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ZEROS OF THE DENSITIES OF INFINITELY DIVISIBLE MEASURES

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We consider an infinitely divisible measure μ on a locally compact Abelian group. If $\mu \ll \lambda$ (Haar measure), and if the semigroup generated by the support of the corresponding Lévy measure ν is the closure of an angular semigroup, then $\mu \sim \lambda$ over the support of μ. In particular, if $\int |\chi(x) - 1|\nu(dx) < \infty$, for all characters χ, or if $\nu \ll \lambda$ then $\mu \ll \lambda$ implies $\mu \sim \lambda$ over the support of μ.

1. Introduction and Summary. Hudson and Tucker (1975a) proved the following result: if F is an absolutely continuous infinitely divisible probability distribution on $\mathbb{R}^1$ with density f, then $f(x) > 0$ a.e. $[\lambda]$ on the support $S(F)$ of F and $S(F)$ is of the form $(-\infty, a]$, or $[a, \infty)$, or $\mathbb{R}$. Here λ is Lebesgue measure. Their method of proof involved admissible translates and used properties unique to the real line, and hence could not be extended even to $\mathbb{R}^2$. The support statements of Hudson and Tucker were generalized in the $\mathbb{R}^1$ setting by Tucker (1975) and Brockett (1977). Brockett showed that for continuous infinitely divisible measures on Hilbert spaces the support is the translate of an angular semigroup when the corresponding Lévy measure ν satisfies $\int_{\|x\|<1}\|x\|d\nu(x) < \infty$, and is of the form $(\mathbb{S} + A)$ with $\mathbb{S}$ angular and A closed in the contrary situation. This result generalized the corresponding result by Hudson and Mason (1975) for ν absolutely continuous on $\mathbb{R}^n$. Related results (but not equivalent ones) were obtained by Kallenberg (1976, Theorem 6.8) for supports of infinitely divisible random measures (or point processes.).

In our present work we shall extend the equivalence result of Hudson and Tucker (1975a) to locally compact groups. In the equivalence problem, we obtain the surprising result that it is the *geometric* character of the support of the Lévy measure which enters into the proof of equivalence with Haar measure rather than the analytical character of the Lévy measure. This is in contrast to the $\mathbb{R}^1$ case in which Hudson and Tucker used analytical properties of the Lévy measure to obtain their admissible translates proof.

Let G be a separable locally compact Abelian group with Haar measure λ. We shall consider infinitely divisible measures μ without Gaussian component since the Gaussian case has been adequately discussed previously (see Heyer 1977). Such measures are known to have characteristic functions of the form

$$\hat{\mu}(\chi) = \chi(x_0)\exp\{\int(\chi(x) - 1 - ig(\chi, x))d\nu(x)\} \tag{1.1}$$

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for each character $\chi \in G^*$ (see Parthasarathy 1967). The measure ν is called the Lévy measure of μ and satisfies $\int \mathrm{Re}(\chi(x) - 1)d\nu(x) < \infty$ for all characters χ, Re being "real part of". The function $g(\chi, x)$ is a centering term which insures the convergence of the integral in (1.1) when $\int|\chi(x) - 1|d\nu(x) = \infty$. It may be omitted in the case $\int|\chi(x) - 1|d\nu(x) < \infty$ in which case the formula

$$\hat{\mu}(\chi) = \chi_1(x)\exp\{\int(\chi(x) - 1)d\nu(x)\} \tag{1.2}$$

obtains. Measures with finite Lévy measures ν are the compound Poisson measures.

2. Equivalence properties. Let μ be infinitely divisible with representation given by (1.1), and let $\mathbb{S}$ denote the semigroup generated by the support of ν, i.e., $\mathbb{S} = (\cup_{k=0}^{\infty} S(\nu^{*k}))^-$ where $S(\alpha)$ denotes the support of the measure α. Using the techniques of Brown (1976) coupled with the support results of Brockett (1977) for generalized compound Poisson measures we may obtain the following lemma.

