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LIMIT THEOREMS FOR MULTIPLY INDEXED MIXING RANDOM VARIABLES, WITH APPLICATION TO GIBBS RANDOM FIELDS[1]

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If d is a fixed positive integer, let Λ_N be a finite subset of Z^d, the lattice points of $\mathbb{R}^d$, with $\Lambda_N \uparrow Z^d$ and satisfying certain regularity properties. Let $(X_{N,Z})_{Z \in \Lambda_N}$ be a collection of random variables which satisfy a mixing condition and whose partial sums $X_N = \sum_{Z \in \Lambda_N} X_{N,Z}$ have uniformly bounded variances. Limit theorems, including a central limit theorem, are obtained for the sequence X_N. The results are applied to Gibbs random fields known to satisfy a sufficiently strong mixing condition.

1. Notation and definitions. In [14] Philipp discusses the central limit problem for a triangular array of random variables $(X_{N,n})$ where the probabilistic dependence of $X_{N,n}$ and $X_{N,n+k}$ approaches zero as $N, k \to \infty$. If we think of the integers Z as defining sites in some physical system and $X_{N,n}$ as describing what is happening at site n, this sort of dependence simply means that distant sites have little effect on one another. Here we consider a model of a system exhibiting a similar type of dependence among sites, where the set of sites is Z^d for some fixed $d \geqq 1$. By Theorem (4.24) in [7] our results are applicable to certain Gibbs random fields.

We will let $|\cdot|$ denote cardinality of a set in Z^d, $d(\cdot, \cdot)$ denote Euclidean distance in Z^d, $[\cdot]$ denote the greatest integer function, and, for $\Lambda \subset Z^d$, $\partial\Lambda = \{Z \in \Lambda : \exists\, Z' \in Z^d - \Lambda \text{ with } d(Z, Z') = 1\}$.

We will need some simple estimates of $|\Lambda|$ for certain types of subsets Λ of Z^d. If Λ is a d-dimensional cube of side t, $([t])^d \leqq |\Lambda| \leqq ([t] + 1)^d$ and $|\partial\Lambda| \leqq L_d([t] + 1)^{d-1}$, where L_d is a positive constant depending only on the dimension d. If Λ is a d-dimensional sphere of radius t, by considering the inscribed and circumscribed cubes, we see that $|\Lambda| = O(t^d)$. Finally if Λ is a d-dimensional annulus with radii t and $t + k$, $|\Lambda|$ is on the order of

$$(t + k)^d - t^d = \sum_{n=1}^{d} \binom{d}{n} t^{d-n} k^n \leqq kd(t + k)^{d-1} .$$

For $k = 1$ this reduces to $d(t + 1)^{d-1} = O(t^{d-1})$.

Let $H \subset \mathbb{R}$ be the set of possible distances in Z^d, arranged in increasing order of size. Note that for fixed $Z_0 \in Z^d$ and $h \in H$, $|\{Z \in Z^d : d(Z, Z_0) = h\}| = O(h^{d-1})$.

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We will wish to consider an increasing sequence of sets (Λ_N) which have certain properties.

(1.1) DEFINITION. The sequence (Λ_N) of finite subsets of Z^d will be called a *regular sequence* if $\Lambda_N \uparrow Z^d$ and if there exists a sequence of positive integers B_N, $B_N \uparrow \infty$, and positive constants L_1 and L_2 such that

$$L_1 B_N{}^d \leqq |\Lambda_N| \leqq L_2 B_N{}^d \,, \tag{1.2}$$

$$|\partial \Lambda_N| = O(B_N{}^{d-1}) \,, \tag{1.3}$$

and

$$\max_{Z,Z' \in \Lambda_N} d(Z, Z') = O(B_N) \,. \tag{1.4}$$

For convenience, we will assume Λ_N is a d-dimensional sphere of radius N, centered at $0 \in Z^d$.

We suppose that for each N we are given a probability space $(\Omega_N, \mathscr{F}_N, P_N)$ and random variables $(X_{N,Z})_{Z \in \Lambda_N}$. For convenience we assume $E_N(X_{N,Z}) = 0$ for all $Z \in \Lambda_N$, where $E_N(\cdot)$ denotes expected value with respect to P_N.

(1.5) DEFINITION. If Λ_N is a regular sequence, any collection of random variables

$$(X_{N,Z})_{Z \in \Lambda_N} \,, \quad N \geqq 1$$

with $E_N(X_{N,Z}) = 0$ for all $Z \in \Lambda_N$, $N \geqq 1$, will also be called *regular*.

