-axis and the graph of the function $f_\theta(t)$ defined as follows*:

$$\begin{aligned} f_\theta(t) &= 2^{m-J(m,\theta)} && \text{if } t \in [0, 2^{J(m,\theta)}) , \\ &= 2^{m-J(m,\theta)-1} && \text{if } t \in [2^{J(m,\theta)}, 2^{J(m,\theta)+1}), \cdots, \\ &= 2^{m-I(m,\theta)+1} && \text{if } t \in [2^{I(m,\theta)-2}, 2^{I(m,\theta)-1}) , \\ &= 2^{m-I(m,\theta)} && \text{if } t \in [2^{I(m,\theta)-1}, 2^{I(m,\theta)}) , \\ &= 0 && \text{if } t \geqq 2^{I(m,\theta)} . \end{aligned}$$

Let $\phi(t)$ be a nonnegative, increasing function defined on R_+, such that $\liminf_{t\to\infty} \phi(t)/t > 0$. *If for every X in L_ϕ, the martingale $(E(X \mid \mathscr{F}_m^* \otimes \mathscr{F}_n^*))_{(m,n)\in K_2}$ converges* a.e. *in T_2^θ, then there exists a number $C(\theta)$ such that for every $m \in N$*:

$$|A_m| \leqq C(\theta)\phi(2^m) ,$$

where $|A_m|$ designates the measure of A_m.

PROOF. The proof is carried on in the same way as in [4]. The important thing to point out is that the sets A_i^k now belong to σ-algebras $\mathscr{F}_m^* \otimes \mathscr{F}_n^*$ with $(m, n) \in T_2^\theta$. This is a consequence of the definition of $f_\theta(t)$.

Lemma 2 ([4], page 13) now becomes:

LEMMA 9. *Let $\phi(t)$ be a nonnegative, increasing function defined on R_+, such that* $\liminf_{t\to\infty} \phi(t)/t > 0$. *If for every X in L_ϕ, the martingale $(E(X \mid \mathscr{F}_m^* \otimes \mathscr{F}_n^*))_{(m,n)\in K_2}$ converges* a.e. *in T_2^θ, then there exists a number $D(\theta)$ such that*:

$$2^m \log 2^m \leqq D(\theta)\phi(2^m) ,$$

for every $m \in N$.

PROOF. For m large enough, we have:

$$\begin{aligned} |A_m| &\geqq \int_{2^{J(m,\theta)}}^{2^{I(m,\theta)}} \frac{2^{m-1}}{t}\, dt \\ &\geqq \frac{I^-(\theta) - J^+(\theta)}{2}\, 2^m \log 2^m , \end{aligned}$$

where $I^-(\theta) = \liminf_{m\to\infty} (I(m,\theta)/m)$ and $J^+(\theta) = \limsup_{m\to\infty} (J(m,\theta)/m)$. According to the preceding lemma, $|A_m| \leqq C(\theta)\phi(2^m)$ for every $m \in N$, and the proof is complete.

The first part of Theorem 7 follows from these lemmas exactly in the same way as in [4]. We have actually proved a stronger statement: the counter-example can be chosen in the class of martingales with respect to $(\mathscr{F}_m^* \otimes \mathscr{F}_n^*)_{(m,n)\in K_2}$.

To treat the case of the reversed martingales we will use the stochastically convex classes introduced by Burkholder in [3]. We first recall the essential elements of this theory.

