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CONTINUOUS PARAMETER MARKOV PROCESSES

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In this paper, we obtain a continuous parameter generalization of Doeblin's and Harris' theory of Markov processes.

0. Introduction. In 1940 Doeblin (1940) introduced a theory of discrete parameter Markov processes which involved the movement of the processes. Chung (1963), Jain and Jamison (1967), and Winkler (1975) have shown the involvement of Doeblin's theory with Harris' theory of Markov processes. In this paper we introduce a theory of continuous parameter Markov processes analogous to Doeblin's theory and show that the theory can be reduced to considering the discrete parameter processes induced by the continuous parameter process which greatly simplifies the study of continuous parameter Markov processes. A similar situation occurs in continuous time Markov branching processes which induce discrete time Galton-Watson processes (see, e.g., Harris (1963), Athreya and Ney (1972)).

There are several important advantages to our approach. We neither assume that we have a Hunt process with its inherent assumptions of the strong Markov property and quasi-left continuity (both of which are difficult to verify for a given Markov process) nor do we assume the existence of a reference measure (see, e.g., Blumenthal and Getoor (1968), pages 196–197).

The first section is devoted to notation, definitions, and preliminary results. In the second section we prove the main decomposition theorem which has discrete parameter analogues (see, e.g., Chung (1963), Jain and Jamison (1967), and Winkler (1975)).

1. Notation and preliminary results. Let $(E, \mathscr{E}, \mathscr{E}')$ be a locally compact separable topological measure space; that is, $\mathscr{E}'$ is locally compact, Hausdorff topological space with countable base and $\mathscr{E}$ is the σ-field generated by $\mathscr{E}'$. Let X_t, $0 \leqq t < \infty$, be a Markov process taking values in E. For each $t > 0$, the transition probability $P_t(\cdot, \cdot)$ has the following properties:

(i) for each $x \in E$, $P_t(x, \cdot)$ is a probability measure on $\mathscr{E}$;
(ii) for each $B \in \mathscr{E}$, $P_t(\cdot, B)$ is an $\mathscr{E}$-measurable function.

In addition, the transition probabilities $P_t(\cdot, \cdot)$, $0 \leqq t < \infty$, satisfy the Chapman-Kolmogorov equations:

$$P_{t+s}(x, A) = \int P_t(x, dy) P_s(y, A)$$

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for all s and $t \in \mathbb{R}_+$, $x \in E$, and $A \in \mathscr{E}$. We make the assumption that $P_0(x, B) = \delta_x(B)$ where

$$\delta_x(B) = 1 \quad \text{if} \quad x \in B$$
$$\qquad = 0 \quad \text{if} \quad x \notin B\,.$$

For each $t \in \mathbb{R}_+$, set $E_t = E$ and $\mathscr{E}_t = \mathscr{E}$. Let $(\Omega, \mathscr{B}) = (\prod_t E_t, \bigotimes_t \mathscr{E}_t)$ be the product space of $\{(E_t, \mathscr{E}_t)\colon t \in \mathbb{R}_+\}$ where $\bigotimes_t \mathscr{E}_t$ is the σ-field generated by the finite cylinder sets. For each probability measure μ on $\mathscr{E}$ we assume that we have a probability measure $\mathbb{P}_\mu$ on $(\Omega, \mathscr{B})$ such that for $0 = t_1 < t_2 < \cdots < t_n$ and $B_i \in \mathscr{E}_i = \mathscr{E}$, $i = 1, 2, \cdots, n$, we have

$$\mathbb{P}_\mu(X_{t_1} \in B_1, X_{t_2} \in B_2, \cdots, X_{t_n} \in B_n)$$
$$= \int_{B_1} \mu(dx_{t_1}) \int_{B_2} P_{t_2 - t_1}(x_{t_1}, dx_{t_2}) \cdots \int_{B_n} P_{t_n - t_{n-1}}(x_{t_{n-1}}, dx_{t_n})$$

(see, e.g., Neveu (1965), page 83 or Rosenblatt (1971), page 236). In the particular case in which $\mu(\cdot) = \delta_x(\cdot)$ we write $\mathbb{P}_x$ for $\mathbb{P}_\mu$.

