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STOCHASTIC PARTIAL ORDERING

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A probability measure P on a partially ordered Polish space E is called stochastically smaller than Q (notation: $P \leqslant Q$) if $\int f\,dP \leqslant \int f\,dQ$ holds for all bounded increasing measurable f. We investigate the question when for a stochastically increasing family $\{P_t, t \in \mathbb{R}\}$ there exists an increasing process $\{X_t, t \in \mathbb{R}\}$ with 1-dimensional marginal distributions P_t. A sufficient condition, satisfied, e.g., for $E = \mathbb{R}^{\mathbb{N}}$, for compact E and for spaces E of Lipschitz-functions, is the compactness of all intervals $\{z \in E : x \leqslant z \leqslant y\}$; but for general countable E such an increasing E-valued process $\{X_t\}$ need not exist.

Let E be a complete, separable metric space and "$\leqslant$" a closed partial order relation on E. Such a space shall be called a p.o. Polish space. We shall use some terminology and notation from [2]. A probability measure P on a partially ordered Polish space E is called stochastically smaller than Q (notation: $P \leqslant Q$) if $\int f\,dP \leqslant \int f\,dQ$ holds for all bounded increasing measurable f. In addition we put

$$B^{\uparrow} := \{x \in E : y \leqslant x \text{ for some } y \in B\}$$

$$B^{\downarrow} := \{x \in E : y \geqslant x \text{ for some } y \in B\}.$$

LEMMA 1. *Let $\mathcal{P}$ be a tight family of probability measures on the σ-algebra $\mathcal{F}$ of Borel sets in E. Then there exists a countable family $\mathcal{C}$ of increasing closed sets in E such that, for $P, Q \in \mathcal{P}$, $P = Q$ if $P(C) = Q(C)$ for all $C \in \mathcal{C}$.*

PROOF. Let $\mathcal{U}$ be a countable open base in E. Let $K_n(n = 1, 2, \cdots)$ be compact sets with $\inf\{P(K_n) : P \in \mathcal{P}\} \geqslant 1 - n^{-1}$, and let $\mathcal{D} = \{(\bar{U} \cap K_n)^{\uparrow} : U \in \mathcal{U}, n = 1, 2, \cdots\}$, where $\bar{U}$ is the closure of U. $\mathcal{D}$ consists of closed increasing sets. Let $\mathcal{C}$ be the minimal family which contains $\mathcal{D}$ and is closed under finite unions and finite intersections. Let $E_0 = \cup_{n=1}^{\infty} K_n$, then $P(E_0) = 1$ for all $P \in \mathcal{P}$. Take any two different points $x, y \in E_0$, then either $x \leqslant y$ or $y \leqslant x$ is false. Assume that $y \leqslant x$ is false. Since the partial order relation in E is closed, there exists a neighbourhood V of y such that $z \leqslant x$ is false for any $z \in V$. Let $U \in \mathcal{U}$ be such that $y \in U$ and $\bar{U} \subset V$. If, for some $n, y \in K_n$, then $(\bar{U} \cap K_n)^{\uparrow}$ contains y but does not contain x. Therefore $\mathcal{C}$ separates points in E_0. For P and Q in $\mathcal{P}$, $P(C) = Q(C)$ for all $C \in \mathcal{C}$ implies $P(B) = Q(B)$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the σ-algebra generated by $\mathcal{C}$, since $\mathcal{C}$ is closed under intersections. As $\mathcal{B}$ has a countable basis separating points in E_0, $\mathcal{B}$ restricted to E_0 coincides with the restriction of $\mathcal{F}$ to E_0. Now $P(E_0) = Q(E_0) = 1$ implies $P = Q$. $\square$

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The next result is implicit in the standard terminology "stochastic partial ordering," but does not seem to appear in print anywhere:

THEOREM 2. *The relation "$\leqslant$" on the space of probability measures on $(E, \mathfrak{F})$ with the topology of weak convergence is a closed partial order relation.*

PROOF. Clearly, the relation "$\leqslant$" satisfies the transitivity. Assume $P \leqslant Q$ and $Q \leqslant P$. Let $\mathcal{C}$ be the family of increasing closed sets constructed in the proof of Lemma 1 for the family $\mathcal{P} = \{P, Q\}$ which is clearly tight. Then $P = Q$ follows from Lemma 1. Thus "$\leqslant$" is a partial order relation. It has been shown in [2] that "$\leqslant$" is a closed relation. $\square$

LEMMA 3. *If Y is a topological space with countable base and a closed partial order and φ is an increasing function from $\mathbb{R}$ into Y, then φ has only countably many discontinuity points.*

