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EXISTENCE OF INDEPENDENT COMPLEMENTS IN REGULAR CONDITIONAL PROBABILITY SPACES

BY D. RAMACHANDRAN
Indian Statistical Institute

Let $(X, \mathcal{Q}, P)$ be a probability space and $\mathcal{B}$ a sub-σ-algebra of $\mathcal{Q}$. Some results on regular conditional probabilities given $\mathcal{B}$ are proved. Using these, when $\mathcal{Q}$ is separable and $\mathcal{B}$ is a countably generated sub-σ-algebra of $\mathcal{Q}$ such that there is a regular conditional probability given $\mathcal{B}$, necessary and sufficient conditions for the existence of an independent complement for $\mathcal{B}$ are given.

0. Introduction and notation. Let $(X, \mathcal{Q}, P)$ be a probability space and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{Q}$. A σ-algebra $\mathcal{B}^* \subset \mathcal{Q}$ is said to be an independent complement of $\mathcal{B}$ if

(i) $\mathcal{B}$ and $\mathcal{B}^*$ are independent and

(ii) for every $A \in \mathcal{Q}$ there exists $A_1 \in \sigma\{\mathcal{B}, \mathcal{B}^*\}$ such that $P(A \Delta A_1) = 0$, i.e., $\mathcal{B} \vee \mathcal{B}^* = \mathcal{Q}$ a.s.$[P]$.

It follows by (i) that $\mathcal{B} \cap \mathcal{B}^* = \{X, \varnothing\}$ a.s.$[P]$.

The problem of existence of independent complements has been studied by Rohlin (1949) and later by Rosenblatt (1959). In this paper, we study this problem in detail for the case when $\mathcal{Q}$ is separable and $\mathcal{B}$ is countably generated. First we collect some facts about regular conditional probabilities. With regular conditional probabilities as our main tool, we next employ Rohlin's techniques in his treatment of the problem for Lebesgue spaces to give necessary and sufficient conditions for the existence of an independent complement.

Let $\mathcal{Q}$ be a σ-algebra of subsets of a set X. $\mathcal{Q}$ is said to be countably generated if there exists a sequence $\{A_n, n \geqslant 1\} \subset \mathcal{Q}$ such that $\mathcal{Q} = \sigma\{A_n\}$. $\mathcal{Q}$ is said to be separable if $\mathcal{Q}$ is countably generated and contains all singletons. $A \in \mathcal{Q}$ is said to be an $\mathcal{Q}$-atom if no nonempty proper subset of A belongs to $\mathcal{Q}$. $\mathcal{Q}$ is said to be atomic if every set in $\mathcal{Q}$ is a union of $\mathcal{Q}$-atoms. A probability Q on $(X, \mathcal{Q})$ where $\mathcal{Q}$ is separable is called continuous if $Q(\{x\}) = 0$ for every $x \in X$. A one-to-one map f from a measurable space $(X, \mathcal{Q})$ onto a measurable space $(Y, \mathcal{B})$ is called bimeasurable if both f and f^{-1} are measurable. For other terminology used in this paper refer to Neveu (1965).

1. On regular conditional probabilities. Let $(X, \mathcal{Q}, P)$ be a probability space and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{Q}$. A function $\mu(x, A)$ defined on $X \times \mathcal{Q}$ is called a

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regular conditional probability given $\mathcal{B}$ if

(CP 1) $\mu(x, \cdot)$ is a probability on $\mathcal{A}$ for each fixed x in X,

(CP 2) $\mu(\cdot, A)$ is $\mathcal{B}$-measurable for each fixed A in $\mathcal{A}$, and

(CP 3) $P(A \cap B) = \int_B \mu(x, A)\, dP$ for every $A \in \mathcal{A}, B \in \mathcal{B}$.

A regular conditional probability $\mu(x, A)$ is said to be proper at $x_0 \in X$ if

(CP 4) $\mu(x_0, B) = 1$ whenever $x_0 \in B \in \mathcal{B}$.

$\mu(x, A)$ is said to be everywhere proper if it is proper at every $x \in X$. Note that if $\mu(x, A)$ is proper at x_0 and if $\mathcal{B}$ is atomic then (CP 4) implies that $\mu(x_0, \cdot)$ is concentrated on the $\mathcal{B}$-atom containing x_0.

In this section we prove certain results about regular conditional probabilities, some of which are of independent interest, while others will be needed in later sections.

Let $(X, \mathcal{A}, P)$ be a probability space and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{A}$. The following proposition deals with the existence of regular conditional probability given $\mathcal{B}$ in $(X, \mathcal{A}, P_1)$ where P_1 is absolutely continuous with respect to P.

