The Annals of Probability
1978, Vol. 6, No. 1, 128–132

A GENERALIZATION OF MARKOV PROCESSES

BY ABEL KLEIN[1]

University of California, Irvine

Osterwalder-Schrader (OS) positive symmetric stationary stochastic processes are discussed. A natural construction is given for the associated positive semigroup structure. Conversely, OS-positive symmetric stationary stochastic processes are constructed from positive semigroup structures. OS-positive processes are seen to be the natural generalization of Markov processes, positive semigroup structures being the natural generalization of positivity preserving semigroups. The inheritability of OS-positivity is discussed.

1. Symmetric stationary stochastic processes. Let $\{X_t\}_{t\in\mathbb{R}}$ be a *stochastic process*, i.e., for each $t \in \mathbb{R}$, X_t is a random variable on a probability space (Q, Σ, μ), the base space, with values in the measurable space $(E, \mathscr{E})$, the state space, where E is a compact Hausdorff space and $\mathscr{E}$ is the Baire σ-algebra. Let us assume the process is *stationary*, i.e., the processes $\{X_t\}_{t\in\mathbb{R}}$ and $\{X_{t+s}\}_{t\in\mathbb{R}}$ are equivalent (e.g., [9]) for all $s \in \mathbb{R}$, and *symmetric*, i.e., the processes $\{X_t\}_{t\in\mathbb{R}}$ and $\{X_{-t}\}_{t\in\mathbb{R}}$ are equivalent. Furthermore let $\{X_t\}_{t\in\mathbb{R}}$ be *weakly stochastically continuous*, in the sense that $\{f \circ X_t\}_{t\in\mathbb{R}}$ is a stochastically continuous process (e.g., [9]) for any real valued continuous function f on E. For $I \subset \mathbb{R}$, Σ_I will denote the σ-algebra generated by $\{X_t\}_{t\in I}$. In particular, we will write Σ_t for $\Sigma_{\{t\}}$, and $\Sigma_+(\Sigma_-)$ for $\Sigma_{[0,\infty)}(\Sigma_{(-\infty,0]})$. We will assume $\Sigma = \Sigma_{\mathbb{R}}$. By E_I we will denote the conditional expectation with respect to Σ_I. Let $U(s)$ and R be the measure preserving transformations corresponding to $X_t \to X_{t+s}$ and $X_t \to X_{-t}$, respectively. $U(s)$ is a one-parameter group of measure preserving automorphisms of $L^\infty(Q, \Sigma, \mu)$, strongly continuous in measure; it follows $U(s)$ is a strongly continuous one-parameter group of isometries in all $L^p(Q, \Sigma, \mu)$, $1 \leqq p < \infty$ [4]. Similarly, R is a measure preserving automorphism of $L^\infty(Q, \Sigma, \mu)$ and an isometry in all $L^p(Q, \Sigma, \mu)$, $1 \leqq p < \infty$, such that $RE_0 = E_0$, $R^2 = I$, and $RU(s) = U(-s)R$.

Given a weakly stochastically continuous symmetric stationary stochastic process we can define a semigroup on $L^2(Q, \Sigma, \mu)$ by $P(t) = E_+U(-t)E_+$ for $t \geqq 0$. In other words,

$$P(t)F(X_{t_1}, \cdots, X_{t_n}) = E(F(X_{t_1-t}, \cdots, X_{t_n-t}) \mid \Sigma_+)\,,$$

where $t, t_1, \cdots, t_n \geqq 0$, and F is a bounded measurable function on E^n. $P(t)$ is a semigroup because $P(t)P(s) = E_+U(-t)E_+U(-s)E_+ = E_+E_{[-t,\infty)}U(-t)U(-s)E_+ = E_+E_{[-t,\infty)}U(-(t+s))E_+ = E_+U(-(t+s))E_+ = P(t+s)$ for $t, s \geqq 0$, as $U(-t)E_+ = E_{[-t,\infty)}U(-t)$ and $E_+E_{[-t,\infty)} = E_+$. It is easy to show that $P(t)$ is a strongly

Received July 7, 1976; revised May 31, 1977.

