The Annals of Probability
1978, Vol. 6, No. 2, 267–278

AMARTS INDEXED BY DIRECTED SETS[1]

BY KENNETH A. ASTBURY
Ohio State University

We prove that an amart indexed by a directed set decomposes into a martingale and an amart which converges to zero in L_1 norm. The main theorem asserts that the underlying family of σ-algebras satisfies the Vitali condition if and only if every L_1 bounded amart essentially converges.

1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let J be a directed set filtering to the right. t and u denote elements of J. Throughout this paper functions, sets, and random variables are considered equal if they are equal almost surely. Let $(\mathscr{F}_t)_{t\in J}$ be an increasing family of sub-σ-algebras of $\mathscr{F}$. A *simple stopping time* of $(\mathscr{F}_t)_{t\in J}$ is a function $\tau : \Omega \to J$, taking only finitely many values, such that $\{\tau = t\} \in \mathscr{F}_t$ for all $t \in J$. Let T be the set of all simple stopping times; under the natural order T is a directed set filtering to the right. σ, τ, and ρ denote elements of T.

Let $(X_t)_{t\in J}$ be a family of random variables *adapted* to $(\mathscr{F}_t)_{t\in J}$, i.e., $X_t : \Omega \to R$ is $\mathscr{F}_t$-measurable for each $t \in J$. For $\tau \in T$ define the random variable X_τ by $X_\tau = X_t$ on $\{\tau = t\}$ and define the σ-algebra $\mathscr{F}_\tau$ by $\mathscr{F}_\tau = \{A \in \mathscr{F} \mid A \cap \{\tau = t\} \in \mathscr{F}_t$ for all $t \in J\}$. The family $(X_t)_{t\in J}$ is called an *amart* for $(\mathscr{F}_t)_{t\in J}$ iff $E|X_t| < \infty$ for all $t \in J$ and the net $(E(X_\tau))_{\tau\in T}$ converges to a finite limit. Clearly, the class of amarts is closed under linear combinations and contains the class of martingales. We show in Section 2 that the L_1 bounded amarts form a lattice. In Section 3 we show by example that supermartingales need not be amarts. A *potential* is an amart such that $\lim_{t\in J} E(1_A X_t) = 0$ for all $A \in \bigcup_{t\in J} \mathscr{F}_t$.

Edgar and Sucheston have given a Riesz decomposition of an amart indexed by $\mathbb{N}$ into a martingale and a potential, for both real-valued [6] and vector-valued [8] amarts. The real-valued Riesz decomposition can be extended to amarts with directed index sets by the usual arguments for extending norm convergence. (See Theorem 4.1.) However, in Section 2, we provide a direct proof of the Riesz decomposition which is also valid for vector-valued amarts.

In Section 3 we prove the main theorem: the family $(\mathscr{F}_t)_{t\in J}$ satisfies the Vitali condition if and only if every potential essentially converges. A weaker form of the Vitali condition (allowing an overlap of the covering sets) is known to be equivalent to the essential convergence of all L_∞ bounded martingales. (See [9], pages 168–170.) From our main theorem and the martingale convergence

---

Received November 30, 1976.

[1] This paper is a part of the author's Ph.D. dissertation being prepared at The Ohio State University under the direction of Professor L. Sucheston.

AMS 1970 *subject classifications.* Primary 60F15, 60G40, 60G45, 60G99; Secondary 46G10.

Key words and phrases. Amart, martingale, potential, directed set, essential convergence, Vitali condition.

theorem we obtain that the (strong) Vitali condition is equivalent to the essential convergence of all L_1 bounded amarts. For the case $J = \mathbb{N}$, the resulting amart convergence theorem was proved by Austin, Edgar, and Ionescu Tulcea [1]; a stronger result was published at the same time by Chacon [3]. In Section 4 we obtain similar results for the descending case. ($(\mathscr{F}_t)_{t\in J}$ is a decreasing family.)

2. The Riesz decomposition. We now prove the Riesz decomposition for real-valued amarts. For the sake of generality, we use in place of the L_1 norm the following equivalent norm $\|\cdot\|$, which, in the vector-valued case, is equivalent to the Pettis norm. (See [8].) We define

$$\|X\| = \sup_{A\in\mathscr{F}} |E(1_A X)| .$$

If X is $\mathscr{F}_\tau$-measurable, then

$$\|X\| = \sup_{A\in\mathscr{F}_\tau} |E(1_A X)| .$$

The conditional expectation contracts this norm:

$$\|E(X|\mathscr{F}_\tau)\| = \sup_{A\in\mathscr{F}_\tau} |E(1_A E(X|\mathscr{F}_\tau))| = \sup_{A\in\mathscr{F}_\tau} |E(1_A X)| \leqq \|X\| .$$

LEMMA 2.1. *Let $(X_t)_{t\in J}$ be an amart for $(\mathscr{F}_t)_{t\in J}$ and let $\varepsilon > 0$. Then there exists $\bar\tau \in T$ such that*

$$\sigma \geqq \tau \geqq \bar\tau \Rightarrow \|X_\tau - E(X_\sigma|\mathscr{F}_\tau)\| \leqq \varepsilon .$$

Consequently for any $\rho \in T$, the net $(E(X_\tau|\mathscr{F}_\rho))_{\tau\in T}$ is Cauchy.