Proposition 1. *Suppose there exists a regular conditional probability $\mu(x, A)$ given $\mathcal{B}$ in $(X, \mathcal{A}, P)$. Then there is a regular conditional probability $\mu(x, A)$ given $\mathcal{B}$ in $(X, \mathcal{A}, P_1)$ where P_1 is any probability absolutely continuous with respect to P. If, further, $\mu(x, A)$ is proper at $x_0 \in X$ then $\mu_1(x, A)$ is also proper at x_0.*

Proof. Let f be a fixed version of dP_1/dP, the Radon-Nikodym derivative of P_1 with respect to P. For every $\mathcal{A}$-measurable function g on X let $\mu(x, g) = \int g(y)\mu(x, dy)$. Then it can be checked that $\mu(x, g)$ is a version of $E(g|\mathcal{B})$, the conditional expectation of g with respect to $\mathcal{B}$. It follows that $\mu(x, f)$ is a version of the Radon-Nikodym derivative of $P_1|_{\mathcal{B}}$ with respect to $P|_{\mathcal{B}}$. Now let $E = \{x : \mu(x, f) > 0\}$. Then $E \in \mathcal{B}$ and $P_1(E) = 1$. Define $\mu_1(x, A)$ on $X \times \mathcal{A}$ by

$$\begin{aligned} \mu_1(x, A) &= \mu(x, f1_A)/\mu(x, f) && \text{if } x \in E \\ &= \mu(x, A) && \text{if } x \in E^c. \end{aligned}$$

Clearly $\mu_1(x, A)$ satisfies (CP 1) and (CP 2). If $B \in \mathcal{B}$ and $A \in \mathcal{A}$ then

$$\begin{aligned} \int_B \mu(x, A)\, dP_1 &= \int_B \mu_1(x, A)\mu(x, f)\, dP \\ &= \int_{B \cap E} \mu(x, f1_A)\, dP \\ &= \int_{B \cap E} f1_A\, dP \\ &= P_1(A \cap B \cap E) = P_1(A \cap B) \end{aligned}$$

and so $\mu_1(x, A)$ satisfies (CP 3). Thus $\mu_1(x, A)$ is a regular conditional probability given $\mathcal{B}$ in $(X, \mathcal{A}, P_1)$.

Finally, if $\mu(x, A)$ is proper at $x_0 \in E^c$ then $\mu_1 x, A)$ is proper at x_0. On the other hand if $\mu(x, A)$ is proper at $x_0 \in E$ and if $x_0 \in B \in \mathcal{B}$ then

$$\mu_1(x, B) = \mu(x_0, f1_B)/\mu(x_0, f) = \mu(x_0, f)/\mu(x_0, f) = 1$$

since $\mu(x_0, B) = 1$. Hence $\mu_1(x, A)$ is proper at x_0.

As a consequence of Proposition 1 we have the following corollary on the existence of regular conditional probability in subspaces of $(X, \mathcal{A}, P)$.

COROLLARY 1. *Let* $X_0 \in \mathcal{A}$ *with* $P(X_0) > 0$. *Consider the subspace* $(X_0, \mathcal{A} \cap X_0, P_0)$ *where* $P_0(A \cap X_0) = P(A \cap X_0)/P(X_0), A \in \mathcal{A}$. *Suppose there exists a regular conditional probability* $\mu(x, A)$ *given* $\mathcal{B}$ *in* $(X, \mathcal{A}, P)$. *Then there exists a regular conditional probability* $\mu_0(x, A \cap X_0)$ *given* $\mathcal{B} \cap X_0$ *in* $(X_0, \mathcal{A} \cap X_0, P_0)$. *If* $\mu(x, A)$ *is proper* a.s. $[P|_{\mathcal{B}}]$ *then* $\mu_0(x, A \cap X_0)$ *is proper* a.s. $[P_0|_{\mathcal{B} \cap X_0}]$.

PROOF. Define P_1 on $\mathcal{A}$ by $P_1(A) = P_0(A \cap X_0), A \in \mathcal{A}$. Then P_1 is absolutely continuous with respect to P. Let $\mu_1(x, A)$ be the regular conditional probability given $\mathcal{B}$ in $(X, \mathcal{A}, P_1)$ obtained by using Proposition 1. By (CP 3), there exists $N_1 \in \mathcal{B}$ with $P_1(N_1) = 0$ such that $\mu_1(x, X_0) = 1$ for all $x \notin N_1$. Define $\mu_0(x, A \cap X_0)$ on $X_0 \times (\mathcal{A} \cap X_0)$ by

$$\begin{aligned} \mu_0(x, A \cap X_0) &= \mu_1(x, A \cap X_0) && \text{if } x \notin N_1 \cap X_0 \\ &= P_0(A \cap X_0) && \text{if } x \in N_1 \cap X_0. \end{aligned}$$

It is easy to check that $\mu_0(x, A \cap X_0)$ satisfies (CP 1) and (CP 2) and that $N_1 \cap X_0 \in \mathcal{B} \cap X_0$ with $P_0(N_1 \cap X_0) = 0$. If $B \in \mathcal{B}$ and $A \in \mathcal{A}$ then

$$\begin{aligned} \textstyle\int_{B \cap X_0} \mu_0(x, A \cap X_0)\, dP_0 &= \textstyle\int_{B \cap N_1^c \cap X_0} \mu_1(x, A \cap X_0)\, dP_0 \\ &= \textstyle\int_{B \cap N_1^c} \mu_1(x, A \cap X_0)\, dP_1 \\ &= P_1(B \cap N_1^c \cap A \cap X_0) = P_0(A \cap B \cap X_0). \end{aligned}$$

Hence $\mu_0(x, A \cap X_0)$ satisfies (CP 3) and thus is a regular conditional probability given $\mathcal{B} \cap X_0$.

Suppose $\mu(x, A)$ is proper at every $x \notin N$ where $N \in \mathcal{B}$ with $P(N) = 0$. Then $P_1(N) = 0$ and by Proposition 1, $\mu_1(x, A)$ is proper at every $x \notin N$. It can be verified that $P_0((N \cup N_1) \cap X_0) = 0$ and $\mu_0(x, A \cap X_0)$ is proper at every $x \notin (N \cup N_1) \cap X_0$.