[1] Supported in part by the National Science Foundation under Grant MCS 76-06332.

AMS 1970 *subject classification*. Primary 60J99.

Key words and phrases. OS-positive symmetric stationary stochastic processes, Markov processes.

continuous contraction semigroup on $L^2(Q, \Sigma_+, \mu)$. This space is, however, too big. We will restrict $P(t)$ to a smaller subspace $\mathscr{H}_0$ of $L_2(Q, \Sigma_+, \mu)$ which contains the function 1 and is left invariant by $P(t)$ and by multiplication by functions in $L^\infty(Q, \Sigma_0, \mu)$. We recall that $L^2(Q, \Sigma_+, \mu)$ is the closed linear span of $\{U(t_1)f_1 U(t_2) \cdots f_n 1 \mid t_1, \cdots, t_n \geqq 0, f_1, \cdots, f_n \in L^\infty(Q, \Sigma_0, \mu)\}$. It is also easy to check that

$$E_+ RE_+ U(t_1)f_1 U(t_2) \cdots f_n 1 = P(t_1)f_1 P(t_2)f_2 \cdots f_n 1 ,$$

for $t_1, \cdots, t_n \geqq 0, f_1, \cdots, f_n \in L^\infty(Q, \Sigma_0, \mu)$. We let $\mathscr{V} = E_+ RE_+$ and take $\mathscr{H}_0$ to be the range of $\mathscr{V}$. It follows $P(t)\mathscr{V}(F) = \mathscr{V}(U(t)F)$ and $f\mathscr{V}(F) = \mathscr{V}(fF)$, where $t \geqq 0$, $F \in L^2(Q, \Sigma_+, \mu)$, and $f \in L^\infty(Q, \Sigma_0, \mu)$.

On $\mathscr{H}_0$ we have a natural sesquilinear form $\langle \cdot \mid \cdot \rangle$, defined by $\langle \mathscr{V}(F) \mid \mathscr{V}(G)\rangle = \langle \mathscr{V}(F), G\rangle$, where $\langle \cdot, \cdot \rangle$ is the L^2 inner product. Equivalently, $\langle \mathscr{V}(F) \mid \mathscr{V}(G)\rangle = \langle RF, G\rangle$. This sesquilinear form is natural in the sense that it makes $P(t)$ self-adjoint, i.e. $\langle \mathscr{V}(F) \mid P(t)\mathscr{V}(G)\rangle = \langle P(t)\mathscr{V}(F) \mid \mathscr{V}(G)\rangle$ for all $F, G \in L^2(Q, \Sigma_+, \mu)$. Furthermore $\langle \mathscr{V}(F) \mid f\mathscr{V}(G)\rangle = \langle \bar{f}\mathscr{V}(F) \mid \mathscr{V}(G)\rangle$ for $f \in L^\infty(Q, \Sigma_0, \mu)$, $L^2(Q, \Sigma_0, \mu) \subset \mathscr{H}_0$, and $\langle \cdot \mid \cdot \rangle$ restricted to $L^2(Q, \Sigma_0, \mu)$ is the L^2 inner product.

2. Osterwalder-Schrader positivity. Let $\{X_t\}_{t \in \mathbf{R}}$ be a weakly stochastically continuous symmetric stationary stochastic process. Such a process is said to be *Markov* if $E_+ E_- = E_+ E_0 E_-$. Equivalently, the process is Markov if and only if $\mathscr{H}_0 = L^2(Q, \Sigma_0, \mu)$, i.e., $P(t)$ leaves $L^2(Q, \Sigma_0, \mu)$ invariant. In this case $P(t)$ is a strongly continuous self-adjoint positivity preserving semigroup on $L^2(Q, \Sigma_0, \mu)$. Conversely, given a strongly continuous self-adjoint positively preserving semigroup, we can construct a weakly stochastically continuous symmetric stationary Markov process (Simon [8], Klein and Landau [4]).