PROOF. By the amart property choose $\bar\tau \in T$ such that

$$\rho, \sigma \geqq \bar\tau \Rightarrow |E(X_\rho) - E(X_\sigma)| < \varepsilon .$$

Let $\sigma \geqq \tau \geqq \bar\tau$. For any $A \in \mathscr{F}_\tau$ define the following simple stopping time

$$\begin{aligned} \rho &= \tau \quad \text{on } A \\ &= \sigma \quad \text{on } A^c . \end{aligned}$$

Then

$$E(1_A(X_\tau - E(X_\sigma|\mathscr{F}_\tau))) = E(1_A X_\tau) - E(1_A X_\sigma) = E(X_\rho) - E(X_\sigma) .$$

Since $\rho \geqq \bar\tau$ we have $|E(1_A(X_\tau - E(X_\sigma|\mathscr{F}_\tau)))| < \varepsilon$, and the first conclusion of the lemma follows. Finally for $\sigma \geqq \tau \geqq \bar\tau, \rho$ we have

$$\begin{aligned} \|E(X_\tau|\mathscr{F}_\rho) - E(X_\sigma|\mathscr{F}_\rho)\| &= \|E(X_\tau - E(X_\sigma|\mathscr{F}_\tau)|\mathscr{F}_\rho)\| \\ &\leqq \|X_\tau - E(X_\sigma|\mathscr{F}_\tau)\| \leqq \varepsilon . \qquad \square \end{aligned}$$

THEOREM 2.1 (Riesz decomposition). *Let $(X_t)_{t\in J}$ be an amart for $(\mathscr{F}_t)_{t\in J}$. Then X_t can be written uniquely as $X_t = Y_t + Z_t$ where $(Y_t)_{t\in J}$ is a martingale for $(\mathscr{F}_t)_{t\in J}$, and $(Z_t)_{t\in J}$ is a potential for $(\mathscr{F}_t)_{t\in J}$. Furthermore, $(Z_\tau)_{\tau\in T}$ converges to zero in L_1 norm.*

PROOF. By the lemma and the completeness of L_1, we know that for every $\rho \in T$, the net $(E(X_\tau|\mathscr{F}_\rho))_{\tau\in T}$ converges in norm $\|\cdot\|$. We denote by n-lim the limit in the $\|\cdot\|$ norm. For each $\rho \in T$ define

$$Y_\rho = n\text{-}\lim_{\tau \in T} E(X_\tau \mid \mathscr{F}_\rho), \qquad Z_\rho = X_\rho - Y_\rho .$$

We now establish that $(Y_\rho)_{\rho \in T}$ is a martingale. Let $\sigma \leqq \rho$; then

$$\begin{aligned} E(Y_\rho \mid \mathscr{F}_\sigma) &= E(n\text{-}\lim_{\tau \in T} E(X_\tau \mid \mathscr{F}_\rho) \mid \mathscr{F}_\sigma) \\ &= n\text{-}\lim_{\tau \in T} E(E(X_\tau \mid \mathscr{F}_\rho) \mid \mathscr{F}_\sigma) \\ &= n\text{-}\lim_{\tau \in T} E(X_\tau \mid \mathscr{F}_\sigma) = Y_\sigma . \end{aligned}$$

Therefore $(Y_t)_{t \in J}$ is a martingale, and $(Z_t)_{t \in J}$ is an amart. For $\sigma \geqq \tau$ we have

$$E(Z_\sigma \mid \mathscr{F}_\tau) = E(X_\sigma - Y_\sigma \mid \mathscr{F}_\tau) = E(X_\sigma \mid \mathscr{F}_\tau) - E(Y_\sigma \mid \mathscr{F}_\tau) = E(X_\sigma \mid \mathscr{F}_\tau) - Y_\tau .$$

Hence

$$\lim_{\sigma \in T} \|E(Z_\sigma \mid \mathscr{F}_\tau)\| = 0 .$$

For all $\tau \in T$ we have

$$\|Z_\tau\| \leqq \|Z_\tau - E(Z_\sigma \mid \mathscr{F}_\tau)\| + \|E(Z_\sigma \mid \mathscr{F}_\tau)\| .$$

Applying Lemma 2.1 yields $\lim_{\tau \in T} \|Z_\tau\| = 0$. In order to establish uniqueness, assume that we have another decomposition $X_t = \bar{Y}_t + \bar{Z}_t$ with the required properties. Then for all $A \in \mathscr{F}_t$ we have

$$E(1_A Y_t) = \lim_{u \in J} E(1_A Y_u) = \lim_{u \in J} E(1_A X_u) = \lim_{u \in J} E(1_A \bar{Y}_u) = E(1_A \bar{Y}_t) .$$

Hence $Y_t = \bar{Y}_t$ for all $t \in J$. □

The Riesz decomposition extends some known martingale convergence properties to amarts. We recover a result of Edgar and Sucheston [6]: if $(X_t)_{t \in J}$ is an amart and $\sup_{t \in J} E|X_t| < \infty$, then $(X_t)_{t \in J}$ converges in probability. We also obtain that if $(X_t)_{t \in J}$ is a uniformly integrable amart, then $(X_t)_{t \in J}$ converges in L_1 norm.

COROLLARY 2.1. *Let $(X_t)_{t \in J}$ be an amart for $(\mathscr{F}_t)_{t \in J}$. Then exactly one of the following conditions holds*:

(i) $\lim_{\tau \in T} E(X_\tau^+) = \lim_{\tau \in T} E|X_\tau| = \infty$;
(ii) $(X_t^+)_{t \in J}$ *and* $(|X_t|)_{t \in J}$ *are amarts for* $(\mathscr{F}_t)_{t \in J}$.