PROOF. The traces of the topology and the order of Y on $\varphi\mathbb{R}$ make $\varphi\mathbb{R}$ a totally ordered topological space with a countable base. If $y \in \varphi\mathbb{R}$ is a value at a discontinuity point of φ, then there is an open neighbourhood U of y in $\varphi\mathbb{R}$ such that $U \cap \{y\}^{\downarrow} = \{y\}$ or $U \cap \{y\}^{\uparrow} = \{y\}$. For all open U in $\varphi\mathbb{R}$ such that $U \cap \{y\}^{\downarrow} = \{y\}$ we can select a V_y from the induced base on $\varphi\mathbb{R}$ such that $V_y \cap \{y\}^{\downarrow} = \{y\}$ and these V_y must be different for different y. So there are only countably many y that have such a U. The case $U \cap \{y\}^{\uparrow} = \{y\}$ is treated similarly. $\square$

LEMMA 4. *If $D \subset \mathbb{R}$ is countable, and $\{P_t, t \in D\}$ is a stochastically increasing family of probability measures on $(E, \mathfrak{F})$, then there exists an E-valued process $\{X_t, t \in D\}$ such that*

(i) *X_t is distributed according to P_t for any $t \in D$, and*
(ii) *for all ω, $X_t(\omega)$ is an increasing function of t.*

PROOF. By a theorem of Nachbin-Strassen (Theorem 1 in [2]) there exists for any pair s and t in D with $s < t$ an "upward kernel" $k_{s,t}$ such that $P_t = P_s^{k_{s,t}}$. If k, k' are two kernels, denote by $k \cdot k'$ the kernel defined by

$$(k \cdot k')(x, S) = \textstyle\int_E k(x, dy)k'(y, S).$$

Let $D_1 \subset D_2 \subset \cdots$ be an increasing family of finite sets such that $D = \cup_{n=1}^{\infty} D_n$. For any pair s and t in D with $s < t$ and for $n = 1, 2, \cdots$ define a kernel $k_{s,t}(n)$ by

$$k_{s,t}(n) = k_{s,t_1} \cdot k_{t_1,t_2} \cdot \; \cdots \; \cdot k_{t_{m-1},t_m} \cdot k_{t_m,t}$$

where $\{t_1 < t_2 < \cdots < t_m\} = D_n \cap (s, t)$. It is clear that $P_t = P_s^{k_{s,t}(n)}$. Also, note that if $s < t < u$ are in D and $t \in D_n$, then $k_{s,t}(n) \cdot k_{t,u}(n) = k_{s,u}(n)$. By the diagonal argument we can find a sequence $h_1 < h_2 < \cdots$ of positive integers such that

$$P_{t_1} * k_{t_1,t_2}(h_n) * k_{t_2,t_3}(h_n) * \; \cdots \; * k_{t_{m-1},t_m}(h_n)$$

converges weakly as $n \to \infty$ for any finite sequence $t_1 < t_2 < \cdots < t_m$ in D. This follows from the fact that the 1-dimensional marginal distributions of these measures with h_n replaced by n and n large enough are independent of n, so that this family is tight. Let the above limit be $Q_{(t_1, \cdots, t_m)}$, it is a probability measure on $E_{t_1} \times E_{t_2} \times \cdots \times E_{t_m}$, where $E_t = E$ for any $t \in D$. Since $P_t = P_s^{k_{s,t}(n)}$ and $k_{s,t}(n) \cdot k_{t,u}(n) = k_{s,u}(n)$ for sufficiently large n, where $s < t < u$ are in D, it is easy to see that the family of measures

$$\{ Q_{(t_1, \cdots, t_m)} : m = 1, 2 \cdots ; t_1 < t_2 < \cdots < t_m \text{ are in } D \}$$

is a consistent family. Therefore there exists a probability measure μ on $E^D = \Pi_{t \in D} E_t$ which is an extension of the measures in this family. For $s \in D$, let X_s be the projection $E^D \to E_t$. For any s the random variable X_s on the probability space $(\Omega = E^D, \mu)$ is distributed according to P_s. Let $s < t$ be in D. Since the joint distribution of (X_s, X_t) is the weak limit of $P_s * k_{s,t}(n)$ and each $k_{s,t}(n)$ is an upward kernel, $X_s \leqslant X_t$ holds with probability 1. Since D is countable we can complete the proof by eliminating a nullset. □

