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SOME L_p VERSIONS FOR THE CENTRAL LIMIT THEOREM

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Let $\bar{F}_n(x)$ denote the distribution of the normalized partial sum of independent, identically distributed random variables with finite second moment, and write $\Delta_n(x) = |\bar{F}_n(x) - \Phi(x)|$, where $\Phi(x)$ is the standard normal distribution. In this paper, the necessary and sufficient conditions for the validity of $\|(1 + |x|)^{2-1/p}\Delta_n(x)\|_p = O(n^{-\delta/2})$ and of $\sum n^{-1+\delta/2}\|(1 + |x|)^{2-1/p}\Delta_n(x)\|_p < \infty$, $0 < \delta < 1$, $1 \leqq p \leqq \infty$, are given. Furthermore, in the case where the underlying random variables $\{X_k\}$ are independent but not necessarily identically distributed, it is shown that $E|X_k|^{2+\delta} < \infty$ implies $\|(1 + |x|)^{2+\delta-1/p}\Delta_n(x)\|_p \leqq Cs_n^{-(2+\delta)} \sum_{k=1}^n E|X_k|^{2+\delta}$, $0 < \delta < 1$, $1 \leqq p \leqq \infty$.

1. Let $\{X_k, k = 1, 2, \cdots\}$ be a sequence of independent, identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$, and distribution function $F(x)$. Write $S_n = \sum_{k=1}^n X_k$, $\bar{F}_n(x) = P(S_n \leqq n^{\frac{1}{2}}x)$ and let $\Phi(x)$ denote the standard normal distribution. Furthermore, for any function $a(x)$, write $\|a(x)\|_p = (\int |a(x)|^p\, dx)^{1/p}$, $1 \leqq p < \infty$, and $\|a(x)\|_\infty = \sup_x |a(x)|$.

Ibragimov [4] and Heyde [3] have studied the necessary and sufficient conditions for the convergence rates of $\|\bar{F}_n(x) - \Phi(x)\|_p \to 0$ as $n \to \infty$. In this paper we shall introduce other L_p versions for the central limit theorem which include the ordinary L_p version and the nonuniform estimate, and we shall study the convergence rates in them.

Put $\Delta_n(x) = |\bar{F}_n(x) - \Phi(x)|$. We first prove the following theorems.

THEOREM 1. *Let* $0 < \delta < 1$ *and* $1 \leqq p \leqq \infty$. *Then, the following statements are equivalent*:

(a) $\int_{|x|>z} x^2\, dF(x) = O(z^{-\delta})$ *as* $z \to \infty$,
(b) $\|(1 + |x|)^{2-1/p}\Delta_n(x)\|_p = O(n^{-\delta/2})$ *as* $n \to \infty$.

THEOREM 2. *Let* $0 < \delta < 1$ *and* $1 \leqq p \leqq \infty$. *Then, the following statements are equivalent*:

(c) $E|X_1|^{2+\delta} < \infty$,
(d) $\sum n^{-1+\delta/2}\|(1 + |x|)^{2-1/p}\Delta_n(x)\|_p < \infty$.

Ibragimov [4] proved that (a) in Theorem 1 is equivalent to

$$\|\Delta_n(x)\|_p = O(n^{-\delta/2}), \qquad 0 < \delta < 1, \tag{1}$$

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and recently Heyde [3] showed that (c) in Theorem 2 is equivalent to

$$\sum n^{-1+\delta/2}\|\Delta_n(x)\|_p < \infty\,, \qquad 0<\delta<1\,. \tag{2}$$

Incidentally we have that the statement (b) and (1) are equivalent to each other for independent, identically distributed random variables with $EX_1 = 0$ and $0 < EX_1^2 < \infty$, and also the statement (d) and (2) are equivalent. Now for proofs of Theorems 1 and 2, it suffices only to prove (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) in each theorem, since $\|\Delta_n(x)\|_p \leqq \|(1+|x|)^{2-1/p}\Delta_n(x)\|_p$. Furthermore, we remark here that our theorems with $p=\infty$ are the results of Heyde ([3], Theorem 1 (ii), (iii)).