PROOF. By Theorem 2.1, $X_t = Y_t + Z_t$ where $(Y_t)_{t \in J}$ is a martingale and $(Z_t)_{t \in J}$ is a potential. Furthermore

$$-|Z_\tau| \leqq -Z_\tau^- \leqq X_\tau^+ - Y_\tau^+ \leqq Z_\tau^+ \leqq |Z_\tau| ,$$

and $(E(X_\tau^+) - E(Y_\tau^+))_{\tau \in T}$ converges to 0. Since $(Y_t)_{t \in J}$ is a martingale, it follows that $(E(Y_\tau^+))_{\tau \in J}$ is an increasing net and, hence, converges to either ∞ or a finite limit. The limit of $(E(X_\tau^+))_{\tau \in T}$ determines the limits of $(E(X_\tau^-))_{\tau \in T}$ and $(E|X_\tau|)_{\tau \in T}$ according to the identities $X^+ = X + X^-$ and $|X| = X^+ + X^-$. □

Similarly, if $(X_t)_{t \in J}$ and $(Y_t)_{t \in J}$ are amarts for $(\mathscr{F}_t)_{t \in J}$, then the limits of $(E(X_\tau \vee Y_\tau))_{\tau \in T}$ and $(E(X_\tau \wedge Y_\tau))_{\tau \in T}$ can be obtained from:

$$\begin{aligned} X_\tau \vee Y_\tau &= X_\tau + (Y_\tau - X_\tau)^+ \\ X_\tau \wedge Y_\tau &= X_\tau - (X_\tau - Y_\tau)^+ . \end{aligned}$$

In particular, the class of L^1 bounded amarts is a lattice. For the case $J = \mathbb{N}$ this is due to Austin, Edgar, and Ionescu Tulcea [1] implicitly, and Edgar and Sucheston [6] explicitly.

We conclude this section by extending the Riesz decomposition to the case of vector-valued amarts. Let $\mathscr{E}$ be a Banach space. The adapted net $(X_t)_{t \in J}$ of strongly measurable random variables $X_t \colon \Omega \to \mathscr{E}$ is an $\mathscr{E}$-valued amart for $(\mathscr{F}_t)_{t \in J}$ iff X_t is Pettis integrable for all $t \in J$, and the net $(E(X_\tau))_{\tau \in T}$ converges in the norm of $\mathscr{E}$. An $\mathscr{E}$-valued potential is an $\mathscr{E}$-valued amart such that $(E(1_A X_t))_{t \in J}$ converges to zero in the norm of $\mathscr{E}$ for all $A \in \bigcup_{t \in J} \mathscr{F}_t$. For more complete definitions see [6].

THEOREM 2.2 (Riesz decomposition for vector-valued amarts). *Let $\mathscr{E}$ have the Radon–Nikodym property and let $(X_t)_{t \in J}$ be an $\mathscr{E}$-valued amart for $(\mathscr{F}_t)_{t \in J}$ such that* $\liminf_{t \in J} E|X_t| < \infty$. *Then X_t can be written uniquely as $X_t = Y_t + Z_t$ where $(Y_t)_{t \in J}$ is a martingale for $(\mathscr{F}_t)_{t \in J}$ and $(Z_t)_{t \in J}$ is an $\mathscr{E}$-valued potential for $(\mathscr{F}_t)_{t \in J}$. Furthermore, $(Z_\tau)_{\tau \in T}$ converges to zero in Pettis norm.*

PROOF. The proof is the same as the real-valued case. To complete the proof we show that for each $\rho \in T$ the Cauchy net $(E(X_\tau \mid \mathscr{F}_\rho))_{\tau \in T}$ does converge in $\|\cdot\|$ norm. $\lim_{\tau \in T} E(1_A X_\tau) = \mu(A)$ exists uniformly in $A \in \mathscr{F}_\rho$; hence μ is a countably additive measure on $\mathscr{F}_\rho$. (This argument appears in [4].) In order to establish that μ has finite variation let $A_1, A_2, \cdots, A_n \in \mathscr{F}_\rho$ be disjoint and let $M = \liminf_{t \in J} E|X_t|$. For any $\varepsilon > 0$ choose $t \in J$ such that

$$|E(1_{A_i} X_t) - \mu(A_i)| \leqq \varepsilon/n \qquad \text{for} \quad i = 1, 2, \cdots, n\,,$$

and

$$E|X_t| \leqq M + \varepsilon\,.$$

Then

$$\begin{aligned}\textstyle\sum_{i=1}^n |\mu(A_i)| &\leqq \textstyle\sum_{i=1}^n |\mu(A_i) - E(1_{A_i} X_t)| + \sum_{i=1}^n |E(1_{A_i} X_t)| \\ &\leqq \varepsilon + E|X_t| \leqq M + 2\varepsilon\,.\end{aligned}$$

Therefore variation $\mu \leqq M$. (This argument appears in [8].) Then $(E(X_\tau \mid \mathscr{F}_\rho))_{\tau \in T}$ converges to the Radon–Nikodym derivative of μ in $\|\cdot\|$ norm.

3. Essential convergence and the Vitali condition.

For the remainder of this paper we consider only real-valued amarts. We recall that the *essential supremum* and *essential infimum* of a family of random variables $(X_t)_{t \in J}$ are the unique random variables $\operatorname{ess\,sup}_{t \in J} X_t$ and $\operatorname{ess\,inf}_{t \in J} X_t$ such that for all random variables X we have:

(i) $X_t \leqq X$ for all $t \in J \Leftrightarrow \operatorname{ess\,sup}_{t \in J} X_t \leqq X$,
(ii) $X_t \geqq X$ for all $t \in J \Leftrightarrow \operatorname{ess\,inf}_{t \in J} X_t \geqq X$.

The family of random variables $(X_t)_{t \in J}$ is said to *essentially converge* iff

$$\operatorname{ess\,lim\,sup}_{t \in J} X_t = \operatorname{ess\,lim\,inf}_{t \in J} X_t\,,$$

where

$$\text{ess lim sup}_{t\in J} X_t = \text{ess inf}_{t\in J} (\text{ess sup}_{u\geqq t} X_u)$$
$$\text{ess lim inf}_{t\in J} X_t = \text{ess sup}_{t\in J} (\text{ess inf}_{u\geqq t} X_u) .$$

Analogous definitions exist for families of measurable sets $(A_t)_{t\in J}$.

The following proposition, due to Austin, Edgar, and Ionescu Tulcea [1] in the case $J = \mathbb{N}$, shows that amarts are a considerable generalization of martingales.