One may wonder whether, in Corollary 1, if we start with a regular conditional probability $\mu(x, A)$ which is proper everywhere, $\mu_0(x, A \cap X_0)$ can also be chosen to be proper everywhere. To show that such a choice is not always possible we shall give an example using the following result.

PROPOSITION 2. *There exists an everywhere proper regular conditional probability* $\mu_1(x, A)$ *given* $\mathcal{B}$ *if and only if there exist* (i) $a[P|_{\mathcal{B}}]$ – *almost everywhere proper regular conditional probability* $\mu(x, A)$ *given* $\mathcal{B}$, *and* (ii) *an everywhere proper transition function* $Q(x, A)$ *given* $\mathcal{B}$.

PROOF. Since $\mu_1(x, A)$ is a transition function on $X \times \mathcal{A}$ given $\mathcal{B}$ the "only if" part follows.

To prove the "if" part, let $N \in \mathcal{B}$ with $P(N) = 0$ be such that $\mu(x, A)$ is proper at every $x \notin N$. Then $\mu_1(x, A)$ defined by

$$\begin{aligned} \mu_1(x, A) &= \mu(x, A) \qquad \text{if } x \notin N \\ &= Q(x, A) \qquad \text{if } x \in N \end{aligned}$$

is an everywhere proper regular conditional probability given $\mathfrak{B}$.

EXAMPLE 1. Let X be the unit square and $\mathcal{Q}$ its Borel σ-algebra. Let $\mathfrak{B}$ be the σ-algebra of vertical cylinders. Let X_0 be a Borel subset of X which does not contain any graph and whose vertical sections are all nonempty (see Blackwell (1968)). Let P_0 be any probability on $(X_0, \mathcal{Q} \cap X_0)$ and let P on $\mathcal{Q}$ be defined by $P(A) = P_0(A \cap X_0)$. It is known (see Theorem 5 of [1]) that there is a regular conditional probability given $\mathfrak{B}$ which is proper a.s. $[P|_{\mathfrak{B}}]$. The function $Q(x, A) = 1_A(f(x))$, where f on X is the projection to the first coordinate, is an everywhere proper transition function given $\mathfrak{B}$. Thus there is an everywhere proper regular conditional probability given $\mathfrak{B}$ by Proposition 2.

In order to show that a regular conditional probability $\mu_0(x, A \cap X_0)$ on $X_0 \times (\mathcal{Q} \cap X_0)$ given $\mathfrak{B} \cap X_0$ cannot be chosen to be everywhere proper, it is now enough to show that there is no transition function $Q_0(x, A \cap X_0)$ on $X_0 \times (\mathcal{Q} \cap X_0)$ given $\mathfrak{B} \cap X_0$ which is everywhere proper. By a theorem of Blackwell and Ryll-Nardzewski (1963, Theorem 1) the existence of such a $Q_0(x, A \cap X_0)$ implies the existence of a $\mathfrak{B} \cap X_0$-measurable function g from X_0 into X_0 such that the graph of g is a subset of X_0, which is impossible by the choice of X_0. Hence there is no regular conditional probability on $X_0 \times (\mathcal{Q} \cap X_0)$ given $\mathfrak{B} \cap X_0$ which is everywhere proper.

2. A decomposition of X. Let $(X, \mathcal{Q}, P)$ be a probability space and let $\mathfrak{B}$ be an atomic sub-σ-algebra of $\mathcal{Q}$. We assume throughout this section the existence of a regular conditional probability $\mu(x, A)$ given $\mathfrak{B}$ which is proper a.s. $[P|_{\mathfrak{B}}]$; that is, there exists $N \in \mathfrak{B}$ with $P(N) = 0$ such that $\mu(x, A)$ is proper at every $x \notin N$.

A set $A \in \mathcal{Q}$ is a called a measurable partial selector for $\mathfrak{B}$ (or simply a partial selector when there can be no confusion) if $A \cap B$ contains at most one point for every $\mathfrak{B}$-atom B.

Analogous to Rohlin's results on Lebesgue spaces (see Section 4, No. 2 of [7]) we have

PROPOSITION 3. *Among the partial selectors for* $\mathfrak{B}$ *there exists one of maximal measure.*

PROOF. Let $\mathcal{Q}_0 = \{A \in \mathcal{Q} : A \text{ is a partial selector for } \mathfrak{B}\}$ and let $\beta = \sup \{P(A) : A \in \mathcal{Q}_0\}$. Let $\{A_n, n \geq 1\} \subset \mathcal{Q}_0$ be a sequence of partial selectors such that $P(A_n) \to \beta$. We shall construct a sequence $\{C_n, n \geq 1\}$ of sets such that (i) $C_n \in \mathcal{Q}_0$ for all n, (ii) $\limsup C_n \in \mathcal{Q}_0$ and (iii) $P(C_n) \geq P(A_n)$ for all n. It will follow that $P(\limsup C_n) = \beta$.