We assume that $\{X_t\}$, $0 \leqq t < \infty$, is right continuous and has left limits (that is, $\lim_{t\to s+} X_t(\omega) = X_s(\omega)$ for $\omega \in \Omega$ and $\lim_{t\to s-} X_t(\omega)$ exists for $\omega \in \Omega$). The right continuity of $\{X_t\}$, $0 \leqq t < \infty$, implies that $X_t(\omega)$ is bimeasurable in (t, ω). In addition we assume that

$$(\mathscr{C}) \qquad \text{for each} \quad x \in E\,, \quad B \in \mathscr{E}\,, \quad s > 0\,, \quad \lim_{t\to s} P_t(x, B) = P_s(x, B)\,.$$

If the state space is countable, then $(\mathscr{C})$ is true without loss of generality if $X_t(\omega)$ is assumed to be bimeasurable in (t, ω). Indeed, by a remark on page 41 of Blumenthal and Getoor (1965), $P_t(x, \{y\})$ is bimeasurable in (t, x) for every $y \in E$, and by Theorem 1 of Chung (1967), page 120, for every $x \in E$, $B \in \mathscr{E}$, and $\delta > 0$, $P_t(x, B)$ is uniformly continuous on $[\delta, \infty)$.

For each $\alpha > 0$, $x \in E$, and $B \in \mathscr{E}$, we set

$$L_\alpha(x, B) \equiv \mathbb{P}_x(\textstyle\bigcup_{n=1}^\infty \{X_{n\alpha} \in B\})$$

and

$$Q_\alpha(x, B) \equiv \mathbb{P}_x(\textstyle\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \{X_{k\alpha} \in B\})\,.$$

$L_\alpha(x, B)$ is the probability that, starting at x, the process $\{X_{n\alpha}\}_{n=1}^\infty$ ever hits B and $Q_\alpha(x, B)$ is the probability that, starting at x, the process $\{X_{n\alpha}\}_{n=1}^\infty$ hits B infinitely often.

DEFINITION 1. A nonempty set $B \in \mathscr{E}$ is called *P_α-stochastically closed, or P_α-closed*, if $P_\alpha(x, B) = 1$ for $x \in B$. A P_α-closed set $B \in \mathscr{E}$ is called *P_α-indecomposable* if it does not contain a disjoint pair of P_α-closed sets; otherwise it is called *P_α-decomposable*.

One of the principal advantages of considering P_α-closed sets is that the Markov process $\{X_{n\alpha}\}_{n=1}^\infty$ restricted to a P_α-closed set $C \in \mathscr{E}$ agrees with the original Markov process (see, e.g., Rosenblatt (1971), pages 50–53).

DEFINITION 2. A set $B \in \mathscr{E}$ such that $Q_\alpha(x, B) = 0$ for all $x \in E$ is called P_α-*inessential*, otherwise it is called P_α-*essential*. A P_α-essential set which is a countable union of P_α-inessential sets is called P_α-*improperly essential*; otherwise it is called P_α-*absolutely essential*.

DEFINITION 3. Let C be a P_α-closed set and let Φ be a σ-finite measure on $(C, \mathscr{E})$ with $\Phi(C) > 0$. Then C is called Φ-α-*recurrent* if $Q_\alpha(x, F) = 1$ for all $x \in C$ whenever $F \subseteqq C$ and $\Phi(F) > 0$. If C is Φ-α-recurrent for some Φ, then C is called *α-recurrent in the sense of Harris* (see Orey (1971), Harris (1956)).

THEOREM 1.1 (Harris (1956)). *Let C be a P_α-closed set such that C is Φ-α-recurrent. Then there exists a unique (up to multiplicative constant) σ-finite measure π on C satisfying*

(i) $\Phi \ll \pi(\pi(B) = 0$ *implies* $\Phi(B) = 0)$,
(ii) $\pi(B) = \int_C \pi(dx) P_{\alpha n}(x, B)$ *for* $B \subseteqq C$ *and* $n \geqq 0$, *and*
(iii) $\pi(B) > 0$ *implies* $Q_\alpha(x, B) = 1$ *for all* $x \in C$.

The measure π is called the α-Harris measure on C and has the following property:

LEMMA 1.1 (Jain (1966), page 209). *Let H be α-recurrent in the sense of Harris and π be the α-Harris measure on H. Let $B \subseteqq H$. $\pi(B) = 0$ if and only if B is P_α-inessential.*

Lemma 1.1 immediately implies that H can contain only P_α-absolutely essential sets or P_α-inessential sets. If D is a P_α-absolutely essential subset of H, then, by Theorem 1.1 and Lemma 1.1, $Q_\alpha(x, D) = 1$ for all $x \in H$.

If a set H is α-recurrent in the sense of Harris, then H is P_α-absolutely essential and P_α-indecomposable (see, e.g., Orey (1971), page 36). A set D which is α-recurrent in the sense of Harris is said to be *maximal α-Harris* if D_1 is any set which is α-recurrent in the sense of Harris and which contains D, then $D = D_1$. The next four lemmas follow from basic facts from Doeblin's theory of Markov processes (see, e.g., Orey (1971), Chung (1964)).