PROPOSITION 3.1. *Let $(X_t)_{t\in J}$ be a family of random variables adapted to $(\mathscr{F}_t)_{t\in J}$. If $(X_t)_{t\in J}$ essentially converges and $E(\text{ess sup}_{t\in J} |X_t|) < \infty$, then $(X_t)_{t\in J}$ is an amart for $(\mathscr{F}_t)_{t\in J}$.*

PROOF. $(X_\tau)_{\tau\in T}$ also essentially converges, say to X, and $\text{ess sup}_{\tau\in T} |X_\tau| = \text{ess sup}_{t\in J} |X_t|$. By the dominated convergence theorem (which also holds for directed index sets), we have $(E(X_\tau))_{\tau\in T}$ converges to $E(X)$, a finite number. □

Our next result depends upon the structure of the σ-algebras $(\mathscr{F}_t)_{t\in J}$. The family $(\mathscr{F}_t)_{t\in J}$ satisfies the *Vitali condition* if the following holds. (See [11], page 99.)

Vitali condition. For every $A \in \mathscr{F}$, for every family of $A_t \in \mathscr{F}_t$ $(t \in J)$ such that $A \subseteq \text{ess lim sup}_{t\in J} A_t$, and for every $\varepsilon > 0$, there exist finitely many indices $t_1, t_2, \cdots, t_n \in J$ and sets $B_i \in \mathscr{F}_{t_i}$ $(i = 1, 2, \cdots, n)$ such that

$$B_i \subseteq A_{t_i} \qquad \text{for} \quad i = 1, 2, \cdots, n ,$$
$$B_i \cap B_j = \varnothing \qquad \text{for} \quad i \neq j ,$$

and

$$P(A \backslash \textstyle\bigcup_{i=1}^n B_i) \leqq \varepsilon .$$

We remark that if J is totally ordered, then the Vitali condition holds. (See [11], page 100.)

THEOREM 3.1. *The family $(\mathscr{F}_t)_{t\in J}$ satisfies the Vitali condition if and only if every potential essentially converges (to zero).*

PROOF. *If.* Let $A \in \mathscr{F}$ and $A_t \in \mathscr{F}_t (t \in J)$ be such that $A \subseteq \text{ess lim sup}_{t\in J} A_t$. Define

$$\begin{aligned}\mathscr{G} = \{\{(t_i, B_i)\}_{i=1,2,\cdots,n} \mid n \ \text{nonnegative};\\ t_i \in J,\ B_i \in \mathscr{F}_{t_i},\ B_i \subseteq A_{t_i} \ \text{for} \ i = 1, 2, \cdots, n;\\ B_i \cap B_j = \varnothing \ \text{for} \ i \neq j\} .\end{aligned}$$

For $G \in \mathscr{G}$ we will write $\bigcup G$ for $\bigcup_{(t,B)\in G} B$. Now we define recursively the sequences $G_k \in \mathscr{G}$ and r_k real as follows:

$$G_0 = \varnothing \qquad \text{and} \qquad r_0 = \sup_{G\in\mathscr{G};G\supseteq G_0} P(\textstyle\bigcup (G\backslash G_0)) ;$$

G_k is any element of $\mathscr{G}$ such that $G_k \supseteq G_{k-1}$ and

$$P(\textstyle\bigcup (G_k\backslash G_{k-1})) \geqq \frac{1}{2} r_{k-1} \qquad \text{and} \qquad r_k = \sup_{G\in\mathscr{G};G\supseteq G_k} P(\textstyle\bigcup (G\backslash G_k)) .$$

Therefore for $G \supseteqq G_k$ we have

$$r_{k-1} \geqq P(\bigcup (G \backslash G_{k-1})) = P(\bigcup (G \backslash G_k)) + P(\bigcup (G_k \backslash G_{k-1}))$$
$$\geqq P(\bigcup (G \backslash G_k)) + \tfrac{1}{2} r_{k-1} ;$$

hence

$$\tfrac{1}{2} r_{k-1} \geqq \sup_{G \in \mathscr{G}; G \supseteqq G_k} P(\bigcup (G \backslash G_k)) = r_k .$$

Thus $r_k \leqq (\frac{1}{2})^k r_0 \leqq (\frac{1}{2})^k$.

Let $\bar{G} = \bigcup_{k=0}^{\infty} G_k$ and denote $\bigcup_{(t,B) \in \bar{G}} B$ by $\bigcup \bar{G}$. Define $C_t \in \mathscr{F}_t$ and the adapted net of random variables $(Z_t)_{t \in J}$ by

$$C_t = A_t \backslash \bigcup_{(u,B) \in \bar{G}; u \leqq t} B ,$$
$$Z_t = 1_{C_t} .$$

We show that $(Z_t)_{t \in J}$ is a potential. Let k be any positive integer. Choose $\bar{t} \in J$ such that $\bar{t} \geqq t$ for all $(t, B) \in G_k$. Let $\tau \in T$, $\tau \geqq \bar{t}$. τ takes finitely many values, say $t_1, t_2, \cdots, t_n$. Define

$$G = G_k \bigcup \{(t_i, \{\tau = t_i\} \cap C_{t_i})\}_{i=1,2,\cdots,n} .$$

It is easy to check that $G_k \subseteqq G \in \mathscr{G}$ and $Z_\tau = 1_{\bigcup(G \backslash G_k)}$. Hence

$$E|Z_\tau| = P(\bigcup (G \backslash G_k)) \leqq r_k \leqq (\tfrac{1}{2})^k .$$

Therefore $(Z_t)_{t \in J}$ is a potential and, by hypothesis, $(Z_t)_{t \in J}$ essentially converges (to zero because $\lim_{t \in J} E|Z_t| = 0$). Hence

$$\emptyset = \operatorname{ess\,lim\,sup}_{t \in J} C_t \supseteqq \operatorname{ess\,lim\,sup}_{t \in J} (A_t \backslash \bigcup \bar{G})$$
$$= (\operatorname{ess\,lim\,sup}_{t \in J} A_t) \backslash \bigcup \bar{G} \supseteqq A \backslash \bigcup \bar{G} .$$

Therefore $A \subseteqq \bigcup \bar{G}$, and for every $\varepsilon > 0$ we can find a finite subclass of $\bar{G}$ which satisfies the requirements of the Vitali condition.