PROOF OF THEOREM 1. As was remarked above, the statement that (a) $\Rightarrow$ (b) with $p=\infty$ has been shown by Heyde [3]. Therefore, if we can show

$$\|(1+|x|)\Delta_n(x)\|_1 = O(n^{-\delta/2})\,, \tag{3}$$

then we see that for $1<p<\infty$,

$$\begin{aligned}&\|(1+|x|)^{2-1/p}\Delta_n(x)\|_p\\ &\qquad \leqq \|(1+|x|)^2\Delta_n(x)\|_\infty^{(p-1)/p}\|(1+|x|)\Delta_n(x)\|_1^{1/p} = O(n^{-\delta/2})\,.\end{aligned}$$

Hence, it is sufficient to show (3) for our purpose. We need here the following estimate due to Bikyalis ([1], Theorem 4).

$$\Delta_n(x) \leqq Cn^{-\frac{1}{2}}(1+|x|)^{-3}\int_0^{n^{\frac{1}{2}}(1+|x|)} L(z)\,dz\,, \tag{4}$$

where $L(z) = \int_{|x|>z} x^2\,dF(x)$. Here and in what follows C denotes a positive constant which may differ from one inequality to another. From (4), we have

$$\begin{aligned}\|(1+|x|)\Delta_n(x)\|_1 &\leqq Cn^{-\frac{1}{2}}\int_{-\infty}^{\infty}(1+|x|)^{-2}\,dx\int_0^{n^{\frac{1}{2}}(1+|x|)}L(z)\,dz\\ &= Cn^{-\frac{1}{2}}\int_0^{n^{\frac{1}{2}}}L(z)\,dz\int_{-\infty}^{\infty}(1+|x|)^{-2}\,dx\\ &\qquad + Cn^{-\frac{1}{2}}\int_{n^{\frac{1}{2}}}^{\infty}L(z)\,dz\int_{|x|\geqq zn^{-\frac{1}{2}}-1}(1+|x|)^{-2}\,dx\\ &= Cn^{-\frac{1}{2}}\int_0^{n^{\frac{1}{2}}}L(z)\,dz + C\int_{n^{\frac{1}{2}}}^{\infty}z^{-1}L(z)\,dz = O(n^{-\delta/2})\end{aligned} \tag{5}$$

because of the condition (a). This completes the proof of Theorem 1.

PROOF OF THEOREM 2. When $p=\infty$, it follows from Theorem 1 (ii) of Heyde [3] that

$$\sum n^{-1+\delta/2}\|(1+|x|)^2\Delta_n(x)\|_\infty < \infty\,. \tag{6}$$

Suppose that

$$\sum n^{-1+\delta/2}\|(1+|x|)\Delta_n(x)\|_1 < \infty\,. \tag{7}$$

Then we find that for $1<p<\infty$,

$$\begin{aligned}&\sum n^{-1+\delta/2}\|(1+|x|)^{2-1/p}\Delta_n(x)\|_p\\ &\qquad\leqq \sum n^{-1+\delta/2}\|(1+|x|)^2\Delta_n(x)\|_\infty^{(p-1)/p}\|(1+|x|)\Delta_n(x)\|_1^{1/p}\\ &\qquad= \sum (n^{-1+\delta/2}\|(1+|x|)^2\Delta_n(x)\|_\infty)^{(p-1)/p}(n^{-1+\delta/2}\|(1+|x|)\Delta_n(x)\|_1)^{1/p}\\ &\qquad\leqq (\sum n^{-1+\delta/2}\|(1+|x|)^2\Delta_n(x)\|_\infty)^{(p-1)/p}(\sum n^{-1+\delta/2}\|(1+|x|)\Delta_n(x)\|_1)^{1/p}\end{aligned}$$