Let $C_1 = A_1 \in \mathcal{Q}_0$. Suppose $C_1, C_2, \cdots, C_{n-1}$ have been defined for $n > 1$ such that $C_i \in \mathcal{Q}_0$, $1 \leq i \leq n - 1$. Let $B_{n-1} = \{x : \mu(x, A_n) > \mu(x, C_{n-1})\}$ and let $C_n =$

$(B_{n-1}^c \cap C_{n-1}) \cup (B_{n-1} \cap A_n)$. Since, by (CP 2), $B_{n-1} \in \mathfrak{B}$ we have $C_n \in \mathcal{Q}_0$. Let $\{C_n, n \geqslant 1\}$ be constructed as above. We note that

$$(*)\quad \begin{aligned} x \in B_{n-1} - N \qquad & \Rightarrow \mu(x, C_n) = \mu(x, C_n \cap B_{n-1}) \\ & = \mu(x, A_n \cap B_{n-1}) = \mu(x, A_n) > \mu(x, C_{n-1}) \\ x \in B_{n-1}^c - N \qquad & \Rightarrow \mu(x, C_n) = \mu(x, C_n \cap B_{n-1}^c) \\ & \mu(x, C_{n-1} \cap B_{n-1}^c) = \mu(x, C_{n-1}) \geqslant \mu(x, A_n) \end{aligned}$$

and hence for all $x \notin N$ we have $\mu(x, C_n) \geqslant \mu(x, C_{n-1})$. For every x, let $B(x)$ denote the $\mathfrak{B}$-atom containing x. *Let* $x \notin N$. Then $\{B(x) \cap C_n\}$ is a sequence of subsets of $B(x)$ each containing at most one point and such that $\mu(x, B(x) \cap C_n) \geqslant \mu(x, B(x) \cap C_{n-1})$. Further from $(*)\mu(x, B(x) \cap C_m) = \mu(x, B(x) \cap C_{m-1}) \Rightarrow x \in B_{m-1}^c - N \Rightarrow B(x) \cap C_m = B(x) \cap C_{m-1}$. Hence $\{B(x) \cap C_n\}$ contains only a finite number of distinct points. Hence there is a natural number $n(x)$ such that for all $n \geqslant n(x)$, $B(x) \cap C_n = B(x) \cap C_{n(x)}$.

Let $M_1 = \cup_{x \notin N} B(x) \cap C_{n(x)}$. Then $M_1 = \lim\sup C_n \in \mathcal{Q}$. M_1 is clearly a partial selector and so $M_1 \in \mathcal{Q}_0$. From $(*)$ we have $\mu(x, C_n) \geqslant \mu(x, A_n)$ for all $x \notin N$. Hence, by (CP 3), $P(C_n) = \int_{N^c} \mu(x, C_n)\, dP \geqslant \int_{N^c} \mu(x, A_n)\, dP = P(A_n)$, for every n.

We now prove the main result of this section.

THEOREM 1. *There is a decomposition of X of the form $X = M_0 \cup M_1 \cup M_2 \cup \cdots \cup M_n \cup \cdots$ where*

(i) *M_n is a measurable partial selector for $\mathfrak{B}$ for every $n \geqslant 1$,*

(ii) *M_n is a set of maximal measure among all measurable partial selectors for $\mathfrak{B}$ which are subsets of $X_{n-1} = (\cup_{i=1}^{n-1} M_i)^c$ for every $n \geqslant 1$ and*

(iii) *M_0 contains no partial selector for $\mathfrak{B}$ of positive measure.*

PROOF. Let M_1 be constructed according to Proposition 3. Suppose $M_1, M_2, \cdots, M_{n-1}$ have been constructed satisfying (i) and (ii) for some $n > 1$. Let $X_{n-1} = (\cup_{i=1}^{n-1} M_i)^c$. If $P(X_{n-1}) = 0$ take $M_n = \varnothing$. If $P(X_{n-1}) > 0$ then by Corollary 1, there is a regular conditional probability on $X_{n-1} \times (\mathcal{Q} \cap X_{n-1})$ given $\mathfrak{B} \cap X_{n-1}$ which is proper a.s. $[P_{n-1}]$ where P_{n-1} on $\mathcal{Q} \cap X_{n-1}$ is defined by $P_{n-1}(A \cap X_{n-1}) = P(A \cap X_{n-1})/P(X_{n-1})$. Now we apply Proposition 3 in the subspace $(X_{n-1}, \mathcal{Q} \cap X_{n-1}, P_{n-1})$ to the class of measurable partial selectors for $\mathfrak{B} \cap X_{n-1}$ to get a set of maximal measure. Plainly M_{n-1} is an $\mathcal{Q}$-measurable partial selector for $\mathfrak{B}$ and also has maximal measure if one considers only partial selectors contained in X_{n-1}. Having defined $\{M_n, n \geqslant 1\}$ in this fashion we set $M_0 = (\cup_1^\infty M_i)^c$. If $A \in \mathcal{Q}$ is a partial selector for $\mathfrak{B}$ contained in M_0 then $P(A) \leqslant P(M_n)$ for all n and hence $P(A) = 0$.

In what follows we call any decomposition of X of the form $X = M_0 \cup M_1 \cup M_2 \cup \cdots$ satisfying (i), (ii) and (iii) of Theorem 1, a maximal decomposition of X.

3. A sufficient condition. For our future considerations we need the following theorem of Rohlin (see Section 4, No. 3 of [7]).

THEOREM 2 (Rohlin). *Suppose X is the unit interval, $\mathcal{Q}$ its Borel σ-algebra and P a probability on $(X, \mathcal{Q})$. If $\mathcal{B}$ is a countably generated sub-σ-algebra of $\mathcal{Q}$ such that there is no measurable partial selector for $\mathcal{B}$ of positive measure, then $\mathcal{B}$ has an independent complement.*

Rohlin's proof of the above theorem uses properties of a version of regular conditional probability given $\mathcal{B}$ which is proper a.s. $[P|_{\mathcal{B}}]$. The existence of such a regular conditional probability is guaranteed in the setup of Theorem 2 (see Theorem 5 of [1], for instance). Using the notion of Marczewski function we obtain the following generalisation of Rohlin's theorem.