Only if. Let $(Z_t)_{t \in J}$ be a potential. By Theorem 2.1 we have $\lim_{\tau \in T} E|Z_\tau| = 0$. Let $a > 0$ and $A = \operatorname{ess\,lim\,sup}_{t \in J} \{|Z_t| > a\} \in \mathscr{F}$. Given $\varepsilon > 0$, choose $\bar{t} \in J$ such that

$$\tau \geqq \bar{t} \Rightarrow E|Z_\tau| < \varepsilon .$$

Define $A_t \in \mathscr{F}_t$ by

$$A_t = \{|Z_t| > a\} \qquad \text{for } t \geqq \bar{t}$$
$$= \emptyset \qquad \text{elsewhere.}$$

Clearly $A \subseteqq \operatorname{ess\,lim\,sup}_{t \in J} A_t$. By the Vitali condition there exist finitely many indices $t_1, t_2, \cdots, t_n \in J$ and sets $B_i \in \mathscr{F}_{t_i}$ $(i = 1, 2, \cdots, n)$ such that

$$B_i \subseteqq A_{t_i} \qquad \text{for } i = 1, 2, \cdots, n ,$$
$$B_i \cap B_j = \emptyset \qquad \text{for } i \neq j ,$$

and

$$P(A \backslash \bigcup_{i=1}^{n} B_i) \leqq \varepsilon .$$

Choose $t_{n+1} \in J$ such that $t_{n+1} \geqq t_i$ for all $i = 1, 2, \cdots, n$. Define $\tau \in T$ by

$$\begin{aligned} \tau &= t_i \quad \text{on } B_i \text{ for } i = 1, 2, \cdots, n, \\ &= t_{n+1} \quad \text{elsewhere.} \end{aligned}$$

$\tau \geqq \bar{t}$ and hence

$$\varepsilon \geqq E|Z_\tau| \geqq \sum_{i=1}^n E|1_{B_i} Z_{t_i}| \geqq aP(\bigcup_{i=1}^n B_i) \geqq a(P(A) - \varepsilon).$$

Taking ε arbitrarily small yields $A = \emptyset$ for all $a > 0$. Therefore $(Z_t)_{t \in J}$ essentially converges to zero. □

We now state our results in terms of amarts. A family of random variables $(X_t)_{t \in J}$ is said to be L_1 bounded iff $\sup_{t \in J} E|X_t| < \infty$.

THEOREM 3.2. *The family $(\mathscr{F}_t)_{t \in J}$ satisfies the Vitali condition if and only if every L_1 bounded amart essentially converges.*

PROOF. *Only If.* By Theorem 2.1, an L_1 bounded amart $(X_t)_{t \in J}$ decomposes into a potential $(Z_t)_{t \in J}$, which essentially converges by Theorem 3.1, and a martingale $(Y_t)_{t \in J}$. Now

$$\sup_{t \in J} E|Y_t| = \lim_{t \in J} E|Y_t| = \lim_{t \in J} E|X_t - Z_t| = \lim_{t \in J} E|X_t| \leqq \sup_{t \in J} E|X_t|.$$

Therefore $(Y_t)_{t \in J}$ is L_1 bounded and essentially converges by the martingale convergence theorem, due to Krickeberg [10]. (See, e.g., [11], page 99.)

If. Theorem 3.1. □

L_1 bounded supermartingales which do not essentially converge are known to exist, even when the Vitali condition holds. Therefore Theorem 3.2 implies that supermartingales exist which are not amarts. We give a simple example of such a supermartingale. We also give an example of a supermartingale which converges to zero in L_1 norm and essentially, but is not an amart, even though the Vitali condition holds.

EXAMPLE 3.1. Let $(\Omega, \mathscr{F}, P)$ be the unit interval (0, 1] with Lebesgue measure. Let $s(i) = 1 + 2 + \cdots + i$. Define the directed set J by

$$J = \{(i, j) \mid i, j \in \mathbb{N},\ 1 \leqq j \leqq 2^{s(i)}\}$$

with the ordering

$$(i, j) > (m, n) \quad \text{iff} \quad i > m.$$

For each $(i, j) \in J$ define $\mathscr{F}_{(i,j)}$ to be the sub-σ-algebra of $\mathscr{F}$ generated by the intervals

$$\left(\frac{k-1}{2^{s(i)}}, \frac{k}{2^{s(i)}}\right], \qquad 1 \leqq k \leqq 2^{s(i)}.$$

The family of σ-algebras $(\mathscr{F}_{(i,j)})_{(i,j) \in J}$ satisfies the Vitali condition because each pair of such σ-algebras is comparable, even though their indices may not be comparable. Define the adapted family of random variables $(X_{(i,j)})_{(i,j)\in J}$ by

$$X_{(i,j)} = 1 \qquad \text{on} \quad \left(\frac{j-1}{2^{s(i)}}, \frac{j}{2^{s(i)}}\right]$$
$$= \frac{1}{2^i} \qquad \text{elsewhere.}$$

In order to establish that $(X_{(i,j)})_{(i,j)\in J}$ is a supermartingale let $(i,j) > (m,n)$ and let A be an atom of $\mathscr{F}_{(m,n)}$:

$$A = \left(\frac{k-1}{2^{s(m)}}, \frac{k}{2^{s(m)}}\right].$$

Then

$$E(1_A X_{(i,j)}) < \frac{1}{2^{s(i)}} + \frac{1}{2^i}\frac{1}{2^{s(m)}}$$
$$\leqq \frac{1}{2^{s(m)+m+1}} + \frac{1}{2^{m+1}}\frac{1}{2^{s(m)}}$$
$$= \frac{1}{2^m}\frac{1}{2^{s(m)}} \leqq E(1_A X_{(m,n)}).$$