For $\Lambda \subset Z^d$ let $\mathscr{M}_\Lambda^N$ be the σ-algebra generated by $(X_{N,Z})_{Z \in \Lambda}$.

2. Mixing inequalities.

(2.1) DEFINITION. Suppose $\alpha_N \colon [1, \infty) \to \mathbb{R}$ is a sequence of continuous positive functions such that for fixed N, $\alpha_N(t) \downarrow 0$ as $t \uparrow \infty$ and for fixed t, $\alpha_M(t) \leqq \alpha_N(t)$ if $M \geqq N$. Then for $i = 1, 2$, we say the collection $(X_{N,Z})$ *satisfies the mixing condition* (M_i) if whenever $\Lambda^1, \Lambda^2 \subset \Lambda_N$ with $d(\Lambda^1, \Lambda^2) = h$ and $A \in \mathscr{M}_{\Lambda^1}^N$, $B \in \mathscr{M}_{\Lambda^2}^N$

$$|P_N(AB) - P_N(A)P_N(B)| \leqq \alpha_N(h) P_N(A) \,, \tag{M_1}$$

$$|P_N(AB) - P_N(A)P_N(B)| \leqq \alpha_N(h) |\Lambda^1|\,|\Lambda^2| \tag{M_2}$$

and

$$\alpha_N(h) h^{d-1} \downarrow 0 \,.$$

It is clear that (M_1) is stronger than (M_2).

Now conditions (M_1) and (M_2) imply the following inequalities:

(2.2) LEMMA. *Suppose $\Lambda^1, \Lambda^2 \subset \Lambda_N$ with $d(\Lambda^1, \Lambda^2) = h$ and $f \in \mathscr{M}_{\Lambda^1}^N$ and $g \in \mathscr{M}_{\Lambda^2}^N$ are such that $E_N|f|^p < \infty$ and $E_N|g|^q < \infty$. If (M_1) holds with $p, q > 1$ and $1/p + 1/q = 1$, then*

$$|E_N(fg) - E_N(f)E_N(g)| \leqq 2\alpha_N{}^{1/p}(h)(E_N|f|^p)^{1/p}(E_N|g|^q)^{1/q} \,. \tag{2.3}$$

If (M_2) holds with $p, q, r > 0$ and $1/p + 1/q + 1/r = 1$, then

$$|E_N(fg) - E_N(f)E_N(g)| \leqq 2\alpha_N{}^{1/r}(h)(E_N|f|^p)^{1/p}(E_N|g|^q)^{1/q}(|\Lambda^1|\,|\Lambda^2|)^{1/r} \,. \tag{2.4}$$

The proofs of these inequalities are essentially the same as the proofs given in [2] and [8]. In these references it is assumed that $d = 1$ and $(X_{N,n}) = X_n$ is a stationary sequence, but neither of these assumptions is necessary.

3. A central limit theorem. We first consider convergence in law, where the random variable X_N is defined on the space $(\Omega_N, \mathscr{F}_N, P_N)$. Clearly, our results apply to the case where $(\Omega_N, \mathscr{F}_N, P_N) = (\Omega_1, \mathscr{F}_1, P_1)$ for all N. We simply replace Λ_N by Z^d and consider $X_N = \sum_{d(Z,0) \leqq N} X_Z$. For convergence in terms of characteristic functions, working with a sequence of probability spaces makes no difference. When we consider almost sure convergence, we will specify one fixed probability space. Our notation for the different modes of convergence is taken from [9].

Set $X_N = \sum_{Z \in \Lambda_N} X_{N,Z}$.

Let $\mathscr{N}(0, 1)$ denote the law of a normally distributed random variable with mean 0 and variance 1. We have the following theorem:

(3.1) THEOREM. *Suppose $(X'_{N,Z})$ is regular. Suppose*

$$E_N(X'_{N,Z_1} X'_{N,Z_2}) \geqq 0 \qquad \text{for} \quad Z_1, Z_2 \in \Lambda_N \tag{3.2}$$

and

$$\min_{Z \in \Lambda_N} E_N(X'_{N,Z})^2 \geqq C > 0 .$$

Suppose also that $X'_{N,Z}$ satisfies (M_2) and for all N

$$\int_1^\infty \alpha_N^{\frac{1}{5}}(h) h^{2d-1}\, dh \leqq A < \infty \tag{3.3}$$

and

$$\max_{Z \in \Lambda_N} E_N |X'_{N,Z}|^5 \leqq B < \infty . \tag{3.4}$$

Set

$$X_{N,Z} = X'_{N,Z} / (E_N(\textstyle\sum_{Z \in \Lambda_N} X'_{N,Z})^2)^{\frac{1}{2}} .$$