THEOREM 3. *Let $(Z, \mathcal{C}, Q)$ be a probability space where $\mathcal{C}$ is separable. Let $\mathcal{C}_0$ be an atomic sub-σ-algebra of $\mathcal{C}$ such that there is no partial selector for $\mathcal{C}_0$ of positive measure. Then $\mathcal{C}_0$ has an independent complement.*

PROOF. By taking a generator $\{C_n, n \geq 1\}$ of $\mathcal{C}$ and by using the Marczewski function of $\{C_n\}$ (see [5]) defined by $f(x) = \sum_{n=1}^{\infty} 1_{C_n}(x)(2/3^n)$ we can assume without loss of generality that $Z \subset [0, 1]$ and $\mathcal{C} = \{A \cap Z : A \in \mathcal{B}_{[0,1]}\}$ where $\mathcal{B}_{[0,1]}$ denotes the Borel σ-algebra of [0, 1]. Define P on $\mathcal{B}_{[0,1]}$ by $P(A) = Q(A \cap Z)$, $A \in \mathcal{B}_{[0,1]}$. Let $\mathcal{C}_0' = \sigma\{C_n'\}$ be a countably generated sub-σ-algebra of $\mathcal{C}_0$ such that $\mathcal{C}_0' = \mathcal{C}_0$ a.s. $[Q]$ (choice of $\mathcal{C}_0'$ is possible since $\mathcal{C}$ is countably generated). Let, for every n, $A_n \in \mathcal{B}_{[0,1]}$ be such that $A_n \cap Z = C_n'$ and let $\mathcal{B}_0 = \sigma\{A_n\}$.

That there is no partial selector for $\mathcal{C}_0$ of positive Q-measure implies that there is no partial selector for $\mathcal{B}_0$ of positive P-measure. By Theorem 2 there is an independent complement $\mathcal{B}_0^*$ of $\mathcal{B}_0$. It is easy to check that $\mathcal{C}_0^* = \mathcal{B}_0^* \cap Z$ is an independent complement of $\mathcal{C}_0$.

REMARK 1. Theorem 3 for the case when there is a regular conditional probability given $\mathcal{C}_0$ which is proper a.s. $[Q|_{\mathcal{C}_0}]$ can be proved by imitating Rohlin's techniques in his proof of Theorem 2 by utilising the given regular conditional probability. But our proof of Theorem 3 is direct and simple. Further, Theorem 3 is really stronger because there is no assumption about the existence of an almost everywhere proper regular conditional probability as illustrated by the following example.

EXAMPLE 2. Let $X_1 = X_2 = [0, 1]$ and let λ on $(X_1, \mathcal{B}_{[0,1]})$ be the Lebesgue measure. Let M be a subset of [0, 1] with outer Lebesgue measure one and inner Lebesgue measure zero. Define a probability P_1 on $\mathcal{Q}_1 = \sigma\{\mathcal{B}_{[0,1]}, M\} = \{(B \cap M) \cup (C \cap M^c) : B, C \in \mathcal{B}_{[0,1]}\}$ by $P_1((B \cap M) \cup (C \cap M^c)) = \lambda(B)$. Then it is well known that there is no regular conditional probability given $\mathcal{B}_{[0,1]}$ on $X_1 \times \mathcal{Q}_1$ (see [4], page 210).

Let $Z = X_1 \times X_2$, $\mathcal{C} = \mathcal{Q}_1 \otimes \mathcal{B}_{[0,1]}$ and let $Q = P_1 \times \lambda$. There is no regular conditional probability given $\mathcal{B}_{[0,1]} \times [0, 1]$ on $Z \times \mathcal{C}$ since there is no regular conditional probability given $\mathcal{B}_{[0,1]}$ on $X_1 \times \mathcal{C}_1$. But by Fubini's theorem there is no partial selector for $\mathcal{B}_{[0,1]} \times [0, 1]$ of positive measure. Thus, in this case there is an independent complement of $\mathcal{B}_{[0,1]} \times [0, 1]$ by Theorem 3, although neither Theorem 2 nor Theorem 3 with the condition of the existence of a regular conditional probability is applicable.

4. Necessary and sufficient conditions. In what follows we let $(X, \mathcal{C}, P)$ be a probability space where $\mathcal{C}$ is separable and let $\mathcal{B}$ be an atomic sub-σ-algebra of $\mathcal{C}$. We assume further that there exists a regular conditional probability $\mu(x, A)$ given $\mathcal{B}$ which is proper a.s. $[P|_{\mathcal{B}}]$; that is, there exists $N \in \mathcal{B}$ with $P(N) = 0$ such that $\mu(x, A)$ is proper at every $x \notin N$.

We shall give a necessary and sufficient condition for the existence of an independent complement $\mathcal{B}^*$ of $\mathcal{B}$. We need the following lemma for later use.

LEMMA 1. (i) *A set $B_1 \in \mathcal{C}$ is independent of $\mathcal{B}$ if and only if $\mu(x, B_1) = P(B_1)$* a.s. $[P|_{\mathcal{B}}]$.