Hence $(X_{(i,j)})_{(i,j)\in J}$ is a supermartingale. It is L_1 bounded; in fact, it is bounded by 1 and converges to 0 in L_1 norm. It is not an amart because for each $(i,j)\in J$ there exists $\tau \geqq (i,j)$ with $X_\tau = 1$, namely τ defined by

$$\tau = (i+1, k) \qquad \text{on} \quad \left(\frac{k-1}{2^{s(i+1)}}, \frac{k}{2^{s(i+1)}}\right], \qquad 1 \leqq k \leqq 2^{s(i+1)}.$$

Proposition 3.1 or a direct argument easily establishes that $(X_{(i,j)})_{(i,j)\in J}$ does not essentially converge.

EXAMPLE 3.2. Let $(\Omega, \mathscr{F}, P)$ be the unit interval $(0,1]$ with Lebesgue measure. Let $s(i) = 1 + 2 + \cdots + i$. Define the directed set J by

$$J = \{(i,j) \mid i, j \in \mathbf{N},\ 1 \leqq j \leqq 2^{2s(i)-i}\}$$

with the ordering

$$(i,j) > (m,n) \qquad \text{iff} \quad i > m.$$

Define recursively the family $(\mathscr{F}_{(i,j)})_{(i,j)\in J}$ of sub-σ-algebras of $\mathscr{F}$ by:

$\mathscr{F}_{(1,1)} = \mathscr{F}_{(1,2)}$ is generated by the intervals $(0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$.

$\mathscr{F}_{(i,j)}$ is generated by $\mathscr{F}_{(i-1,1)}$ and the intervals

$$\left(\frac{k-1}{2^{2s(i)}}, \frac{k}{2^{2s(i)}}\right], \qquad 1 \leqq k \leqq 2^{2s(i)-i}.$$

The family of σ-algebras $(\mathscr{F}_{(i,j)})_{(i,j)\in J}$ satisfies the Vitali condition because each pair of such σ-algebras is comparable, even though their indices may not be comparable. Furthermore all of the σ-algebras $\mathscr{F}_{(i,j)}$ for $(i,j) \geqq (m,n)$ are identical on the interval $((1/2^m), 1]$ and consequently every supermartingale is eventually a decreasing net of random variables on $((1/2^m), 1]$. Hence every supermartingale essentially converges. Define the adapted family of random variables $(X_{(i,j)})_{(i,j)\in J}$ by

$$X_{(i,j)} = 2^i \qquad \text{on} \quad \left(\frac{j-1}{2^{2s(i)}}, \frac{j}{2^{2s(i)}}\right]$$
$$= \frac{1}{2^i} \qquad \text{elsewhere.}$$

In order to establish that $(X_{(i,j)})_{(i,j)\in J}$ is a supermartingale let $(i,j) > (m,n)$. On the interval $((1/2^m), 1]$ we have

$$X_{(i,j)} = \frac{1}{2^i} < \frac{1}{2^m} = X_{(m,n)}\,.$$

On the interval $(0, (1/2^m)]$ consider an atom of $\mathscr{F}_{(m,n)}$:

$$A = \left(\frac{k-1}{2^{2s(m)}}, \frac{k}{2^{2s(m)}}\right].$$

Then

$$\begin{aligned} E(1_A X_{(i,j)}) &\leqq 2^i \frac{1}{2^{2s(i)}} + \frac{1}{2^i}\frac{1}{2^{2s(m)}} \\ &= \frac{1}{2^{s(i)+s(i-1)}} + \frac{1}{2^i}\frac{1}{2^{2s(m)}} \\ &\leqq \frac{1}{2^{s(m+1)+s(m)}} + \frac{1}{2^{m+1}}\frac{1}{2^{2s(m)}} \\ &= \frac{1}{2^m}\frac{1}{2^{2s(m)}} \leqq E(1_A X_{(m,n)})\,. \end{aligned}$$

Hence $(X_{(i,j)})_{(i,j)\in J}$ is a supermartingale. It is L_1 bounded; in fact, it converges to 0 in L_1 norm. However, it is not an amart because for each $(i,j)\in J$ we can define $\tau\in T$, $\tau \geqq (i,j)$ by

$$\tau = (i+1, k) \qquad \text{on} \quad \left(\frac{k-1}{2^{2s(i+1)}}, \frac{k}{2^{2s(i+1)}}\right], \quad 1 \leqq k \leqq 2^{2s(i+1)-(i+1)}$$
$$= (i+1, 1) \qquad \text{on} \quad \left(\frac{1}{2^{i+1}}, 1\right].$$

Then $X_\tau = 2^{i+1}$ on $(0, 1/2^{i+1}]$ and hence $E(X_\tau) \geqq 1$.

4. The descending case. In this section we assume that $(\mathscr{F}_t)_{t\in J}$ is a decreasing family of sub-σ-algebras of $\mathscr{F}$; i.e., if $t \leqq u$ then $\mathscr{F}_t \supseteqq \mathscr{F}_u$. A potential is an amart such that $\lim_{t\in J} E(1_A X_t) = 0$ for all $A \in \mathscr{F}_\infty = \bigcap_{t\in J}\mathscr{F}_t$. All other definitions remain the same.

For the case $J = \mathbb{N}$, Edgar and Sucheston [6] have shown that if $(X_t)_{t\in\mathbb{N}}$ is an amart then $(X_\tau)_{\tau\in T}$ is pointwise convergent, and uniformly integrable; hence $(X_\tau)_{\tau\in T}$ is L_1 convergent. We extend the L_1 norm convergence to amarts with arbitrary directed index sets by using a standard argument.