by Hölder's inequality. Therefore, (6) and (7) give us $\sum n^{-1+\delta/2}\|(1+|x|)^{2-1/p}\Delta_n(x)\|_p < \infty$ for $1 < p < \infty$, and we need only to show (7). From (5), we have

$$\begin{aligned}\sum n^{-1+\delta/2}&\|(1+|x|)\Delta_n(x)\|_1\\ &\leqq C\sum n^{-(3-\delta)/2}\int_0^{n^{\frac12}} L(z)\,dz + C\sum n^{-1+\delta/2}\int_{n^{\frac12}}^\infty z^{-1}L(z)\,dz\\ &\equiv \Sigma_1+\Sigma_2\,,\end{aligned}$$

say. The statement that $\Sigma_1 \leqq CE|X_1|^{2+\delta} < \infty$ $(0<\delta<1)$ was shown by Heyde [3]. Furthermore, we have

$$\Sigma_2 = C\sum_{n=1}^\infty n^{-1+\delta/2}\sum_{m=n}^\infty \int_{m^{\frac12}}^{(m+1)^{\frac12}} z^{-1}L(z)\,dz\,.$$

Since $L(z)$ decreases as z increases, we have

$$\begin{aligned}\Sigma_2 &\leqq \sum_{n=1}^\infty n^{-1+\delta/2}\sum_{m=n}^\infty m^{-\frac12}((m+1)^{\frac12}-m^{\frac12})L(m^{\frac12})\\ &\leqq C\sum_{n=1}^\infty n^{-1+\delta/2}\sum_{m=n}^\infty m^{-1}L(m^{\frac12})\\ &= C\sum_{m=1}^\infty m^{-1}L(m^{\frac12})\sum_{n=1}^m n^{-1+\delta/2}\\ &\leqq C\sum_{m=1}^\infty m^{-1+\delta/2}L(m^{\frac12})\\ &\leqq CE|X_1|^{2+\delta} < \infty\,,\end{aligned}$$

where the last step is found to be true in the proof of Heyde [3]. The proof of Theorem 2 is thus completed.

2. In this section, we assume that $\{X_k,\ k=1,2,\cdots\}$ is a sequence of independent, but not necessarily identically distributed random variables with $EX_k=0$, $EX_k^2=\sigma_k^2<\infty$ and distribution function $F_k(x)$. Write $s_n^2=\sum_{k=1}^n \sigma_k^2$ and $\bar F_n(x) = P(S_n \leqq s_n^{\frac12}x)$ in this section. We are going to show the following theorem of the Berry–Esseen type, which extends a result of Erickson ([2], corollary with $0<\delta<1$).

THEOREM 3. *Let* $0<\delta<1$ *and* $1\leqq p\leqq\infty$. *If* $E|X_k|^{2+\delta}<\infty$, *then we have*

$$\|(1+|x|)^{2+\delta-1/p}\Delta_n(x)\|_p \leqq Cs_n^{-(2+\delta)}\sum_{k=1}^n E|X_k|^{2+\delta}\,. \tag{8}$$

PROOF. This time we use the following inequality for independent random variables, which was shown by Bikyalis ([1], Theorem 4).

$$\begin{aligned}\Delta_n(x) &\leqq Cs_n^{-3}(1+|x|)^{-3}\sum_{k=1}^n\int_0^{s_n(1+|x|)}dv\int_{|u|>v}u^2\,dF_k(u)\\ &= C\{s_n^{-2}(1+|x|)^{-2}\sum_{k=1}^n\int_{|u|>s_n(1+|x|)}u^2\,dF_k(u)\\ &\quad + s_n^{-3}(1+|x|)^{-3}\sum_{k=1}^n\int_{|u|\leqq s_n(1+|x|)}|u|^3\,dF_k(u)\}\,.\end{aligned} \tag{9}$$