REMARK. Let $\{P_t, t \in \mathbb{R}\}$ be a stochastically increasing family of probability measures on E. Using the above results it is not hard to see that there always exists an E-valued process $\{X_t, t \in \mathbb{R}\}$ with 1-dimensional marginals such that for fixed $s \leqslant t$, $X_s \leqslant X_t$ almost surely. Later on Example B shall show that this does not imply the existence of a process with increasing paths. Now we turn to a sufficient condition: We say that E has compact intervals if all intervals $\{x\}^{\uparrow} \cap \{y\}^{\downarrow} = [x, y]$ are compact. Examples of such spaces are compact p.o. Polish spaces, $E = \mathbb{R}^{\mathbb{N}}$ and the space of Lipschitz-functions on $[a, b]$ with a fixed Lipschitz-constant. We shall need:

LEMMA 5. *If E has compact intervals, then any increasing sequence in E which is bounded above converges.*

PROOF. Let $x_1 \leqslant x_2 \leqslant \cdots$ be a sequence bounded by x. As $[x_1, x]$ is compact a subsequence converges to some $y \in [x_1, x]$. If z is another limit point, then we have $z \leqslant y$ and $y \leqslant z$, since "$\leqslant$" is a closed relation. □

Now it shall be easy to derive our main result:

THEOREM 6. *If E is a* p.o. *Polish space with compact intervals and $\{P_t, t \in \mathbb{R}\}$ a stochastically increasing family of probability measures on $(E, \mathfrak{F})$, then there exists an E-valued stochastic process $\{X_t, t \in \mathbb{R}\}$ on a probability space (Ω, μ) such that*

(i) *P_t is the distribution of X_t for any $t \in \mathbb{R}$, and*
(ii) *$X_s(\omega) \leqslant X_t(\omega)$ for all $\omega \in \Omega$, and all $s < t$.*

PROOF. By Theorem 2 and results in [1, Appendix III] we can apply Lemma 3 to the map $\varphi : t \to P_t$. Thus, the set D_0 of discontinuity points of φ is at most countable. Let D be a dense countable set in R containing D_0. Let $\{X_t, t \in D\}$ be the process constructed in Lemma 4, and (Ω, μ) the corresponding probability space. For $s \in \mathbb{R} \setminus D$ define X_s by

$$X_s(\omega) = \lim_{t \to s;\, t \in D,\, t \leqslant s} X_t(\omega).$$

The limit exists by Lemma 5. It is clear that (ii) holds. By the definition of X_s ($s \in \mathbb{R} \setminus D$), X_t converges in law to X_s as $t \to s$ ($t \leqslant s$, $t \in D$). As the distribution of X_t is P_t and s is a point of continuity of φ the distribution of X_s must be P_s. ☐

We finish by giving some examples. The first example is an application of Theorem 6.

EXAMPLE A (Gibbsian random fields with negative pairwise potentials on $\mathbb{Z}^2$). Let $E = \{0, 1\}^{\mathbb{Z}^2}$ with $x \leqslant x'$ iff $x(i) \leqslant x'(i)$ for all $i \in \mathbb{Z}^2$. Let $\varphi : \mathbb{Z}^2 \to R$ be a function such that for any $i \in \mathbb{Z}^2$

(i) $\varphi(i) \geqslant 0$;
(ii) $\varphi(i) = \varphi(-i)$; and
(iii) $c \equiv 2^{-1}\Sigma_i \varphi(i) < \infty$.

A probability measure P on E is called an equilibrium state at $t \in \mathbb{R}$ if for any finite set $V \subset \mathbb{Z}^2$ and $\omega \in \{0, 1\}^V$ and $\omega' \in \{0, 1\}^{V^c}$ the conditional probability has the form

$$P(\omega|\omega') = K^{-1} \exp\{2^{-1}\Sigma_{i,j \in V}\varphi(i-j)\omega(i)\omega(j) + \Sigma_{i \in V; j \in V^c}\varphi(i-j)\omega(i)\omega'(j) + t\Sigma_{i \in V}\omega(i)\},$$

where $K = K(\omega')$ is the normalizing constant. Let $\mathcal{G}_t$ denote the set of equilibrium states at t. It is well known [3] that $|\mathcal{G}_t| = 1$ for $t \neq -c$. Let $\mathcal{G}_t = \{P_t\}$ ($t \neq -c$) and let P_{-c} be any element in $\mathcal{G}_{-c}$. It is known [3] that $\{P_t, t \in R\}$ is stochastically increasing. Theorem 6 yields the existence of an increasing process with marginals P_t.

The next example shows that the compactness-condition in Theorem 6 cannot be dropped.