PROPOSITION 4.1. *If $(X_t)_{t\in J}$ is an amart then $(X_\tau)_{\tau\in T}$ converges in L_1 norm.*

PROOF. Let $(X_t)_{t \in J}$ be an amart and assume that $(X_\tau)_{\tau \in T}$ does not converge in L_1 norm. Then $(E(X_\tau))_{\tau \in T}$ converges, say to M, and $(X_\tau)_{\tau \in T}$ is not L_1 Cauchy. We can choose $\tau_1 \leqq \tau_2 \leqq \tau_3 \leqq \cdots$ such that

$$\tau \geqq \tau_i \Rightarrow |E(X_\tau) - M| < \frac{1}{i}$$

and

$$(X_{\tau_i})_{i \in \mathbb{N}} \quad \text{is not} \quad L_1 \quad \text{Cauchy.}$$

Then $(X_{\tau_i})_{i \in \mathbb{N}}$ is an amart for $(\mathscr{F}_{\tau_i})_{i \in \mathbb{N}}$ but $(X_{\tau_i})_{i \in \mathbb{N}}$ is not L_1 convergent. This contradicts the known results for $J = \mathbb{N}$. □

The L_1 limit of an amart is $\mathscr{F}_\infty$-measurable and the L_1 limit of a potential is zero. Clearly, every amart can be written as the sum of its L_1 limit and a potential. Therefore in the following theorem it suffices to prove that (i) and (ii) are equivalent.

THEOREM 4.1. *The following are equivalent.*

(i) *The family* $(\mathscr{F}_t)_{t \in J}$ *satisfies the Vitali condition.*
(ii) *Every potential essentially converges* (*to zero*).
(iii) *Every amart essentially converges.*

PROOF. (ii) $\Rightarrow$ (i). Let $A_t \in \mathscr{F}_t (t \in J)$. For $\tau \in T$ define $A_\tau = \bigcup_{t \in J} (A_t \cap \{\tau = t\}) \in \mathscr{F}_\tau$. Let $\delta = \lim\sup_{\tau \in T} P(A_\tau)$. Choose $\tau_1 \leqq \tau_2 \leqq \tau_3 \leqq \cdots$ such that

$$\sup_{\tau \geqq \tau_i} P(A_\tau) \leqq \delta + \frac{1}{i} \qquad \text{and} \qquad P(A_{\tau_i}) \geqq \delta - \frac{1}{i}.$$

For $\tau, \tau' \in T$, $\tau \geqq \tau'$ there exists $\rho \in T$, $\rho \geqq \tau'$, namely

$$\begin{aligned} \rho &= \tau \qquad \text{on} \quad A_\tau \in \mathscr{F}_\tau \\ &= \tau' \qquad \text{on} \quad A_\tau^c \in \mathscr{F}_\tau \subseteqq \mathscr{F}_{\tau'}, \end{aligned}$$

such that

$$A_\rho = A_\tau \cup (A_\tau^c \cap A_{\tau'}) = A_\tau \cup A_{\tau'}.$$

Applying this repeatedly, for any $m, n \in N$, $m \leqq n$, one obtains that there exists $\rho \in T$, $\rho \geqq \tau_m$ such that $A_\rho = \bigcup_{i=m}^n A_{\tau_i}$. By the definition of τ_m, we have $P(\bigcup_{i=m}^n A_{\tau_i}) \leqq \delta + 1/m$.

For each $m \in \mathbb{N}$, define $C_m = \bigcup_{i=m}^\infty A_{\tau_i}$. Then $(C_m)_{m \in \mathbb{N}}$ is a decreasing sequence of sets and $\delta \leqq P(C_m) \leqq \delta + 1/m$ for all $m \in \mathbb{N}$. Define $C = \bigcap_{m \in \mathbb{N}} C_m$. $P(C) = \delta$. Denote set symmetric difference by Δ. Then

$$\begin{aligned} P(C \,\Delta\, A_{\tau_m}) &\leqq P(C \,\Delta\, C_m) + P(C_m \,\Delta\, A_{\tau_m}) \\ &\leqq \frac{1}{m} + \frac{2}{m} = \frac{3}{m}. \end{aligned}$$

Let $t \in J$. For any $m \in \mathbb{N}$, choose $\sigma_1, \sigma_2, \cdots \in T$ satisfying the same conditions as $\tau_1, \tau_2, \cdots$ as well as $\sigma_i = \tau_i$ $(i = 1, 2, \cdots, m)$ and $\sigma_{m+1} \geqq t$. Defining $D \in \mathscr{F}$

analogously to C, we have $D \in \mathscr{F}_t$ and

$$P(C \,\Delta\, D) \leqq P(C \,\Delta\, A_{\tau_m}) + P(A_{\tau_m} \,\Delta\, D) \leqq \frac{3}{m} + \frac{3}{m} = \frac{6}{m}.$$

Since m is arbitrary, it follows that $C \in \mathscr{F}_t$. Therefore $C \in \mathscr{F}_\infty$.

Define the adapted family of random variables $(Z_t)_{t \in J}$ by $Z_t = 1_{A_t \cap C^c}$. Then $Z_\tau = 1_{A_\tau \cap C^c}$ and $E|Z_\tau| = P(A_\tau \cap C^c)$. For $\tau \geqq \tau_m$ we have

$$\begin{aligned} P(A_\tau \cap C^c) + \delta - \frac{1}{m} &\leqq P(A_\tau \cap C^c) + P(A_{\tau_m}) \\ &\leqq P(A_\tau \cap C^c) + P(A_{\tau_m} \cap C) + \frac{3}{m} \\ &\leqq P(A_\tau \cup A_{\tau_m}) + \frac{3}{m} \\ &\leqq P(A_\rho) + \frac{3}{m}, \qquad \text{some} \quad \rho \geqq \tau_m \\ &\leqq \delta + \frac{1}{m} + \frac{3}{m} = \delta + \frac{4}{m}. \end{aligned}$$