Then

$$\mathscr{L}(X_N) \to \mathscr{N}(0, 1) .$$

PROOF. For $Z \in Z^d$

$$E_N(\textstyle\sum_{Z_1 \in \Lambda_N} X'_{N,Z} X'_{N,Z_1}) \geqq C > 0$$

and

$$\mathscr{S}_N^2 = E_N(\textstyle\sum_{Z \in \Lambda_N} X'_{N,Z})^2 \geqq CN^d . \tag{3.5}$$

Thus by (3.4)

$$\max_{Z \in \Lambda_N} E_N(X_{N,Z})^4 = O(N^{-2d}) , \tag{3.6}$$

and

$$\sigma_N^2 = \max_{Z \in \Lambda_N} E_N(X_{N,Z})^2 = O(N^{-d}) .$$

Also (3.6) and (2.4) with $r = 2$, $p = q = 4$ imply

$$\mathscr{S}_N^2 = O(N^d) .$$

Now we proceed as in [14], replacing intervals by annuli concentric about $0 \in Z^d$. For each N we have, using (2.4) and (3.6),

$$\Sigma_N^2 = E_N(\sum_{z \in \Lambda_N} X_{N,z})^2 = 1 .$$

Choose ε satisfying

$$0 < \varepsilon < \tfrac{1}{2}$$

and set $\alpha = d - \varepsilon/2$. Define disjoint annuli

$$A_{N,1},\ A'_{N,1},\ A_{N,2},\ A'_{N,2},\ \cdots,\ A_{N,l_N},\ A'_{N,l_N},\ A'_{N,l_N+1}\ ,$$

and random variables

$$y_{N,j} = \sum_{z \in A_{N,j}} X_{N,z}$$

and

$$v_{N,j} = \sum_{z \in A'_{N,j}} X_{N,z}$$

by choosing $A_{N,j}$ to be the largest annulus outside $A'_{N,j-1}$ satisfying

$$E_N(y^2_{N,j}) \leqq N^{\alpha-d}$$

and $A'_{N,j}$ to be the annulus outside $A_{N,j}$ with radius

$$k_N = N^{1-\varepsilon} .$$

Because the $X'_{N,z}$ satisfy (M_2) and (3.3) we can show that $v_{N,j}$ is small enough to be neglected in the computation of $E_N(\exp(itX_N))$, while the $v_{N,j}$ separate the $y_{N,j}$ enough so that the $y_{N,j}$ behave approximately as independent random variables. In fact, a proof similar to that of Lemma (5) in [14] gives

$$E_N(y^2_{N,j}) = N^{\alpha-d}(1 + o(1)) ,$$
$$\sum_{j \leq l_N} E_N(y^2_{N,j}) = l_N N^{\alpha-d}(1 + o(1)) ,$$
$$l_N = O(N^{d-\alpha})$$

and

$$E_N(\sum_{j \leq l_N+1} v_{N,j})^2 = o(\Sigma_N^2) .$$

Therefore, setting

$$Y_N = \sum_{j \leq l_N} y_{N,j}$$

and following the proof of Lemma (6) in [14], we obtain

$$E_N(\exp(itY_N)) = \prod_{j \leq l_N} E_N(\exp(ity_{N,j})) + o(1)$$

and

$$E_N(\exp(itX_N)) = E_N(\exp(itY_N)) + o(1) .$$

So it suffices by Liapounov's theorem ([9], page 275) to show

$$\sum_{j \leq l_N} E_N(y^4_{N,j}) \to 0 .$$

Letting p_j be the number of summands in $y_{N,j}$, we can use (2.4) and Hölder's inequality to show

$$E_N(y^4_{N,j}) = O(p_j^{\,2} N^{-2d}) , \qquad p_j = O(N^\alpha) .$$

Thus

$$\sum_{j \leq l_N} E_N(y^4_{N,j}) = O(N^{d-\alpha} N^{2\alpha} N^{-2d}) = o(1) .$$

From results in [3], a result similar to (3.1) follows for the strictly stationary case. The method described here, however, works for processes where the mixing condition contains a factor related to the dimension d (see, e.g., [7], [10]). The method can also be used to obtain the analogue of Theorem 1 in [14], which states that if $(X_{N,Z})$ is regular and satisfies (M_1) and certain moment conditions then the partial sums X_N have as possible limits only those laws defined in [9, Theorem A, page 293] ([12]).