LEMMA 1.2. *Let H be α-recurrent in the sense of Harris. Then $D \equiv \{x: Q_\alpha(x, H) = 1\}$ is maximal α-Harris.*

LEMMA 1.3. *Let H be α-recurrent in the sense of Harris and let C be a P_α-closed subset of H. Then C is α-recurrent in the sense of Harris and consequently, P_α-absolutely essential.*

LEMMA 1.4. *Let H_1 and H_2 be two maximal α-Harris sets. Either $H_1 = H_2$ or $H_1 \cap H_2 = \emptyset$.*

LEMMA 1.5. *Let C be a P_α-closed set contained in a set D such that $x \in D \backslash C$ implies $L_\alpha(x, C) = 1$. Then $D \backslash C$ is P_α-inessential.*

REMARK 1.1. Let H be α-recurrent in the sense of Harris and let C be a P_α-closed set containing H such that $C \backslash H$ is P_α-inessential. Then C is α-recurrent in the sense of Harris.

Doeblin (1940) proved (see also Orey (1971), page 46):

THEOREM 1.2. *Let $(X_n)_{n=1}^{\infty}$ be a discrete parameter Markov process and let there exist a finite measure $m(\cdot)$ on $\mathscr{E}$ such that if $C \in \mathscr{E}$ is P_1-closed, then $m(C) > 0$. Then there exists a disjoint decomposition $E = (\bigcup_{n=1}^{\infty} C_n) \cup I$ such that I is either P_1-inessential or P_1-improperly essential and each C_n is a P_1-absolutely essential and P_1-indecomposable closed set.*

REMARK 1.2. Doeblin's theorem was proved on an arbitrary measure space $(E, \mathscr{E})$ where the σ-field $\mathscr{E}$ satisfies no topological or separability conditions. Under the additional assumption that the σ-field $\mathscr{E}$ is separable (i.e., $\mathscr{E}$ is countably generated), Jain and Jamison ((1967), page 29), proved that each $C_n = H_n \cup J_n$ where each H_n is P_1-recurrent in the sense of Harris and J_n is P_1-improperly essential or P_1-inessential. Thus, the decomposition of Theorem 1.2 becomes $E = J \cup (\bigcup_{n=1}^{\infty} H_n)$ where $J \equiv I \cup (\bigcup_{n=1}^{\infty} J_n)$.

If we assume that there exists a σ-finite measure $m(\cdot)$ on $\mathscr{E}$ such that, for some fixed $\alpha > 0$, if C is P_α-closed, then $m(C) > 0$, then we have the following condition:

($\mathscr{N}$) For some fixed $\alpha > 0$, there exists no uncountable disjoint collection of P_α-closed sets.

Winkler ((1975), Theorem 2) proved:

THEOREM 1.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a discrete parameter Markov process. If there exists no uncountable disjoint collection of P_1-closed sets, then there exists a disjoint decomposition $E = I \cup (\bigcup_{n=1}^{\infty} D_n)$ such that I is either P_1-inessential or P_1-improperly essential and each D_n is a P_1-absolutely essential and P_1-indecomposable closed set.*

REMARK 1.3. Under the additional assumption that the σ-field $\mathscr{E}$ is separable, we obtain the disjoint decomposition $E = J \cup (\bigcup_{n=1}^{\infty} H_n)$ mentioned in Remark 1.2.

REMARK 1.4. If C is any P_α-closed set and condition ($\mathscr{N}$) holds on subsets of C (in particular ($\mathscr{N}$) holds if there exists a σ-finite measure π which assigns positive measure to all P_α-closed subsets of C), then, by considering the Markov process restricted to C, C may be decomposed in the following manner: $C = I^\alpha \cup (\bigcup_{n=1}^{\infty} K_n{}^\alpha)$ where I^α is either P_α-inessential or P_α-improperly essential and each $K_n{}^\alpha$ is α-recurrent in the sense of Harris.

REMARK 1.5. Under condition ($\mathscr{C}$) if there exists a σ-finite measure μ on $(E, \mathscr{E})$ such that $\mu(A) = 0 \Leftrightarrow \int_0^\infty P_t(x, A)\, dt = 0$ for all $x \in E$, then condition ($\mathscr{N}$) holds. Indeed, if C is P_α-closed, then for any $x \in C$, $\int_0^\infty P_t(x, C)\, dt > 0$ which implies that $\mu(C) > 0$. For our purposes the above condition is effectively equivalent to the existence of a reference measure (see, e.g., Blumenthal and Getoor (1968), pages 196–197 or Meyer (1962), page 160).

DEFINITION 4. We say that a set $T \in \mathscr{E}$ is *P_α-transient* if there exists a real number $M = M(T)$ such that $\sum_{k=1}^{\infty} P_{k\alpha}(x, T) \leqq M$ for all $x \in E$. We say that a set $S \in \mathscr{E}$ is *σ-P_α-transient* if it is contained in a countable union of P_α-transient sets.