Therefore $(Z_t)_{t \in J}$ is a potential and, assuming (ii), $(Z_t)_{t \in J}$ essentially converges (to zero because $\lim_{t \in J} E|Z_t| = 0$). Hence, $(\text{ess lim sup}_{t \in J} A_t) \cap C^c = \text{ess lim sup}_{t \in J} (A_t \cap C^c) = \varnothing$, and $\text{ess lim sup}_{t \in J} A_t \subseteqq C$. Given $\varepsilon > 0$ choose A_{τ_m} such that $P(C \backslash A_{\tau_m}) < \varepsilon$ and define $t_i \in J$ $(i = 1, 2, \cdots, n)$ to be the range of τ_m and $B_i \in \mathscr{F}_{t_i}$ $(i = 1, 2, \cdots, n)$ by $B_i = A_{t_i} \cap \{\tau_m = t_i\}$.

(i) $\Rightarrow$ (ii). The proof is the same as that of Theorem 3.1 (only if) with the modification: define $t_{n+1} = \bar{t}$. □

We now give an example of a descending amart which fails to essentially converge.

EXAMPLE 4.1. Let $(I, \mathscr{L}, m)$ be the interval $[0, 1)$ with Lebesgue measure. Define the directed set J by

$$J = \{(i, j) \mid i, j \in N, i \geqq 3, 2 \leqq j \leqq 2^i\}$$

with the ordering $(i, j) > (m, n)$ iff $i > m$. Let $G_{(i,j)} = [0, 1/2^i) \cup [(j-1)/2^i, j/2^i)$. Let $\mathscr{I}$ and $\mathscr{G}_{(i,j)}$ be the sub-σ-algebras of $\mathscr{L}$ generated respectively by I and $G_{(i,j)}$. Define

$$\begin{aligned} (\Omega, \mathscr{F}, P) &= \textstyle\prod_{n=3}^{\infty} (I, \mathscr{L}, m) \\ \mathscr{F}_{(i,j)} &= (\textstyle\prod_{n=3}^{i-1} \mathscr{I}) \times \mathscr{G}_{(i,j)} \times (\textstyle\prod_{n=i+1}^{\infty} \mathscr{L}) \\ A_{(i,j)} &= (\textstyle\prod_{n=3}^{i-1} I) \times G_{(i,j)} \times (\textstyle\prod_{n=i+1}^{\infty} I) \\ X_{(i,j)} &= 1_{A_{(i,j)}}. \end{aligned}$$

For fixed i and for sets (possibly empty) $B_{(i,j)} \in \mathscr{F}_{(i,j)}$ $(j = 2, 3, \cdots, 2^i)$ with

$B_{(i,j)} \subseteqq A_{(i,j)}$, $B_{(i,j)} \cap B_{(i,k)} = \emptyset$ $(j \neq k)$ we can write

$$B_{(i,j)} = (\prod_{n=3}^{i-1} I) \times G_{(i,j)} \times (C_{(i,j)})\,, \qquad C_{(i,j)} \in \prod_{n=i+1}^{\infty} \mathscr{L}$$

where the $C_{(i,j)}$ are pairwise disjoint because the $G_{(i,j)}$ are pairwise overlapping. Hence $\sum_{j=2}^{2i} P(C_{(i,j)}) \leqq 1$ and $\sum_{j=2}^{2i} P(B_{(i,j)}) \leqq m(G_{(i,j)}) = 1/2^{i-1}$. Therefore if $\tau \in T$, $\tau \geqq (n, 2)$, then

$$E|X_\tau| = \sum_{(i,j) \in J} P(A_{(i,j)} \cap \{\tau = (i,j)\}) \leqq \sum_{i=n}^{\infty} \frac{1}{2^{i-1}} = \frac{1}{2^{n-2}}\,,$$

and $(X_t)_{t \in J}$ is a potential. $(X_t)_{t \in J}$ does not essentially converge because ess lim $\inf_{t \in J} X_t = 0$ and ess lim $\sup_{t \in J} X_t = 1_\Omega$.

REFERENCES

[1] AUSTIN, D. G., EDGAR, G. A. and IONESCU TULCEA, A. (1974). Pointwise convergence in terms of expectations. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30** 17–26.

[2] BRUNEL, A. and SUCHESTON, L. (1976). Sur les amarts faibles à valeurs vectorielles. *C. R. Acad. Sci. Paris Ser. A* **282** 1011–1014.

[3] CHACON, R. V. (1974). A "stopped" proof of convergence. *Advances in Math.* **14** 365–368.

[4] CHACON, R. V. and SUCHESTON, L. (1975). On convergence of vector-valued asymptotic martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **33** 55–59.

[5] CHOW, Y. S. (1960). Martingales in a σ-finite measure space indexed by directed sets. *Trans. Amer. Math. Soc.* **97** 254–285.

[6] EDGAR, G. A. and SUCHESTON, L. (1976). Amarts: a class of asymptotic martingales. *J. Multivariate Anal.* **6** 193–221.

[7] EDGAR, G. A. and SUCHESTON, L. (1976). Les amarts: une classe de martingales asymptotiques. *C. R. Acad. Sci. Paris Ser. A* **282** 715–718.

[8] EDGAR, G. A. and SUCHESTON, L. (1976). The Riesz decomposition for vector-valued amarts. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **36** 85–92.

[9] HAYES, C. A. and PAUC, C. Y. (1970). *Derivation and Martingales.* Springer-Verlag, New York.

[10] KRICKEBERG, K. (1956). Convergence of martingales with a directed index set. *Trans. Amer. Math. Soc.* **83** 313–337.

[11] NEVEU, J. (1975). *Discrete-Parameter Martingales.* North Holland, Amsterdam.

231 WEST 18TH AVENUE
COLUMBUS, OHIO 43210