4. Convergence to a degenerate limit. In this section we give some conditions which imply X_N converges to 0.

(4.1) PROPOSITION. *Suppose* $(X'_{N,Z})$ *is any collection satisfying*

(4.2) (M_1) *holds with* $\int_1^\infty h^{d-1}\alpha_N^{\frac{1}{2}}(h)\,dh \leqq A < \infty$
and
$\max_{Z\in\Lambda_N} E_N(X'_{N,Z})^2 \leqq B < \infty$ *uniformly in* N,

or

(4.3) (M_2) *holds with* $\int_1^\infty h^{d-1}\alpha_N^{\frac{1}{2}}(h)\,dh \leqq A < \infty$
and
$\max_{Z\in\Lambda_N} E_N(X'_{N,Z})^4 \leqq B < \infty$ *uniformly in* N.

Fix $k > \frac{1}{2}$ *and set* $X_{N,Z} = X'_{N,Z}/|\Lambda_N|^k$. *Then* $\mathscr{L}(\sum_{Z\in\Lambda_N} X_{N,Z}) \to \mathscr{L}(0)$.

PROOF. We use Chebyshev's inequality and the standard fact that convergence in probability implies convergence in law. Thus it suffices to show that

$$E_N(\textstyle\sum_{Z\in\Lambda_N} X_{N,Z})^2 \to 0\,,$$

which is easily proved.

As in the independent case, if $X_N \to_P 0$ it is natural to ask whether $X_N \to_{\text{a.s.}} 0$.

(4.4) DEFINITION. Suppose (Λ_N) is a regular sequence. Set $\mathscr{F}_\infty = \bigcap_N \mathscr{F}_{\Lambda_N c}$.

In the independent case it is easily shown that $\mathscr{F}_\infty$ is trivial. The same result holds here.

(4.5) LEMMA. *Suppose* $(X_Z)_{Z\in Z^d}$ *is a multidimensional sequence of random variables defined on a probability space* $(\Omega, \mathscr{F}, P)$ *and satisfying* (M_2) *with* $\alpha_N(t) \equiv \alpha_1(t)$. *Let* (Λ_N) *be a regular sequence. Then the* σ*-field* $\mathscr{F}_\infty$ *defined by* (Λ_N) *is trivial.*

PROOF. The proof in the independent case is easily modified for the mixing case (see, e.g., [9], page 230).

Thus if (X_Z) is a multidimensional sequence satisfying (M_2) with $E(X_Z) = 0$ for $Z \in Z^d$ and $b_N \uparrow \infty$,

$$(\textstyle\sum_{Z\in\Lambda_N} X_Z)/b_N$$

converges or diverges almost surely. We give some sufficient conditions for convergence.

(4.6) PROPOSITION. *Suppose $(X_Z)_{Z\in Z^d}$ is a multidimensional sequence of random variables with $E(X_Z) = 0$ and $\max_{Z\in Z^d} E(X_Z^2) \leqq B$. Suppose (4.2) or (4.3) holds for a regular sequence (Λ_N) and $\alpha_N(h) = \alpha_1(h)$ for all N. Suppose $k \geqq 1$ if $d = 1$ and $k > \frac{1}{2}(1 + 1/d)$ if $d > 1$. Then*

$$X_N = (\textstyle\sum_{Z\in\Lambda_N} X_Z)/|\Lambda_N|^k \to_{\text{a.s.}} 0 .$$

PROOF. We may assume $|\Lambda_N| = N^d$. Proceeding as in the proof of (4.1), we find $E(X_N^2) = O(N^{-d(2k-1)})$. If $k > \frac{1}{2}(1 + 1/d)$, then $d(2k - 1) > 1$, and so

$$\textstyle\sum_{N=1}^{\infty} E(X_N^2) = E(\sum_{N=1}^{\infty} X_N^2) < \infty$$

and $\sum_{N=1}^{\infty} X_N^2$ is finite a.s., which implies $X_N \to 0$ a.s. For the case $d = 1$, $k = 1$ we are considering

$$X_N = \textstyle\sum_{j=-[N/2]}^{j=[N/2]} X_j/N .$$

It suffices to show $\sum_{j=0}^{[N/2]} X_j/N \to_{\text{a.s.}} 0$ and $\sum_{j=-[N/2]}^{-1} X_j/N \to_{\text{a.s.}} 0$.