PROPOSITION 1.1 (Abrahamse (1971), page 220). *For some $\alpha > 0$, let I be a P_α-inessential set. Then I is σ-P_α-transient.*

COROLLARY 1.1. *Let $T \in \mathscr{E}$. Then T is σ-P_α-transient if and only if T is not P_α-absolutely essential.*

PROOF. Since every P_α-transient set is P_α-inessential, the conclusion of Corollary 1.1 follows from Proposition 1.1 and Definition 2.

By Lemma 1.2, Lemma 1.4, and Corollary 1.1, the decomposition of the state space E as given in Remark 1.2 may be considered in the following form:

($\mathscr{H}$) For some $\alpha > 0$, there exists a countable disjoint collection $\{H_n^\alpha : n \geqq 1\}$ each of which is maximal α-Harris such that $E = J^\alpha \cup (\bigcup_{n=1}^\infty H_n^\alpha)$ where J^α is σ-P_α-transient.

In Section 2 we will prove:

THEOREM 2.1. *Assume ($\mathscr{C}$) holds. Then ($\mathscr{N}$) $\Rightarrow$ ($\mathscr{H}$) and the decomposition is independent of $\alpha > 0$.*

REMARK 1.6. This theorem generalizes the decomposition of Jain and Jamison mentioned in Remark 1.2. In addition, it generalizes results for Markov chains given by Chung ((1967), Section II.10).

2. Main decomposition theorem. The important feature of Theorem 2.1 is not that the decomposition ($\mathscr{H}$) exists, but that the decomposition is independent of $\alpha > 0$. Throughout this section we will assume that condition ($\mathscr{N}$) holds. The proof that the decomposition ($\mathscr{H}$) is independent of $\alpha > 0$ will contain three key steps:

(I) if T is σ-P_α-transient, then T is σ-P_β-transient for all $\beta > 0$;
(II) if H is maximal α-Harris, then H is $P_{\alpha/n}$-closed for all $n \geqq 1$, and
(III) if H is maximal α-Harris, then H is maximal t-Harris for all $t > 0$.

LEMMA 2.1 (Chung (1964), page 239). *Let $\beta > 0$ and let k be a positive integer. A set B is $P_{k\beta}$-inessential, $P_{k\beta}$-improperly essential, or $P_{k\beta}$-absolutely essential according to whether B is P_β-inessential, P_β-improperly essential, or P_β-absolutely essential.*

LEMMA 2.2. *If T is P_α-transient, then T is σ-P_β-transient for all $\beta > 0$.*

PROOF. Let $\alpha > 0$ be fixed and T be P_α-transient. For each $x \in E$, $j \geqq 0$, and $r > 0$, we have that

$$P_{j\alpha+r}(x, T) = \int P_r(x, dy) P_{j\alpha}(y, T) . \tag{2.1}$$

Using (2.1) we have for all positive integers N that

$$\begin{aligned} \int_0^{(N+1)\alpha} P_t(x, T)\, dt &= \sum_{j=0}^N \int_0^\alpha \int P_r(x, dy) P_{j\alpha}(y, T)\, dr \\ &= \int \left(\int_0^\alpha P_r(x, dy)\, dr\right)\left(\sum_{j=0}^N P_{j\alpha}(y, T)\right) . \end{aligned} \tag{2.2}$$

From (2.2) we obtain that

$$\int_0^\infty P_t(x, T)\,dt = \int \left(\int_0^\alpha P_r(x, dy)\,dr\right)\left(\sum_{j=0}^\infty P_{j\alpha}(y, T)\right) \tag{2.3}$$
$$\leqq \int \left(\int_0^\alpha P_r(x, dy)\,dr\right) M(T) \leqq \alpha M(T) .$$

We note that for each $x \in E$ $\int_0^\alpha P_r(x, dy)\,dr$ is a well-defined measure uniformly bounded by α.

Let $\beta > 0$ and let $C \equiv \{x \mid L_\beta(x, T) = 0\}$. If $x \in C^c$, then $\sum_{k=1}^\infty P_{k\beta}(x, T) > 0$ and using condition ($\mathscr{C}$) we obtain that $\int_0^\infty P_t(x, T)\,dt > 0$. Since $T = (T \cap C) \cap (T \cap C^c)$ and $T \cap C$ is trivially P_β-transient, we must show that $T \cap C^c$ is σ-P_β-transient. For $n = 0, 1, 2, \cdots$, let $R_n \equiv \{x \in T \cap C^c \mid \int_0^{n\beta} P_t(x, T)\,dt = 0$ and $\int_0^{(n+1)\beta} P_t(x, T)\,dt > 0\}$. Since $T \cap C^c = \bigcup_{n=0}^\infty R_n$ we must show that each R_i is σ-P_β-transient. Let $S_n = \{x \in T \cap C^c \mid \int_0^\beta P_t(x, T)\,dt \geqq 1/n\}$ for $n = 1, 2, \cdots$. Then $R_0 = \{x \in T \cap C^c\colon \int_0^\infty P_t(x, T)\,dt > 0\} = \bigcup_{n=1}^\infty S_n$. Using previous reasoning we have that