EXAMPLE B. $E := ([0, 1] \times [0, 1]) \setminus \{(x, x) : x \in [0, 1]\}$ with the induced topology from $\mathbb{R}^2$, and the partial order restricted to horizontal lines: $(x_1, x_2) \leqslant (y_1, y_2)$ iff $x_2 = y_2$ and $x_1 \leqslant y_1$. Let P_t be Lebesgue-measure on $\{t\} \times ([0, 1] \setminus \{t\})$. Assume there exists an E-valued increasing process $\{X_t(\omega), t \in [0, 1], \omega \in \Omega\}$ with marginals P_t, defined on a space (Ω, μ). We may write $X_t(\omega) = (X_t^{(1)}(\omega), X_t^{(2)}(\omega))$ with $X_t^{(i)}(\omega) \in [0, 1]$. Almost surely for all $s, t \in [0, 1] \cap \mathbb{Q}$ $X_t^{(1)}(\omega) = t$ and $X_t^{(2)}(\omega) = X_s^{(2)}(\omega)$. Eliminate the exceptional nullset. For the remaining points ω $X_t^{(1)}(\omega) = t$ holds for all $t \in [0, 1]$, since X_t is increasing. Thus X_t also takes values in the diagonal $\{(x, x) : x \in [0, 1]\}$, a contradiction.

A much more sophisticated example is necessary to show that the compactness condition cannot even be eliminated if E is countable:

EXAMPLE C. Let α, β be distinct, $E_n^\alpha = \{(\alpha, h_1, \cdots, h_n) : h_i \in \{0, 1\}\ (1 \leqslant i \leqslant n)\}$ ($n \geqslant 1$), $E^\alpha = \{\alpha\} \cup \bigcup_{n=1}^\infty E_n^\alpha$, $E_n^\beta = \{(\beta, h_1, \cdots, h_n) : h_i \in \{0, 1\}\ (1 \leqslant i \leqslant$

$n)\}$ $(n \geqslant 1)$, $E^\beta = \{\beta\} \cup \cup_{n=1}^\infty E_n^\beta$, $E = E^\alpha \cup E^\beta$. Further let $E^* = E \cup \{0, 1\}^{\mathbb{N}}$. A partial ordering is defined in E^* by requiring that for all $(h_1, h_2, \cdots) \in \{0, 1\}^{\mathbb{N}}$ and for all $n \in \mathbb{N}$

$$\alpha \leqslant (\alpha, h_1, \cdots, h_n) \leqslant (\alpha, h_1, \cdots, h_n, h_{n+1}) \leqslant (h_1, h_2, \cdots, h_n, \cdots)$$
$$\leqslant (\beta, h_1, \cdots, h_n, h_{n+1}) \leqslant (\beta, h_1, \cdots, h_n) \leqslant \beta.$$

All elements for which an order relation is not obtained by iterated applications of these inequalities shall be incomparable.

We define a family of probabilities P_t on E by defining a process $\{U_t, t \in \mathbb{R}\}$ on the probability space (Ω, P) where $\Omega = \{0, 1\}^{\mathbb{N}}$ and $P = \mu_0^{\mathbb{N}}$ with $\mu_0(\{0\}) = \mu_0(\{1\}) = \frac{1}{2}$. The process will take values in E^*, but for each $t \in \mathbb{R}$ $P\{U_t \in E\} = 1$ so that $P_t = P \circ U_t^{-1}$ is a family of distributions in E.

For $x = (x_1, x_2, \cdots) \in \Omega$ define $\varphi_1(x) = 4^{-1}(1 + 2x_1)$, $\tau_{n+1}(x) - \tau_n(x) = 4^{-(n+1)}(1 + 2x_{n+1})$, $\tau_\infty(x) = \lim_{n\to\infty}\tau_n(x)$, $c_i(x) = 1 - x_i$, $c(x) = (1 - x_1, 1 - x_2, \cdots) \in \Omega$.

It follows that $\tau_\infty(x) + \tau_\infty(c(x)) = \gamma > 0$ is independent of x. The process is now given by

$$\begin{aligned}
U_t(x) &= \alpha & &(-\infty < t < \tau_1(x))\\
U_t(x) &= (\alpha, x_1, \cdots, x_n) & &(\tau_n(x) \leqslant t < \tau_{n+1}(x))\\
U_t(x) &= x & &(t = \tau_\infty(x))\\
U_t(x) &= (\beta, x_1, \cdots, x_n) & &(\gamma - \tau_{n+1}(c(x)) < t \leqslant \gamma - \tau_n(c(x)))\\
U_t(x) &= \beta & &(\gamma - \tau_1(c(x)) < t < \infty).
\end{aligned}$$

As τ_∞ has a continuous distribution and $U_t(x) \in E^* \setminus E$ only for $t = \tau_\infty(x)$ each P_t has support in E. The family $\{P_t, t \in \mathbb{R}\}$ is increasing.