Let I be a countable set, filtering to the right, and $\mathscr{H}$ be the class of stochastic processes indexed by I. A subclass $\mathscr{C}$ of $\mathscr{H}$, consisting of nonnegative processes, is said to be stochastically convex if for each sequence $(X_\alpha{}^i)_{\alpha\in I}$, $i\in N$, of processes belonging to $\mathscr{C}$, there exists a sequence $(Y_\alpha{}^i)_{\alpha\in I}$, $i\in N$, of independent processes defined on the same probability space, such that:

(a) for each $i\in N$, $(X_\alpha{}^i)_{\alpha\in I}$ and $(Y_\alpha{}^i)_{\alpha\in I}$ are equivalent (i.e., they have the same finite distributions);

(b) if $(a_i)_{i\in N}$ is a sequence of nonnegative real numbers with $\sum_{i=1}^\infty a_i = 1$, then for each $\alpha\in I$, $\sum_{i=1}^\infty a_i Y_\alpha{}^i$ converges a.e. to a rv Z_α, and the process $(Z_\alpha)_{\alpha\in I}$ is equivalent to a process in $\mathscr{C}$.

Our interest in this notion is due to the following theorem.

THEOREM 10. *Let $\mathscr{C}$ be a stochastically convex class. If each process $(X_\alpha)_{\alpha\in I}$ in $\mathscr{C}$ satisfies the condition $P\{\sup_{\alpha\in I} X_\alpha < \infty\} > 0$ (resp. $P\{\lim_{\alpha\to}\sup X_\alpha < \infty\} > 0$), then there exists a constant $B > 0$ such that*

$$\lambda P\{\sup_{\alpha\in I} X_\alpha > \lambda\} \leqq B (\text{resp. } \lambda P\{\lim_{\alpha\to}\sup X_\alpha > \lambda\} \leqq B) ,$$

for each $\lambda > 0$ and for each process $(X_\alpha)_{\alpha\in I}$ in $\mathscr{C}$.

The proof is exactly the same as that given for the case $I = N$ ([3]).

Now let ϕ be a function with properties given in the Theorem 9, and let us introduce the class $\mathscr{C}_\phi$ of stochastic processes $(X_{mn})_{(m,n)\in K_2}$ defined on $(\Omega\otimes\Omega, \mathscr{F}\otimes\mathscr{F}, P\otimes P)$, where $\Omega = [0, 1)$ with the Lebesgue σ-algebra $\mathscr{F}$ and the Lebesgue measure P, such that $(X_{mn})_{(m,n)\in K_2}$ is a nonnegative reversed martingale with respect to a family of σ-algebras of the form $(\mathscr{F}_m\otimes\mathscr{F}_n)_{(m,n)\in K_2}$, and $\sup_{(m,n)\in K_2} E(\phi(X_{mn})) = E(\phi(X_{11})) \leqq 1$. We will prove that $\mathscr{C}_\phi$ is a stochastically convex class. Let $(\Omega_i, \mathscr{F}_i, P_i)$, $i\in N$, be a sequence of independent copies of $(\Omega, \mathscr{F}, P)$ and $(X_{mn}^i)_{(m,n)\in K_2}$, $i\in N$, be a sequence in $\mathscr{C}$. For each $i\in N$, let us designate by $(Y_{mn}^i)_{(m,n)\in K_2}$ the processes defined on $\Omega^N\otimes\Omega^N$, such that

$$Y_{mn}^i(u_1, u_2, \cdots, u_n, \cdots; v_1, v_2, \cdots, v_n, \cdots) = X_{mn}^i(u_i, v_i) ,$$

where u_n, v_n belong to $[0, 1)$ for each n in N. It is clear that $(Y_{mn}^i)_{(m,n)\in K_2}$ is equivalent to $(X_{mn}^i)_{(m,n)\in K_2}$, and the processes $(Y_{mn}^i)_{(m,n)\in K_2}$, $i\in N$, are independent. Furthermore, $(Y_{mn}^i)_{(m,n)\in K_2}$ is a reversed martingale with respect to the family

$$((\otimes_{i=1}^\infty \mathscr{F}_m{}^i)\otimes(\otimes_{i=1}^\infty \mathscr{F}_n{}^i))_{(m,n)\in K_2} = (\mathscr{G}_m\otimes\mathscr{G}_n)_{(m,n)\in K_2} ,$$