(ii) *A σ-algebra $\mathcal{B}_1 \subset \mathcal{C}$ is independent of B if and only if for every $B \in \mathcal{B}_1$, $\mu(x, B_1) = P(B_1)$* a.s. $[P|_{\mathcal{B}}]$.

(iii) *A countably generated σ-algebra $\mathcal{B}_1 \subset \mathcal{C}$ is independent of $\mathcal{B}$ if and only if there is a set $N_1 \in \mathcal{B}$ with $P(N_1) = 0$ such that for every $x \notin N_1$, $\mu(x, B_1) \equiv_{B_1 \in \mathcal{B}_1} P(B_1)$.*

PROOF. (i) follows from (CP 3) and hence (ii) follows. The sufficiency part of (iii) follows form (ii). To prove the necessary part let $\{B_n, n \geqslant 1\} \subset \mathcal{B}_1$ be a countable algebra generating $\mathcal{B}_1$. It is clear using (i) that there exists $N_1 \in \mathcal{B}_1$ with $P(N_1) = 0$ such that for every $x \notin N_1$, $\mu(x, B_n) \equiv P(B_n)$. Now if $\mathcal{B}_2 = \{C \in \mathcal{B}_1 : \mu(x, C) = P(C)$ for all $x \notin N_1\}$ then $\mathcal{B}_2$ being a monotone class containing $\mathcal{B}_1$ we have $\mathcal{B}_2 = \mathcal{B}_1$.

We first prove the following result.

PROPOSITION 4. *The following statements are equivalent.*

(a) *Almost all the measures $\{\mu(x, \cdot)\}$ are continuous.*

(b) *There is no partial selector for $\mathcal{B}$ of positive measure.*

(c) *There is an independent complement $\mathcal{B}^*$ of $\mathcal{B}$ such that $P|_{\mathcal{B}^*}$ is nonatomic.*

PROOF. (a) $\Rightarrow$ (b). If $\mu(x, \cdot)$'s are continuous for every $x \notin N_0$, where $N_0 \in \mathcal{B}$ with $P(N_0) = 0$, then for any partial selector $A \in \mathcal{C}$, by (CP 3),

$$P(A) = \int_{(N \cup N_0)^c} \mu(x, B(x) \cap A)\, dP = 0.$$

(b) $\Rightarrow$ (c). By Theorem 3, there is an independent complement $\mathcal{B}^*$ of $\mathcal{B}$ which can, without loss of generality, be taken to be countably generated. There exists $X_1 \in \mathcal{C}$ with $P(X_1) = 1$ such that $(\mathcal{B} \vee \mathcal{B}^*) \cap X_1 = \mathcal{C} \cap X_1$. For every $\mathcal{B}^*$-atom B^*, $P(B^*) = P(B^* \cap X_1) = 0$ since $B^* \cap X_1$ is a partial selector for $\mathcal{B}$.

(c) $\Rightarrow$ (a). By Lemma 1, there exists $N_1 \in \mathcal{B}$ with $P(N_1) = 0$ such that for all $x \notin N_1$, $\mu(x, B^*) \equiv_{\mathcal{B}^*} P(B^*)$. Let $N_0 = N \cup N_1$. If $x \notin N_0$ and if $y \in B(x)$ then $\mu(x, \{y\}) \leqslant \mu(x, B^*(y)) = P(B^*(y)) = 0$, since $P|_{\mathcal{B}^*}$ is nonatomic where $B(x)$ is the $\mathfrak{B}$-atom containing x and $B^*(y)$ is the $\mathfrak{B}^*$-atom containing y. Thus $\mu(x, \cdot)$ is continuous for every $x \notin N_0$.

We shall now introduce, following Rohlin, a sequence $\{m_n, n \geqslant 1\}$ of functions on X and study their properties, after which we shall prove our main theorem of this section.

Let $x \notin N$. Then $(B(x), \mathcal{C} \cap B(x), \mu(x, \cdot))$ is a probability space where $B(x)$ denotes the $\mathfrak{B}$-atom containing x. Let $y_1, y_2, \cdots, y_k, \cdots$ be an enumeration of points of $B(x)$ of positive $\mu(x, \cdot)$ measure such that for every $k \geqslant 1$, $\mu(x, \{y_k\}) \geqslant \mu(x, \{y_{k+1}\})$. If the sequence $\{y_k\}$ is infinite let

$$m_n(x) = \mu(x, \{y_n\}), \qquad n \geqslant 1$$

and if the sequence $\{y_k\}$ contains only r elements let

$$\begin{aligned} m_n(x) &= \mu(x, \{y_n\}) && \text{if } n \leqslant r \\ &= 0 && \text{if } n > r. \end{aligned}$$

We have thus defined a sequence of functions $\{m_n, n \geqslant 1\}$ on $X - N$. Let m_n, for each $n \geqslant 1$, be defined to be identically zero on N.

Now $\{m_n, n \geqslant 1\}$ is a sequence of functions defined on X such that

$$\text{(a) } m_n \geqslant 0; \text{ (b) } m_n \geqslant m_{n+1}; \text{ and (c) } \Sigma_{n=1}^{\infty} m_n \leqslant 1.$$

Using the decomposition theorem of Section 2 now we shall show that $\{m_n, n \geqslant 1\}$ are measurable functions a.s. $[P|_{\mathfrak{B}}]$. Let $X = M_0 \cup M_1 \cup M_2 \cup \cdots$ be a maximal decomposition of X.