$$\int_0^{(N+1)\beta} P_t(x, T)\,dt = \sum_{j=0}^N \int_0^\beta \int P_{j\beta}(x, dy) P_r(y, T)\,dr$$
$$= \int \left(\int_0^\beta P_r(y, T)\,dr\right)\left(\sum_{j=0}^N P_{j\beta}(x, dy)\right) \tag{2.4}$$
$$\geqq \int_{S_n} \left(\int_0^\beta P_r(y, T)\,dr\right)\left(\sum_{j=0}^N P_{j\beta}(x, dy)\right)$$
$$\geqq \left(\frac{1}{n}\right) \sum_{j=0}^N P_{j\beta}(x, S_n) .$$

From (2.4) we obtain that

$$\sum_{j=0}^\infty P_{j\beta}(x, S_n) \leqq n\alpha M(T) \qquad \text{for all} \quad x \in E . \tag{2.5}$$

Therefore, each S_n is P_β-transient and R_0 is σ-P_β-transient.

For $n = 1, 2, \cdots$, let $T_n = \{x \in R_1 \mid \int_0^{2\beta} P_t(x, T)\,dt \geqq 1/n\}$. Then $R_1 = \bigcup_{n=1}^\infty T_n$. Reasoning as in equation (2.4) we obtain that

$$\int_0^{(N+1)2\beta} P_t(x, T)\,dt \geqq \int_{T_n} \left(\int_0^{2\beta} P_r(y, T)\,dr\right) \sum_{j=0}^N P_{j2\beta}(x, dy)) \tag{2.6}$$
$$\geqq \left(\frac{1}{n}\right) (\sum_{j=0}^N P_{j2\beta}(x, T_n) .$$

From (2.3) and (2.6) we obtain that

$$\sum_{j=0}^\infty P_{j2\beta}(x, T_n) \leqq n\alpha M(T) \qquad \text{for all} \quad x \in E . \tag{2.7}$$

Since

$$\sum_{j=0}^\infty P_{(2j+1)\beta}(x, T_n) = \int P_\beta(x, dy)\left(\sum_{j=0}^\infty P_{j2\beta}(y, T_n)\right) \leqq n\alpha M(T) ,$$

we obtain that

$$\sum_{j=0}^\infty P_{j\beta}(x, T_n) \leqq 2n\alpha M(T) .$$

Therefore, each T_n is P_β-transient and R_1 is σ-P_β-transient. In a similar manner we may show that for each $j \geqq 2$ R_j is σ-P_β-transient.

REMARK 2.1. Using Lemma 2.2 and Corollary 1.1 we obtain that a set is P_α-absolutely essential for some $\alpha > 0$, then the set is P_t-absolutely essential for all $t > 0$. If a set J is P_α-inessential for some $\alpha > 0$, then, by Proposition 1.1

and Lemma 2.2, J is σ-P_t-transient for all $t > 0$. However, we cannot in general conclude that J is P_t-inessential. This distinction shows the importance of Lemma 2.1.

The next two lemmas deal with the aperiodicity of the Markov process and are the key to making the decomposition in Theorem 2.1 independent of $\alpha > 0$. If B is a subset of a set H which is maximal α-Harris and if $B_1 \equiv \{x: Q_\alpha(x, B) = 1\}$, then $B_1 \subseteqq H$. In other words, H contains all points x, starting from which the process $\{X_{k\alpha}\}_{k=1}^\infty$ hits B infinitely often with probability one.