LEMMA 2.1. *There exists infinitely divisible measures μ_1 and μ_2 such that $\mu = \mu_1 * \mu_2$ where $\mu_1 \sim \mu$ and μ_2 is compound Poisson with $S(\mu_2) = \mathbb{S} = (\cup_{k=0}^{\infty} S(\nu^{*k}))^-$*

Before stating our main result we first recall the following definition. An angular semigroup is an open subsemigroup which contains zero in its closure. Further information on angular semigroups is available in Hille and Philips (1957), and the connection between angular semigroups and supports of infinitely divisible measures is found in Hudson and Mason (1975) and Brockett (1977).

THEOREM 2.1. *If μ is infinitely divisible, $\mu \ll \lambda$, and if $\mathbb{S}$ is the closure of an angular semigroup, then $\mu \sim \lambda$ on $S(\mu)$*, i.e., $\lambda\{x \varepsilon S(\mu): d\mu(x)/d\lambda = 0\} = 0$.

PROOF. Assume to the contrary that λ is not absolutely continuous with respect to μ. Then there exists a set $E \subseteq S(\mu)$ such that $\mu(E) = 0$ and $\lambda(E) > 0$. Let E' denote the collection of points in E of metric density 1 (i.e., such that $\lambda(E \cap N_c)/\lambda(N_c) \to 1$ as the neighborhood N_c of c collapses down to $\{c\}$). It is known that $\lambda(E \setminus E') = 0$ (see McShane 1947, pages 222–224). For $c \varepsilon E'$, take B as an open neighborhood about c.

Now, using Lemma 2.1 we may write $\mu = \mu_1 * \mu_2$ where $\mu \sim \mu_1$ and $S(\mu_2) = \mathbb{S}$. Then

$$0 = \mu(E) = \int \mu_1(E - t)d\mu_2(t)$$

so that $\mu_1(E - t) = 0$ for μ_2 a.e. t. Take a countable dense subset T of $S(\mu_2) = \mathbb{S}$ such that $\mu_1(E - t) = 0$ (and hence $\mu(E - t) = 0$) for $t \varepsilon T$. In particular, $\mu(B \cap E' - T) = 0$. Let us now define $F = [B \cap E' - \mathbb{S}] \setminus [B \cap E' - T]$.
Using an idea similar to that of Brown (1976) we prove:

CLAIM. $\lambda(F) = 0$.

PROOF. Suppose $\lambda(F) > 0$ and let $b - s$ be a point of metric density 1 of F where $b \varepsilon B \cap E'$ and $s \varepsilon \mathbb{S}$. Let $\mathfrak{N}(b - s)$ denote the family of neighborhoods of the point $b - s$. Take $\mathfrak{U} \subseteq \mathfrak{N}(b - s)$ such that if $U' \varepsilon \mathfrak{U}$ then $\lambda(U' \cap F) > (1 -$

$\varepsilon)\lambda(U')$. This can be done since $b - s$ is a point having metric density one. Since $b\varepsilon B \cap E'$ and B is open, there exists $V \varepsilon \mathcal{N}(b)$ such that $V \subseteq B$, $V - s\varepsilon \mathcal{U}$ and $\lambda(V \cap E') > (1 - \varepsilon)\lambda(V)$.

Since T is dense in $\mathbb{S}$, we choose $t_n \varepsilon T$, $t_n \to s$ and note that $\lambda(V \cap E' - t_n \Delta V - s) \leqslant \lambda(V \cap E' - t_n \Delta V - t_n) + \lambda(V - t_n \Delta V - s) = \lambda(V \cap E' \Delta V) + \lambda(V - t_n \Delta V - s)$. This second term converges to 0 as $n \to \infty$, hence is less than $\varepsilon\lambda(V)$ for n large. From our choice of V, $\lambda(V \cap E' \Delta V) < \varepsilon\lambda(V)$ so for n large $\lambda[V \cap E' - t_n \Delta(V - s)] < 2\varepsilon\lambda(V)$. Now $\lambda(V - s \cap F) \leqslant \lambda[(V - s) \setminus (V \cap E' - t_n)] + \lambda[(V \cap E' - t_n) \cap F]$. However, $V \cap E' \subseteq B \cap E'$ so $(V \cap E' - t_n) \cap F = \varnothing$ and hence

$$\lambda(V - s \cap F) \leqslant \lambda[(V - s) \setminus (V \cap E' - t_n)] \leqslant \lambda[(V - s)\Delta(V \cap E' - t_n)] < 2\varepsilon\lambda(V). \tag{2.1}$$