Let us now consider a weakening of the Markov property. Instead of requiring that $L^2(Q, \Sigma_0, \mu) = \mathscr{H}_0$, we only require that $\mathscr{H}_0$ with the sesquilinear form $\langle \cdot \mid \cdot \rangle$ is a pre-Hilbert space, i.e., $\langle \cdot \mid \cdot \rangle$ is positive definite. In other words, we require the *Osterwalder–Schrader positivity condition* (Osterwalder and Schrader [6]): $\langle RF, F\rangle \geqq 0$ for all $F \in L^2(Q, \Sigma_+, \mu)$, i.e.,

$$\int \bar{F}(X_{-t_1}, \cdots, X_{-t_n})F(X_{t_1}, \cdots, X_{t_n})\, d\mu \geqq 0$$

for $t_1, \cdots, t_n \geqq 0$ and F a bounded measurable function on E^n. We will say that such a process is *OS-positive*. We can then complete $\mathscr{H}_0$ into a Hilbert space $\mathscr{H}$. Moreover $\|P(t)\mathscr{V}(F)\|_{\mathscr{H}} \leqq \|F\|_{L^2(Q,\Sigma_+,\mu)}$ for all $t \geqq 0$ so $P(t)$ is a contraction on $\mathscr{H}_0$ (Osterwalder and Schrader [6]; also Klein [3]) and thus $P(t)$ extends by continuity to a strongly continuous self-adjoint contraction semigroup on $\mathscr{H}$. Furthermore, if $f \in L^\infty(Q, \Sigma_0, \mu)$ let us denote by $\tilde{f}$ the operator on $\mathscr{H}_0$ corresponding to multiplication by f, i.e., $\tilde{f}\mathscr{V}(F) = \mathscr{V}(fF)$, then $\tilde{f}$ extends by continuity to a bounded operator on $\mathscr{H}$ with $\|\tilde{f}\| = \|f\|_\infty$, and $\mathfrak{A} = \{\tilde{f} \mid f \in L_\infty(Q, \Sigma_0, \mu)\}$ is a commutative von Neumann algebra of operators on $\mathscr{H}$ having $\Omega = \mathscr{V}(1)$ as a separating vector (Klein [3]). Moreover, if $t_1 \leqq t_2 \leqq \cdots \leqq t_n$,

$f_1, \cdots, f_n \in L^\infty(Q, \Sigma_0, \mu)$, $f_{t_i} = U(t_i)f_i$ for $i = 1, \cdots, n$, then

$$\int f_{t_1} f_{t_2} \cdots f_{t_n} \, d\mu = \langle \Omega \,|\, \tilde{f}_1 P(t_2 - t_1) \tilde{f}_2 \cdots P(t_n - t_{n-1}) \tilde{f}_n \Omega \rangle .$$

We call $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ the *associated semigroup structure.*

A *positive semigroup structure* $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ consists of

(i) a Hilbert space $\mathscr{H}$;
(ii) a strongly continuous self-adjoint contraction semigroup $P(t)$ or $\mathscr{H}$;
(iii) a commutative von Neumann algebra $\mathfrak{A}$ of operators on $\mathscr{H}$;
(iv) a unit vector $\Omega \in \mathscr{H}$;

such that

(v) $P(t)\Omega = \Omega$ for all $t \geqq 0$;
(vi) Ω is a cyclic vector for the algebra generated by $\mathfrak{A} \cup \{P(t) \,|\, t \geqq 0\}$, i.e., the linear span of $\{P(t_1) f_1 P(t_2) \cdots P(t_n) f_n \Omega \,|\, f_1, \cdots, f_n \in \mathfrak{A}, \; t_1, \cdots, t_n \geqq 0\}$ is dense in $\mathscr{H}$;
(vii) for all $f_1, \cdots, f_n \in \mathfrak{A}^+ = \{f \in \mathfrak{A} \,|\, f \geqq 0\}$ and $t_1, \cdots, t_n \geqq 0$,

$$\langle \Omega \,|\, P(t_1) f_1 P(t_2) \cdots P(t_n) f_n \Omega \rangle \geqq 0 .$$