LEMMA 2.3. *Fix $\gamma > 0$ and assume that H is maximal γ-Harris. Then H is $P_{\gamma/n}$-closed for all positive integers n.*

PROOF. Fix $n \geqq 1$ and $x_1 \in H$. Let $\frac{1}{2} > \varepsilon > 0$. Using condition ($\mathscr{C}$), we have that there exists $\delta > 0$ such that $|h| < \delta$ implies

$$|P_\gamma(x_1, H) - P_{\gamma+h}(x_1, H)| < \varepsilon . \tag{2.8}$$

Choose a multiple m of n sufficiently large so that $\gamma/m < \delta$. For notational convenience we set $\beta = \gamma/m$. For $i = 0, 1, \cdots, m-1$, set $N_i^\infty \equiv \bigcap_{j=1}^\infty \bigcup_{k=j}^\infty \{X_{(km+i)\beta} \in H\}$ and set $A_i = \{x: \mathbb{P}_x(N_i^\infty) = 1\}$. Since $A_0 = H$ we have that $A_0 \neq \varnothing$. Let $y \in A_0$. Then

$$\begin{aligned} 1 = \mathbb{P}_y(N_0^\infty) &= \int P_\beta(y, dz)\mathbb{P}_z(\textstyle\bigcap_{j=1}^\infty \bigcup_{k=j}^\infty \{X_{(km-1)\beta} \in H\}) \\ &= \int P_\beta(y, dz)\mathbb{P}_z(N_{m-1}^\infty) \\ &= \int_{E\setminus A_{m-1}} P_\beta(y, dz)\mathbb{P}_z(N_{m-1}^\infty) + \int_{A_{m-1}} P_\beta(y, dz)\mathbb{P}_z(N_{m-1}^\infty) . \end{aligned}$$

If $P_\beta(y, E\setminus A_{m-1}) \neq 0$, then the sum of the last two integrals would be strictly less than 1. Hence, $P_\beta(y, E\setminus A_{m-1}) = 0$ and $P_\beta(y, A_{m-1}) = 1$. In a similar manner we can show that if $y \in A_i$, $1 \leqq i \leqq m-1$, then $P_\beta(y, A_{i-1}) = 1$. Each A_i is P_γ-closed and for each $y \in A_0$, we have that $P_{\gamma+\beta}(y, A_{m-1}) = 1$.

Using (2.8) and the fact that $\beta = \gamma/m < \delta$ we have that

$$|P_\gamma(x_1, A_0) - P_{\gamma+\beta}(x_1, A_0)| < \varepsilon . \tag{2.9}$$

Now (2.9) and the fact that $P_\gamma(x_1, A_0) = 1$ imply that $P_{\gamma+\beta}(x_1, A_0) > \frac{1}{2} > 0$. If $A_{m-1} \cap A_0 = \varnothing$, then

$$P_{\gamma+\beta}(x_1, A_{m-1} \cup A_0) = P_{\gamma+\beta}(x_1, A_{m-1}) + P_{\gamma+\beta}(x_1, A_0) > 1 ,$$

which is a contradiction. Hence $A_0 \cap A_{m-1} \neq \varnothing$. Now $A_0 \cap A_{m-1}$ is a P_γ-closed subset of $A_0 = H$ which is γ-recurrent in the sense of Harris. Fix $x \in A_0$. Then $Q_\gamma(x, A_0 \cap A_{m-1}) = 1$ which implies that $Q_\gamma(x, A_{m-1}) = 1$.

Let $\tau = \inf\{k \geqq 0: X_{k\gamma} \in A_{m-1}\}$. Then $\mathbb{P}_x(\tau < \infty) = 1$. Now we have that

$$\begin{aligned} \mathbb{P}_x(N_{m-1}^\infty) &= \textstyle\sum_{k=0}^\infty \mathbb{P}_x(N_{m-1}^\infty \cap \{\tau = k\}) \\ &= \textstyle\sum_{k=0}^\infty \mathbb{P}_x(N_{m-1}^\infty \cap \{X_{k\gamma} \in A_{m-1}, X_{(k-1)\gamma} \in A_{m-1}^c, \cdots, X_0 \in A_{m-1}^c\}) \\ &= \textstyle\sum_{k=0}^\infty \int_{\{y_k \in A_{m-1}, \cdots, y_0 \in A_{m-1}^c\}} \mathbb{P}_x(N_{m-1}^\infty \mid X_{k\gamma} = y_k, \cdots, X_0 = y_0) \\ &\qquad \times \mathbb{P}_x(X_{k\gamma} \in dy_k, \cdots, X_0 \in dy_0) \\ &= \textstyle\sum_{k=0}^\infty \int_{\{y_k \in A_{m-1}, \cdots, y_0 \in A_{m-1}^c\}} 1 \cdot \mathbb{P}_x(X_{k\gamma} \in dy_k, \cdots, X_0 \in dy_0) \\ &= \textstyle\sum_{k=0}^\infty \mathbb{P}_x(\tau = k) = 1 . \end{aligned}$$