Moreover, this inequality readily gives us that for $0<\delta<1$,

$$\Delta_n(x) \leqq Cs_n^{-(2+\delta)}(1+|x|)^{-(2+\delta)}\sum_{k=1}^n E|X_k|^{2+\delta}\,,$$

if $E|X_k|^{2+\delta}<\infty$ (Bikyalis [1], Corollary 1 of Theorem 4). Therefore, (8) holds for $p=\infty$. Thus, if we can show

$$\|(1+|x|)^{1+\delta}\Delta_n(x)\|_1 \leqq Cs_n^{-(2+\delta)}\sum_{k=1}^n E|X_k|^{2+\delta}\,, \tag{10}$$

then we have that for $1 < p < \infty$,

$$\begin{aligned}&\|(1+|x|)^{2+\delta-1/p}\Delta_n(x)\|_p\\ &\quad\leqq \|(1+|x|)^{2+\delta}\Delta_n(x)\|_\infty^{(p-1)/p}\|(1+|x|)^{1+\delta}\Delta_n(x)\|_1^{1/p}\\ &\quad\leqq Cs_n^{-(2+\delta)}\sum_{k=1}^n E|X_k|^{2+\delta}\,.\end{aligned}$$

Accordingly, it suffices to prove (10). Making use of (9), we have

$$\begin{aligned}&\|(1+|x|)^{1+\delta}\Delta_n(x)\|_1\\ &\quad\leqq C\{s_n^{-2}\sum_{k=1}^n \int_{-\infty}^{\infty}(1+|x|)^{-1+\delta}\,dx\int_{|u|>s_n(1+|x|)}u^2\,dF_k(u)\\ &\qquad+s_n^{-3}\sum_{k=1}^n\int_{-\infty}^{\infty}(1+|x|)^{-2+\delta}\,dx\int_{|u|\leqq s_n(1+|x|)}|u|^3\,dF_k(u)\}\\ &\quad\equiv \textstyle\int_1+\int_2\,,\end{aligned}\tag{11}$$

say. We have

$$\begin{aligned}\textstyle\int_1 &= Cs_n^{-2}\sum_{k=1}^n\int_{|u|>s_n}u^2\,dF_k(u)\int_{|x|<|u|s_n^{-1}-1}(1+|x|)^{-1+\delta}\,dx\\ &\leqq Cs_n^{-2}\sum_{k=1}^n(2/\delta)s_n^{-\delta}\int_{|u|>s_n}|u|^{2+\delta}\,dF_k(u)\\ &\leqq Cs_n^{-(2+\delta)}\sum_{k=1}^n E|X_k|^{2+\delta}\,.\end{aligned}\tag{12}$$

Furthermore, we have

$$\begin{aligned}\textstyle\int_2 &= Cs_n^{-3}\sum_{k=1}^n\int_{|u|\leqq s_n}|u|^3\,dF_k(u)\int_{-\infty}^{\infty}(1+|x|)^{-2+\delta}\,dx\\ &\qquad+Cs_n^{-3}\sum_{k=1}^n\int_{|u|>s_n}|u|^3\,dF_k(u)\int_{|x|\geqq|u|s_n^{-1}-1}(1+|x|)^{-2+\delta}\,dx\\ &\leqq Cs_n^{-3}\sum_{k=1}^n\int_{|u|\leqq s_n}|u|^3\,dF_k(u)\\ &\qquad+Cs_n^{-3}\sum_{k=1}^n(2/(1-\delta))\int_{|u|>s_n}|u|^3(|u|s_n^{-1})^{-1+\delta}\,dF_k(u)\\ &\leqq Cs_n^{-3}\sum_{k=1}^n s_n^{1-\delta}\int_{|u|\leqq s_n}|u|^{2+\delta}\,dF_k(u)\\ &\qquad+Cs_n^{-(2+\delta)}\sum_{k=1}^n\int_{|u|>s_n}|u|^{2+\delta}\,dF_k(u)\\ &\leqq Cs_n^{-(2+\delta)}\sum_{k=1}^n E|X_k|^{2+\delta}\,.\end{aligned}\tag{13}$$

Thus, (11)—(13) complete the proof of the theorem.

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