Proposition 5. (i) $\mu(x, \cdot)|_{M_0}$ *is continuous* a.s. $[P|_{\mathfrak{B}}]$ *and*
(ii) $\mu(x, M_n) \geqslant \mu(x, M_{n+1})$ a.s. $[P|_{\mathfrak{B}}]$ *for every* $n \geqslant 1$.

Proof. (i) If $P(M_0) = 0$ then the first assertion is trivial. If $P(M_0) > 0$ then consider the subspace $(M_0, \mathcal{C} \cap M_0, P_{M_0} = P(\cdot \cap M_0)/P(M_0))$. Clearly in this space there is no partial selector for $\mathfrak{B} \cap M_0$ of positive measure. Further, by Corollary 1, there is a regular conditional probability $\mu_0(x, A \cap M_0)$ on $M_0 \times (\mathcal{C} \cap M_0)$ given $\mathfrak{B} \cap M_0$ which is proper a.s. $[P_{M_0}|_{\mathfrak{B} \cap M_0}]$ such that

$$\mu_0(x, A \cap M_0) = \mu(x, A \cap M_0)/\mu(x, M_0) \text{ a.s. } \left[P_{M_0}|_{\mathfrak{B} \cap M_0}\right].$$

By Proposition 4, $\mu_0(x, \cdot)$ is continuous a.s. $[P_{M_0}|_{\mathfrak{B} \cap M_0}]$. It follows that $\mu(x, \cdot)|_{M_0}$ is continuous a.s. $[P|_{\mathfrak{B}}]$.

(ii) Let $N_n = \{x : \mu(x, M_n) < \mu(x, M_{n+1})\}$ and let $M_n' = (N_n^c \cap M_n) \cup (N_n \cap M_{n+1})$. Then $M_n' \in \mathcal{C}$ and is a partial selector contained in $X_{n-1} = \left(\cup_{i=1}^{n-1} M_i\right)^c$. If $P(N_n) > 0$ then

$$\begin{aligned} P(M_n') &= \textstyle\int \mu(x, M_n')\, dP \\ &= \textstyle\int_{N_n^c} \mu(x, M_n)\, dP + \int_{N_n} \mu(x, M_{n+1})\, dP \\ &> \textstyle\int \mu(x, M_n)\, dP = P(M_n) \end{aligned}$$

which is impossible since M_n is a set of maximal measure among measurable partial selectors contained in X_{n-1}. Hence $P(N_n) = 0$.

PROPOSITION 6. *There exists* $N_0 \in \mathfrak{B}$ *with* $P(N_0)=0$ *such that for all* $x \notin N_0 \cup N$

$$\mu(x, M_n) = m_n(x), \qquad n = 1, 2, \cdots.$$

In other words for all $n \geqslant 1$, m_n *is* $\mathfrak{B}$*-measurable* a.s. $[P|_{\mathfrak{B}}]$.

PROOF. Using Proposition 5, we can get a set $N_0 \in \mathfrak{B}$ with $P(N_0) = 0$ such that for every $x \notin N_0$, $\mu(x, \cdot)|_{M_0}$ is continuous and $\mu(x, M_n) \geqslant \mu(x, M_{n+1})$ for all $n \geqslant 1$. Thus, for every $x \notin N_0 \cup N$ the sequence $\{B(x) \cap M_n\}$ consists of sets containing at most one point and so arranged that their measures form a nonincreasing sequence. Further this sequence contains every singleton which has positive measure. So by definition of $\{m_n\}$ we have

$$m_n(x) = \mu(x, B(x) \cap M_n) = \mu(x, M_n)$$

for all n and for every $x \notin N_0 \cup N$.

REMARK 2. Observe that the assertion in Proposition 6 is independent of the choice of maximal decomposition of X. Thus if $X = M_0 \cup M_1 \cup M_2 \cup \cdots = M_0' \cup M_1' \cup M_2' \cup \cdots$ are two maximal decompositions of X, then by Proposition 6, $\mu(x, M_n) = \mu(x, M_n')$ for all $n \geqslant 1$.

The following theorem is the main result of this section.

THEOREM 4. *The following statements are equivalent.*

(a) $\mathfrak{B}$ *has an independent complement.*

(b) *The functions* $\{m_n, n \geqslant 1\}$ *are constants* a.s. $[P|_{\mathfrak{B}}]$.

(c) *Every maximal decomposition of* X *consists of sets independent of* $\mathfrak{B}$.

(d) *There is a maximal decomposition of* X *consisting of sets independent of* $\mathfrak{B}$.

PROOF. (a) $\Rightarrow$ (b). Since $\mathcal{Q}$ is countably generated we can assume without loss of generality that the independent complement $\mathfrak{B}^*$ of $\mathfrak{B}$ given by (a) is countably generated.

Let $B_1^*, B_2^*, \cdots$ be an enumeration of $\mathfrak{B}^*$-atoms of positive measure such that $P(B_1^*) \geqslant P(B_2^*) \geqslant \cdots$ and let $B_0^* = (U_n B_n^*)^c$. By Lemma 1, there exists $N_1 \in \mathfrak{B}$ with $P(N_1) = 0$ such that $x \notin N_1$ and $B^* \in \mathfrak{B}^* \Rightarrow \mu(x, B^*) = P(B^*)$. Since $\mathfrak{B} \vee \mathfrak{B}^* = \mathcal{Q}$ a.s. $[P]$ and since $\mathcal{Q}$ is countably generated there exists $X_1 \in \mathcal{Q}$ with $P(X_1) = 1$ such that $(\mathfrak{B} \vee \mathfrak{B}^*) \cap X_1 = \mathcal{Q} \cap X_1$. By (CP 3), $P(X_1) = 1$ implies the existence of $N_2 \in \mathfrak{B}$ with $P(N_2) = 0$ such that for all $x \notin N_2$, $\mu(x, X_1) = 1$.