For the third equality, see Breiman (1968), page 131. The fourth equality follows from the fact that $N_{m-1}^{\infty} = \bigcap_{j=r}^{\infty} \bigcup_{k=j}^{\infty} \{X_{(km-1)\beta} \in H\}$ for all $r \geqq 2$ and the fact that

$$\begin{aligned}\mathbb{P}_x(N_{m-1}^{\infty} \mid X_{k\gamma} = y_k, \cdots, X_0 = y_0) &= \mathbb{P}_x(N_{m-1}^{\infty} \mid X_{k\gamma} = y_k) \\ &= \mathbb{P}_{y_k}(N_{m-1}^{\infty}) = 1 \qquad \text{for all} \quad y_k \in A_{m-1}\,.\end{aligned}$$

Hence, $x \in A_{m-1}$ which implies that $A_0 \subseteqq A_{m-1}$. By inductively reasoning as above we may show that $A_0 \subseteqq A_i$, $1 \leqq i \leqq m-2$. Consequently $H = A_0 = \bigcap_{i=0}^{m-1} A_i$ is P_β-closed.

LEMMA 2.4. *Let H be maximal α-Harris. If, for all $t > 0$, C is a P_t-closed subset of H, then $J \equiv H \backslash C$ is P_t-inessential for all $t > 0$.*

PROOF. Fix t such that $0 < t \leqq \alpha$. Without loss of generality we may assume that $t \leqq \alpha$ since, by Lemma 2.1, J is P_t-inessential if and only if J is P_{mt}-inessential for all integers $m \geqq 1$. Each interval $[kt, (k+1)t]$ contains at most one $n\alpha$.

Let $z \in J$. Since C has positive π_α-measure, we have that $Q_\alpha(z, C) = 1$ which implies that $L_\alpha(z, C) = 1$. Let $\tau_\alpha \equiv \inf\{n \geqq 1 : X_{n\alpha} \in C\}$ and let $\tau_t \equiv \inf\{k \geqq 1 : X_{kt} \in C\}$. τ_α and τ_t are, respectively, the first entrance times of the processes $\{X_{n\alpha}\}_{n=1}^{\infty}$ and $\{X_{kt}\}_{k=1}^{\infty}$ into C.

Fix $n_0 \geqq 1$. Then there exists $k_0 \geqq 1$ such that $n_0\alpha \in [k_0 t, (k_0+1)t]$. Using the fact that C is P_s-closed for all $s > 0$ and the Chapman-Kolmogorov equations we obtain that

$$\mathbb{P}_z(X_{n_0\alpha} \in C) \leqq \mathbb{P}_z(X_{(k_0+1)t} \in C)\,. \tag{2.10}$$

In addition we obtain that

$$\begin{aligned}\mathbb{P}_z(\tau_\alpha \leqq n_0) &= \mathbb{P}_z(X_{n_0\alpha} \in C) \qquad \text{and} \\ \mathbb{P}_z(X_{(k_0+1)t} \in C) &= \mathbb{P}_z(\tau_t \leqq k_0 + 1)\,.\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) and letting both n_0 and $k_0 \nearrow \infty$ we obtain that

$$1 = L_\alpha(z, C) = \mathbb{P}_z(\textstyle\bigcup_{n=1}^{\infty} \{\tau_\alpha = n\}) \leqq \mathbb{P}_z(\bigcup_{k=1}^{\infty} \{\tau_t = k\}) = L_t(z, C)\,.$$

Since $z \in J$ is arbitrary, we have by Lemma 1.5 that J is P_t-inessential.

LEMMA 2.5. *Let H be maximal α-Harris and let π_α be the α-Harris measure on H. Then for each $t > 0$, H is maximal t-Harris.*

PROOF. By Remark 2.1, H is P_t-absolutely essential for all $t > 0$. Fix $t > 0$. First we will show that every P_t-closed subset of H is P_t-absolutely essential. If not, then there exists a P_t-closed subset C of H which is P_t-improperly essential. For each $n \geqq 0$ let $D_n \equiv \{x \in H : Q_{t/2^n}(x, C) = 1\}$. Then we have that $C \subseteqq D_0 \subseteqq D_1 \subseteqq \cdots \subseteqq H$ and $D \equiv \bigcup_{n=0}^{\infty} D_n$ is $P_{t/2^n}$-closed for all $n \geqq 0$ and by Proposition 9 of Chung (1964) each D_n is $P_{t/2^n}$-improperly essential. Thus, D is P_t-improperly essential. Using condition ($\mathscr{C}$) D is P_s-closed for all $s > 0$ and using Lemma 2.4 $H \backslash D$ is P_s-inessential for all $s > 0$. By Remark 2.1 D is not P_α-absolutely essential.

Thus $H = D \cup (H \backslash D)$ is not P_α-absolutely essential, which is a contradiction. Hence, every P_t-closed subset of H is P_t-absolutely essential.