It remains to show that there cannot exist an increasing E-valued process $\{X_t, t \in \mathbb{R}\}$ with distributions P_t. This is done by showing that such a process must essentially look like $\{U_t, t \in \mathbb{R}\}$. For convenience we write $\tau_n(x_1, x_2, \cdots, x_n)$ for $\tau_n(x)$ when $x = (x_1, x_2, \cdots)$. Let $\{X_t, t \in \mathbb{R}\}$ be defined on a probability space $(\Sigma, \mathcal{G}, Q)$. Eliminating a nullset we may assume $X_0 \equiv \alpha$ and $X_\gamma \equiv \beta$. $P_{\tau_1(0)}$ has mass $\frac{1}{2}$ in $(\alpha, 0)$. $P_{\tau_1(1)}$ has mass $\frac{1}{2}$ in $(\alpha, 1)$ and the rest of the mass somewhere above $(\alpha, 0)$. As no path can go from $(\alpha, 0)$ to $(\alpha, 1)$ there must be a set $A_0 \in \mathcal{G}$ with $Q(A_0) = \frac{1}{2}$ such that—except for a nullset—the paths $X_t(\sigma)$, $t \geqslant 0$ for $\sigma \in A_0$ start in α and after time $\tau_1(0)$ go to $(\alpha, 0)$, and the remaining ones go to $(\alpha, 1)$, remain in α for $t < \tau_1(1)$ and go to $(\alpha, 1)$ at time $\tau_1(1)$. The same argument can be repeated on A_0 and on $A_1 = A_0^c$ (after eliminating the disturbing nullset) starting with time $\tau_1(0)$ resp. $\tau_1(1)$. The elements in A_1 cannot contribute for any of the mass in $(\alpha, 0, 1)$ or $(\alpha, 0, 0)$ and higher up except later for mass in β, since the paths are increasing. Thus A_0 splits into two sets A_{00}, A_{01} each of probability $\frac{1}{4}$ so that the paths for $\sigma \in A_{00}$ go to $(\alpha, 0, 0)$ and those of A_{01} go to $(\alpha, 0, 1)$ at just the same time when the U_t-process makes the jumps. Similarly A_1 splits into A_{10} and A_{11} both of measure $\frac{1}{4}$.

This way we can work our way up as long as the process stays in E^α. Similarly, using the marginals P_t with t close to γ and $t \leqslant \gamma$ we can work backwards and find that there exist sets B_0, B_1 of probability $\frac{1}{2}$, B_{00}, B_{01}, B_{10}, B_{11} of probability $\frac{1}{4}$, etc. For $\sigma \in B_{10}$,

$$\begin{aligned} X_t(\sigma) &= \beta \qquad (+\infty > t > \gamma - \tau_1(0)) \\ &= (\beta, 1) \quad (\gamma - \tau_1(0) \geqslant t > \gamma - \tau_2(0, 1)) \\ &= (\beta, 1, 0)\ (t = \gamma - \tau_2(0, 1)). \end{aligned}$$

(Note that here the zeros and ones have to be interchanged in the jump-times $\gamma - \tau_n$.)

Since X_t is increasing with probability 1, both $A_1 \cap B_0$ and $A_0 \cap B_1$ have measure 0. Thus we have $A_0 = B_0$, $A_1 = B_1$ modulo nullsets. Argue similarly with A_{01}, B_{00} and with A_{11}, B_{10} to get $A_{00} = B_{00}$, $A_{01} = B_{01}$, etc. modulo nullsets. Then show $A_{000} = B_{000}$, etc. Eliminate the at most countably many nullsets.

Since there remains at least one point $\sigma \in \Sigma$ not eliminated, there exists a sequence $(i_1, i_2, i_3, \cdots) \in \{0, 1\}^{\mathbb{N}}$ for which $\sigma \in \cap_{n=1}^{\infty} A_{i_1, i_2, \cdots, i_n} = \cap_{n=1}^{\infty} B_{i_1, i_2, \cdots, i_n}$. Look at the path $X_t(\sigma)$. The interval $\{t : 0 \leqslant t \leqslant \gamma : X_t(\sigma) \in E^{\alpha}\}$ is open on the right side and contains 0 as a left endpoint, the interval $\{t : 0 \leqslant t \leqslant \gamma : X_t(a) \in E^{\beta}\}$ is open on the left side and contains γ as a right endpoint. Therefore there exists some $t \in [0, \gamma]$ for which $X_t(\sigma)$ is not in $E = E^{\alpha} \cup E^{\beta}$, a contradiction to the assumption that the process is E-valued.

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