Now by (4.2) or (4.3)

$$E(X_{[N/2]}) \textstyle\sum_{j=0}^{[N/2]} X_j/N = O(1/N) ,$$

and thus the desired result follows from Theorem 2B, page 420 of [13], with the number q defined in the statement of the theorem equal to 1. The same proof also shows

$$\textstyle\sum_{j=-[N/2]}^{-1} X_j/N \to_{\text{a.s.}} 0 .$$

5. Applications to Gibbs random fields. In this section we consider a type of physical model, the so-called Gibbs random field, to which the above results may be applied. Intuitively we may think of each site in Z^d being occupied by a particle which has "spin" $+1$ or -1. We assume some interaction among the sites which leads to a probability measure on the possible configurations in Z^d. Such processes have received a great deal of attention in recent years (see, e.g., [15] and [16]).

We set $\Omega = \{-1, 1\}^{Z^d}$ with $\mathscr{F}$ the σ-algebra generated by finite-dimensional cylinder sets. We let C denote the class of finite subsets of Z^d.

The interaction between the particles in our system will be specified by means of a potential function.

(5.1) DEFINITION. A *potential* is any map $\Phi\colon C \to \mathbb{R}$ such that $\Phi(\phi) = 0$.

Here we will assume

(5.2) $\Phi_\Lambda = \Phi_{\Lambda+Z}$ for $\Lambda \subset Z^d$, $Z \in Z^d$ (Φ is translation invariant)

and

(5.3) there exists an $r > 0$ such that $\Phi_\Lambda = 0$ if $\operatorname{diam}(\Lambda) > r$ (Φ is finite range).

(5.4) DEFINITION. A probability measure P on $(\Omega, \mathscr{F})$ will be called a *Gibbs state* for the potential Φ if for $Z \in Z^d$

$$P[\omega(Z) \mid \mathscr{M}_{Z^d-\{Z\}}] = [1 + \exp(2 \textstyle\sum_{\Lambda\ni Z} \Phi_\Lambda \prod_{y\in\Lambda} \omega_y)]^{-1}$$

is a regular conditional probability distribution for the "spin" at Z given the configuration on $Z^d - \{Z\}$.

Clearly if such a measure P exists and if the potential Φ has range r, then the resulting probability space is an r-Markov random field; that is, events inside of a finite set Λ, conditioned on the σ-field of events generated over the points at distance $\leqq r$ from Λ, are independent of what happens outside Λ. If $d = r = 1$ we are just dealing with a Markov chain with finite state space, and the conditional probabilities defined above ensure the existence of a unique stationary measure P on $(\Omega, \mathscr{F})$ which gives rise to those conditional probabilities. Even for $d = 2$, $r = 1$, however, it is possible that there exist two or more probability measures on $(\Omega, \mathscr{F})$ giving rise to the same conditional probabilities, and in some cases not all these measures need be stationary [16].

For sufficiently regular potentials Φ (see, e.g., [7]) it can be shown that there exists a unique Gibbs state on $(\Omega, \mathscr{F})$.

Let us define the multidimensional sequence $(X_Z)_{Z \in Z^d}$ on the probability space $(\Omega, \mathscr{F}, P)$ by

$$X_Z(\omega) = \omega(Z)\,, \qquad \omega \in \Omega\,. \tag{5.5}$$

It is believed that, at least in cases where the measure P is unique, the averages $\sum_{Z \in \Lambda_N} X_Z$, suitably normalized, should have a distribution which converges to that of a normal random variable with mean 0 and variance 1 as $\Lambda_N \uparrow Z^d$. In [4] and [10] such a theorem has been proved for certain finite range pair potentials ($\Phi_\Lambda = 0$ if $|\Lambda| > 2$), while in [1] and [11] some central limit theorems have been shown to hold under rather stringent conditions. In the central limit theorem proved here, we have relaxed certain of those conditions. We have also made clear a sufficient set of conditions which would guarantee that an arbitrary random field satisfy the central limit theorem.

We use a mixing property of P which is proved in [7].