where $(\mathscr{F}_n{}^i)_{n\in N}$ is the copy of $(\mathscr{F}_n)_{n\in N}$, defined on $(\Omega_i, \mathscr{F}_i, P_i)$. If $(a_i)_{i\in N}$ is a sequence of nonnegative real numbers with $\sum_{i=1}^\infty a_i = 1$, then for each (m, n) in K_2, we have

$$E(\sum_{i=1}^\infty a_i Y_{mn}^i) = \sum_{i=1}^\infty a_i E(Y_{mn}^i) < \infty ,$$

and consequently, $\sum_{i=1}^\infty a_i Y_{mn}^i$ converges a.e. to a rv Z_{mn}. It is easy to see that

$(Z_{mn})_{(m,n)\in K_2}$ is a reversed martingale with respect to $(\mathscr{G}_m \otimes \mathscr{G}_n)_{(m,n)\in K_2}$, and the convexity of ϕ implies that

$$E(\phi(Z_{11})) \leqq \sum_{i=1}^{\infty} a_i E(\phi(Y_{11}^i)) \leqq 1 .$$

In using the isomorphism theorem given in [11], page 173, we see that $(Z_{mn})_{(m,n)\in K_2}$ is equivalent to a reversed martingale in $\mathscr{C}_\phi$. The class $\mathscr{C}_\phi$ is then stochastically convex.

PROOF (second part of Theorem 7). Let T_2^θ be a sector in K_2 and let $\phi(t)$ be a function satisfying the conditions given in Theorem 7. Let us denote by C_ϕ^θ the class of processes in C_ϕ restricted to T_2^θ. It is clear that C_ϕ^θ is stochastically convex. On the other hand, for each integer r, let us define $Q_r = \{(m, n) \in K_2 \mid (m, n) \leqq (r, r)\}$. It is always possible to translate Q_r diagonally so that its image will be contained in T_2^θ. Let (s, s) be the image of $(1, 1)$ by this operation and let us introduce for each increasing family of σ-algebras $(\mathscr{F}_n)_{n\in N}$ defined in $[0, 1)$:

$$\begin{aligned} \tilde{\mathscr{F}}_n &= \mathscr{F}_r && \text{if } n \leqq s , \\ &= \mathscr{F}_{r+s-n} && \text{if } s < n \leqq r+s-1 , \\ &= \mathscr{F}_1 && \text{if } n > r+s-1 . \end{aligned}$$

The family $\tilde{\mathscr{F}}_n$ is decreasing.

Given an arbitrary uniformly integrable martingale $X_{mn} = E(X \mid \mathscr{F}_m \otimes \mathscr{F}_n)$ in $\mathscr{M}_2$, such that $X \geqq 0$ and $E(\phi(X)) \leqq 1$, and given $r \in N$, it is clear that $\tilde{X}_{mn} = (X \mid \tilde{\mathscr{F}}_m \otimes \tilde{\mathscr{F}}_n)$ is a reversed martingale belonging to C_ϕ, and to C_ϕ^θ if restricted to T_2^θ. Let us now suppose that each process in C_ϕ^θ converges a.e. in T_2^θ. Theorem 10 implies the existence of $B > 0$, such that for every $r \in N$ and $\lambda > 0$:

$$P\{\sup_{(m,n)\leqq(r,r)} X_{mn} > \lambda\} \leqq P\{\sup_{(m,n)\in K_2} \tilde{X}_{mn} > \lambda\} \leqq \frac{B}{\lambda} .$$

The number B being independent of r, we conclude that:

$$P\{\sup_{(m,n)\in K_2} X_{mn} > \lambda\} \leqq \frac{B}{\lambda} .$$

This inequality would imply the a.e. convergence of the martingales X_{mn} previously defined, and would then contradict Theorem 2 ([4], page 6). It is clear that this method can be used to prove the first part of Theorem 7. But in doing so we can not prove the stronger statement about the martingales with respect to the family $(\mathscr{F}_m^* \otimes \mathscr{F}_n^*)_{(m,n)\in K_2}$.

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