Let $N_0 = N \cup N_1 \cup N_2$ and let $x \notin N_0$. Then $\mu(x, B(x)) = \mu(x, B(x) \cup X_1) = \sum_{n=0}^{\infty} \mu(x, B(x) \cap B_n^*)$. If $y \in B(x) \cap B_0^*$ then

$$\mu(x, \{y\}) \leqslant \mu(x, B(x) \cap B^*(y)) = \mu(x, B^*(y)) = P(B^*(y)) = 0,$$

where $B^*(y) \subset B_0^*$ is the $\mathfrak{B}^*$-atom containing y. If $y \in B(x) - X_1$ then $\mu(x, \{y\}) \leqslant \mu(x, X_1^c) = 0$. Hence for every $x \notin N_0$,

$$\mu(x, \{y\}) > 0 \Rightarrow y \in B(x) \cap B_n^* \cap X_1 \qquad (n \geqslant 1)$$

and since $\mathcal{Q} \cap X_1 = (\mathfrak{B} \vee \mathfrak{B}^*) \cap X_1$ we have

$$B(x) \cap B_n^* \cap X_1 = \{y\}.$$

Thus for every $x \notin N_0$ and for every $n \geqslant 1$

$$(^{**}) \qquad m_n(x) = \mu(x, B(x) \cap B_n^* \cap X_1) = P(B_n^*)$$

or m_n = constant a.s. $[P|_{\mathfrak{B}}]$.

(Notice that (**) together with Proposition 6 implies that $X = \bar{B}_0^* \cup \bar{B}_1^* \cup \bar{B}_2^* \cup \cdots$ where $\bar{B}_n^* = \bar{B}_n^* \cap X_1$ if $n \geqslant 1$ and $\bar{B}_0^* = (U_{n \geqslant 1}\bar{B}_n^*)^c$ is a maximal decomposition and hence (a) $\Rightarrow$ (d).)

(b) $\Rightarrow$ (c) follows from Lemma 1 and Proposition 6.

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a). Let $X = M_0 \cup M_1 \cup M_2 \cup \cdots$ be a maximal decomposition of X such that M_n is independent of $\mathfrak{B}$ for every $n \geqslant 0$.

Suppose $P(M_0) > 0$. In the subspace $(M_0, \mathcal{Q}_0 \cap M_0, P_{M_0})$ there is no partial selector for $\mathfrak{B} \cap M_0$ of positive measure. By Theorem 3 there exists an independent complement $\mathfrak{B}_0^*$ of $\mathfrak{B} \cap M_0$ in $(M_0, \mathcal{Q} \cap M_0, P_{M_0})$.

By Lemma 1 the given condition implies that $\mu(x, M_n) = P(M_n)$ a.s. $[P|_{\mathfrak{B}}]$. Let

$$\begin{aligned} \mathfrak{B}^* &= \sigma\{\mathfrak{B}_0^*, M_1, M_2, \cdots\} && \text{if } P(M_0) > 0, \\ &= \sigma\{M_1, M_2, \cdots\} && \text{if } P(M_0) = 0. \end{aligned}$$

It can be shown using Lemma 1 that $\mathfrak{B}^*$ is independent of $\mathfrak{B}$. Since each M_n is a partial selector for $\mathfrak{B}$ and since $(\mathfrak{B}_0^* \vee (\mathfrak{B} \cap M_0)) = \mathcal{Q} \cap M_0$ a.s. $[P_{M_0}]$ if $P(M_0) > 0$, it follows that $\mathfrak{B} \vee \mathfrak{B}^* = \mathcal{Q}$ a.s. $[P]$. Thus $\mathfrak{B}^*$ is an independent complement of $\mathfrak{B}$.

REMARK 3. Theorems 1, 3 and 4 are applicable in many situations where no assumption of Lebesgue spaces is made. For instance, consider the mixture problem for probabilities where we have a probability space $(X, \mathcal{Q}, \lambda)$, a Borel space $(Y, \mathfrak{B})$ and a transition function $\mu(x, B)$ given $\mathcal{Q}$ on $X \times \mathfrak{B}$. The measure μ on $\mathfrak{B}$ defined by $\mu(B) = \int \mu(x, B)d\lambda$ is a λ-mixture of $\mu(x, \cdot)$'s. When $\mathcal{Q}$, $\mathfrak{B}$ are separable consider the space $(X \times Y, \mathcal{Q} \otimes \mathfrak{B}, \lambda\mu)$ where $\lambda\mu$ is the product measure. Clearly the transition function gives rise to an everywhere proper regular conditional probability on $(X \times Y) \times (\mathcal{Q} \otimes \mathfrak{B})$ given $\mathcal{Q} \times Y$ and so Theorems 1, 3 and 4 are applicable in this setup. By doing so one can study the properties of the product space and of the mixture measure μ which is a marginal. Using this approach, in a subsequent paper, we will study perfect mixtures of perfect measures.

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STAT.-MATH. DIVISION
INDIAN STATISTICAL INSTITUTE
203, BARRACKPORE TRUNK ROAD
CALCUTTA 700 035, INDIA