(5.6) THEOREM ([7], Theorem (4.24)). *Suppose Φ is a potential satisfying* (5.2), (5.3), *and*

$$|\Delta[1 + \exp(2 \textstyle\sum_{\Lambda \ni 0} J_\Lambda \prod_{Z \in \Lambda} \omega(Z))]^{-1}| < 1\,,$$

where the operator Δ is defined on functions on Ω by

$$\begin{aligned} \hat{\omega}(Z') &= \omega(Z')\,, \quad Z' \neq 0 \\ &= -\omega(0)\,, \quad Z' = 0\,, \\ \Delta f(\omega) &= f(\hat{\omega}) - f(\omega)\,. \end{aligned}$$

Then there exists a unique Gibbs state P with potential Φ. Also, there exist $K > 0$, $B > 0$, such that, for each $\Lambda_1 \in C$ and $\Lambda_2 \subset Z^d - \Lambda_1$, (M_2) is satisfied with

$$\alpha_N(h) = Ke^{-Bh}\,.$$

Now clearly

$$\max_{Z \in \Lambda_N} \|X_Z\|_\infty = 1$$

and so if (Λ_N) is a regular sequence we may apply the results from Sections 3 and 4. We will state these for completeness.

(5.7) THEOREM. *Suppose* Λ_N *is a regular sequence and the hypotheses of* (5.6) *are satisfied. Then*

(i) *for* $k > \frac{1}{2}$,

$$X_N = \left(\textstyle\sum_{Z \in \Lambda_N} (X_Z - EX_Z)\right)/|\Lambda_N|^k \to_P 0\,,$$

(ii) *for any sequence* $b_N \to \infty$,

$$X_N = \left(\textstyle\sum_{Z \in \Lambda_N} (X_Z - EX_Z)\right)/b_N$$

converges or diverges almost surely, and

(iii) *for* $k \geqq 1$ *if* $d = 1$ *and* $k > \frac{1}{2}(1 + 1/d)$ *if* $d > 1$,

$$X_N = \left(\textstyle\sum_{Z \in \Lambda_N} (X_Z - EX_Z)\right)/|\Lambda_N|^k \to_{\text{a.s.}} 0\,.$$

PROOF. The statements are immediate from (4.1), (4.5), and (4.6), respectively.

A different proof of (iii) for the case $k = 1$ is indicated in [16].

To apply our results on convergence to a normal law, we note that the second part of (3.2) follows from the translation invariance of the measure P ([5]). We also need the following, which is explained in detail in [6]:

(5.8) DEFINITION. The potential Φ is a *strongly superadditive potential* (s.s.a.p.) if whenever Λ is a finite subset of Z^d and ω_1 and ω_2 are configurations on Z^d with $\omega_1 \geqq \omega_2$ on $Z^d - \Lambda$, then

$$\Phi(\omega_1') + \Phi(\omega_2') \geqq \Phi(\omega_1) + \Phi(\omega_2)$$

where

$$\begin{aligned} \omega_1' &= \omega_1 \vee \omega_2 && \text{on } \Lambda \\ &= \omega_1 && \text{on } Z^d - \Lambda \end{aligned}$$

and

$$\begin{aligned} \omega_2' &= \omega_1 \wedge \omega_2 && \text{on } \Lambda \\ &= \omega_2 && \text{on } Z^d - \Lambda\,. \end{aligned}$$

For example, if $\Phi_\Lambda \geqq 0$, $|\Lambda| > 1$ and $\Phi_\Lambda = 0$, $|\Lambda| > 2$, then Φ is a s.s.a.p. Or, if for $\Lambda = \{Z_1, Z_2\}$, $Z_1 \neq Z_2$, we have

$$\Phi_\Lambda \geqq \textstyle\sum_{Z \notin \Lambda} |\Phi_{\Lambda \cup \{Z\}}|$$

and $\Phi_\Lambda = 0$, $|\Lambda| > 3$, then Φ is a s.s.a.p. This notion is also discussed in [15], where such a potential is called supermodular.

Now from the FKG inequalities it follows that if Φ is a s.s.a.p. then

$$EX_{Z_1}X_{Z_2} \geqq EX_{Z_1}EX_{Z_2}\,,$$

and therefore (3.2) would hold.

Now by (5.3) X_Z is not a.s. constant and thus the following result on convergence to a normal law is an immediate consequence of Theorem (3.1):

(5.9) THEOREM. *If* (Λ_N) *is a regular sequence and* Φ *is a* s.s.a.p. *satisfying the hypotheses of* (5.6), *then*

$$\mathscr{L}(\sum_{Z\in\Lambda_N}(X_Z - EX_Z)/(E(\sum_{Z\in\Lambda_N}(X_Z - EX_Z)^2))^{\frac{1}{2}}) \to \mathscr{N}(0, 1)\,.$$

PROOF. In the proof of (3.1) we need only set

$$X'_{N,Z} = X_Z - EX_Z$$

for $Z \in \Lambda_N$ and

$$\alpha_N(h) = Ke^{-Bh}$$

for all N.

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