We have thus proved the first part of the following theorem:

THEOREM (Klein [3]): *Let $\{X_t\}_{t \in \mathbf{R}}$ be a weakly stochastically continuous OS-positive symmetric stationary stochastic process. Then its associated semigroup structure $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ form a positive semigroup structure.*

Conversely, let $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ be a positive semigroup structure. Then there exists a weakly stochastically continuous OS-positive symmetric stationary stochastic process $\{X_t\}_{t \in \mathbf{R}}$ such that $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ is its associated semigroup structure.

Let us sketch the proof of the converse. As $\mathfrak{A}$ is a commutative von Neumann algebra, $\mathfrak{A} \approx C(Q_0)$, where Q_0, the spectrum of $\mathfrak{A}$, is a Stonean space (i.e., a compact Hausdorff totally disconnected space, e.g., [7]). Let $Q = \times_{t \in \mathbf{R}} Q_t$, where each Q_t is a copy of Q_0, and Σ_B the Baire σ-algebra on Q. We identify $\{F \in C(Q)\} \,|\,$ there exists $f \in C(Q_0)$ such that $F(q) = f(q_0)$ for all $q = (q_t)_{t \in \mathbf{R}} \in Q\}$ with $C(Q_0)$, and write Σ_0 for the σ-algebra it generates. We define a Baire measure μ on Q by

$$(2.1) \qquad \int f_{t_1} f_{t_2} \cdots f_{t_n} \, d\mu = \langle \Omega \,|\, f_1 P(t_2 - t_1) f_2 \cdots P(t_n - t_{n-1}) f_n \Omega \rangle ,$$

where $t_1 \leqq t_2 \leqq \cdots \leqq t_n$, $f_1, \cdots, f_n \in C(Q_0)$, and $f_{t_i}(q) = f_i(q_{t_i})$ for $i = 1, \cdots, n$. The proof that (2.1) indeed defines a Baire measure involves in a crucial way the fact that Q_0 is a Stonean space and thus finite linear combinations of idempotents are dense in $C(Q_0)$. We now define the stochastic process $\{X_t\}_{t \in \mathbf{R}}$, having base space (Q, Σ, μ) and state space Q_0, by $X_t(q) = q_t$ for $q = (q_t)_{t \in \mathbf{R}} \in Q$. Here Σ is the σ-algebra generated by $\{X_t\}_{t \in \mathbf{R}}$. It can now be shown that $\{X_t\}_{t \in \mathbf{R}}$ is a weakly stochastically continuous OS-positive symmetric stationary stochastic process with $(\mathscr{H}, P(t), \mathfrak{A}, \Omega)$ as its associated semigroup structure.

We can now characterize, by their associated semigroup structures, those OS-positive processes that are actually Markov. To do so let us first notice that if $P(t)$ is a strongly continuous self-adjoint positivity preserving semigroup on $L^2(M)$, M a probability space, then $(L^2(M), P(t), L^\infty(M), 1)$ form a positive semigroup structure. Conversely, a OS-positive process is Markov if and only if its positive semigroup structure can be put in this form. More precisely:

COROLLARY (Klein [2]). *Let $\{X_t\}_{t\in\mathbf{R}}$ be a weakly stochastically continuous OS-positive symmetric stationary stochastic process, and let $(\mathscr{H}, P(t), \mathscr{H}, \Omega)$ be its associated semigroup structure. Then $\{X_t\}_{t\in\mathbf{R}}$ is Markov if and only if Ω is a cyclic vector for $\mathscr{H}$.*

We can thus see that in the semigroup characterization OS-positive processes are the natural generalization of Markov processes. Markov processes correspond to positive semigroup structures in which condition (vi) is replaced by the stronger

(vi)′ Ω is a cyclic vector for $\mathfrak{A}$.