Since π_α assigns positive measure to each P_t-closed subset of H there exists at most a countable disjoint collection of P_t-closed subsets of H. By considering the Markov process restricted to H as mentioned in Remark 1.4 there exists a disjoint decomposition $H = (\bigcup_{i=1}^\infty K_i) \cup J$ where each K_i is t-Harris and J is σ-P_t-transient. If, for instance, $K_1 \neq \varnothing$, then using Lemma 2.3 and condition $(\mathscr{C})$ we could obtain that K_1 is P_s-closed for all $s > 0$. But this contradicts the fact that H is P_α-indecomposable unless only $K_1 \neq \varnothing$. By Lemma 2.4 $H \backslash K_1$ is P_t-inessential, and consequently we may extend the t-Harris measure π_t on K_1 to H by assigning measure 0 to $H \backslash K_1$. If H were not maximal t-Harris, then H would not be maximal α-Harris.

LEMMA 2.6. *For each $t > 0$, let H be maximal t-Harris. Then there exists a fixed t-Harris measure which is independent of $t > 0$.*

PROOF. Fix $t > 0$. Let $\pi_{kt/n}$ be the (kt/n)-Harris measure on H for $k, n \geqq 1$. By Lemma 2.1 a subset B of H is P_t-absolutely essential if and only if it is $P_{kt/n}$-absolutely essential for all $k, n \geqq 1$. By Lemma 1.1 we have that $\pi_{kt/n}$ assigns positive measure only to $P_{kt/n}$-absolutely essential sets. By the uniqueness of Harris measures we obtain, for each $k \geqq 1$ and $n \geqq 1$, that there exists a positive constant $C_{k,n}$ such that $\pi_t = C_{k,n} \cdot \pi_{kt/n}$. For convenience we assume that $C_{k,n} = 1$ for all $k, n \geqq 1$. We note that for all $k \geqq 1$ and $n \geqq 1$ we have that

$$\pi_t(A) = \int_H P_{kt/n}(x, A)\pi_t(dx) \qquad \text{for all} \quad A \in \mathscr{E} .$$

Fix $s > 0$. Let $\{t_j\}_{j=1}^\infty \subseteqq \{kt/n\}_{k,n=1}^\infty$ be a sequence such that $t_j \to s$ as $j \to \infty$. Let $A \in \mathscr{E}$. We wish to show that $\pi_t(A) = \pi_t P_s(A) = \int_H P_s(x, A)\pi_t(dx)$. Since π_t is σ-finite, it is sufficient to show this for $A \subseteqq H$ such that $\pi_t(A) < \infty$. Using Fatou's lemma and $(\mathscr{C})$, we have that

$$\begin{aligned}\pi_t P_s(A) &= \int_H \lim_{j\to\infty} P_{t_j}(x, A)\pi_t(dx) \\ &\leqq \liminf_{j\to\infty} \int_H P_{t_j}(x, A)\pi_t(dx) = \pi_t(A) .\end{aligned}$$

Let π_s be the s-Harris measure. Using Lemmas 1.1 and 2.2, we have that both π_t and π_s assign positive measure only to P_s-absolutely essential subsets of H. By Jain and Jamison ((1967), page 34), the only, up to multiplicative constant, P_s-subinvariant measure supported on H is the s-Harris measure π_s. Thus, there exists a constant $C_{s,t}$ such that $\pi_s = C_{s,t} \cdot \pi_t$. For convenience we assume that $C_{s,t} = 1$ for all $s, t > 0$. Hence, $\pi \equiv \pi_t$ is the unique s-Harris measure for all $s > 0$.

THEOREM 2.1. *Assume that condition $(\mathscr{C})$ and condition $(\mathscr{N})$ hold. Then there exists a disjoint decomposition $E = T \cup (\bigcup_{n=1}^\infty H_n)$ such that T is σ-P_t-transient for all $t > 0$ and each H_n is maximal t-Harris for all $t > 0$.*

PROOF. By Theorem 1.3 and Remark 1.3, $E = T \cup (\bigcup_{n=1}^\infty H_n)$ where T is either P_α-inessential or P_α-improperly essential and each H_n is α-recurrent in the sense of Harris. By Lemmas 1.2 and 1.4 we may assume that each H_n is maximal α-Harris, and, by Corollary 1.1 we may assume that T is σ-P_α-transient. By Lemma 2.2 T is σ-P_t-transient for all $t > 0$, and by Lemma 2.5 each H_n is maximal t-Harris for all $t > 0$.

COROLLARY 2.1. *If $E = H$ and π is the t-Harris measure for all $t > 0$, then we have the following zero-one property*:

(i) *if $\pi(B) > 0$, then $Q_t(x, B) = 1$ for all $t > 0$ and $x \in E$;*
(ii) *if $\pi(B) = 0$, then $Q_t(x, B) = 0$ for all $t > 0$ and $x \in E$.*

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