In this case (vii) is equivalent to

(vii)′ for all $f, g \in \mathfrak{A}^+ = \{f \in \mathfrak{A} \mid f \geqq 0\}$ and $t \geqq 0$, $\langle f\Omega \mid P(t) g\Omega\rangle \geqq 0$.

3. Inheritability of OS-positivity. Unlike the Markov property, OS-positivity is inherited under fairly general conditions, for example:

(i) *Functions of OS-positive processes.* Let $\{X_t\}_{t\in\mathbf{R}}$ be a weakly stochastically continuous symmetric stationary stochastic process with base space (Q, Σ, μ) and state space E. Let E' be another state space, let $\alpha : E \to E'$ be a measurable map, and let us consider the process $\{Y_t\}_{t\in\mathbf{R}}$, where $Y_t = \alpha \circ X_t$. It follows $\{Y_t\}_{t\in\mathbf{R}}$ is a weakly stochastically continuous symmetric stationary stochastic process. Moreover, if $\{X_t\}_{t\in\mathbf{R}}$ is OS-positive, so is $\{Y_t\}_{t\in\mathbf{R}}$.

Functions of Markov processes are not in general Markov, but functions of OS-positive processes are OS-positive. In particular, functions of Markov processes are OS-positive.

(ii) *Linear combinations of OS-positive processes.* Let us consider weakly stochastically continuous OS-positive symmetric stationary stochastic processes with the same base and state space, the state space being a vector space. We can then consider linear combinations of these processes, those are again OS-positive. Again, linear combinations of Markov processes are not Markov in general but they are OS-positive.

4. Comments. The Osterwalder–Schrader positivity condition appeared in Osterwalder and Schrader's Euclidean formulation of Quantum Field Theory [6], where it replaced the Markov property used by Nelson [5] in the reconstruction of relativistic quantum fields. Euclidean fields (given the existence of time-zero fields) are examples of weakly stochastically continuous OS-positive symmetric stationary stochastic processes. Using the characterization of OS-positive processes by positive semigroup structures we have been able to determine which relativistic quantum fields correspond to Euclidean fields (Klein [1, 3]).

Gaussian processes satisfying OS-positivity will be studied in a forthcoming paper (Klein [2]).

REFERENCES

[1] KLEIN, A. (1976). When do Euclidean fields exist? Letters in Mathematical Physics **1** 131–133.

[2] KLEIN, A. (1977). Gaussian OS-positive processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **40** 115–124.

[3] KLEIN, A. (1978). The semigroup characterization of Osterwalder-Schrader path spaces and the construction of Euclidean fields. *J. Functional Analysis*, to appear.

[4] KLEIN, A. and LANDAU, L. (1975). Singular perturbations of positivity preserving semigroups via path space techniques. *J. Functional Analysis* **20** 44–82.

[5] NELSON, E. (1973). Construction of quantum fields from Markoff fields. *J. Functional Analysis* **12** 97–112.

[6] OSTERWALDER, K. and SCHRADER, R. (1973). Axioms for Euclidean green's functions, I. *Comm. Math. Phys.* **31** 83–112; II, *Comm. Math. Phys.* **42** (1975), 281–305.

[7] SAKAI, S. (1971). *C*-Algebras and W*-Algebras.* Springer-Verlag, New York.

[8] SIMON, B. (1973). Positivity of the Hamiltonian semigroup and the construction of Euclidean region fields. *Helv. Phys. Acta.* **46** 686–696.

[9] YEH, J. (1973). *Stochastic Processes and the Wiener Integral.* Dekker, New York.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
IRVINE, CALIFORNIA 92717