Observe however that $V - s$ is a neighborhood of $b - s$ and $V - s\varepsilon\mathcal{U}$ so that (2.1) contradicts $\lambda(U' \cap F) \geqslant (1 - \varepsilon)\lambda(U')$ for all $U'\varepsilon\mathcal{U}$. Thus $\lambda(F) = 0$ as claimed.

Now since $\mu(B \cap E' - T) = 0$ it follows from the claim and the absolute continuity of μ that $\mu(B \cap E' - \mathbb{S}) = 0$. The choice of the neighborhood B was somewhat arbitrary, so that if we cover E' by a countable number of neighborhoods $\{B_n\}$ we have $E' - \mathbb{S} \subseteq \cup_n(B_n \cap E' - \mathbb{S})$ so that $\mu(E' - \mathbb{S}) = 0$.

Using techniques based upon those of Hudson and Mason (1975, Lemma 2.7) we may show that if $\mathbb{S}$ is an angular semigroup, and A is a closed set, then $\lambda(\partial(A + \mathbb{S})) = 0$. From known support properties of convolution and Lemma 2.1, we know $S(\mu) = (A + \mathbb{S})^-$ where A is closed ($A = S(\mu)$) and $\mathbb{S}$ is angular. Thus without loss of generality we may assume the set $E \subseteq S(\mu)$ with $\mu(E) = 0$ and $\lambda(E) > 0$ satisfies $E \subseteq \text{int}(S(\mu)) = S(\mu)^o$. Let $c\varepsilon E'$ and note that $\mu(c - \mathbb{S}^o) = 0$. Since $c - \mathbb{S}^o$ is an open set this implies $S(\mu) \subseteq [c - \mathbb{S}^o]^c$. However, $c\varepsilon S(\mu)^o$ and $\mathbb{S}^o$ contains 0 as a limit point so $S(\mu) \cap [c - \mathbb{S}^o] \neq \varnothing$. This is a contradiction and implies such a set E cannot exist, i.e., $\lambda \sim \mu$ over $S(\mu)$.

COROLLARY 2.1. *If* $\mu \ll \lambda$ *satisfies* $\int|\chi(x) - 1|d\nu(x) < \infty$ *and* ν *is infinite, then* $\mu \sim \lambda$ *over* $S(\mu) = a + \mathbb{S}$.

PROOF. That $S(\mu) = a + \mathbb{S}$ follows from Brockett (1977). Without loss of generality we shall take $a = 0$. To invoke Theorem 2.1 it remains to show $\mathbb{S} = (\cup_{k=0}^{\infty} S(\nu^{*k}))^-$ is the closure of an angular semigroup.

CLAIM 1. $\mathbb{S}^o$ *is angular if, for a decreasing sequence of neighborhoods* $\{B_n\}$ *with* $\cap B_n = \{0\}$, $\lambda(B_n \cap \mathbb{S}) > 0$.

PROOF. The sum of two sets of positive measure contains an open set. Thus if $\lambda(B_n \cap \mathbb{S}) > 0$, then $(B_n \cap \mathbb{S}) + (B_n \cap \mathbb{S})$ contains an open set. But $\{B_n\}$ may be chosen such that $B_n + B_n \subseteq B_{n-1}$, hence $B_{n-1} \cap \mathbb{S}$ contains an open set implying $\mathbb{S}^o$ has 0 as a limit point.

CLAIM 2. $\lambda(B_n \cap \mathbb{S}) > 0$ *for all* n.

PROOF. Decompose $\mu = \mu_1 * \mu_2$ where $\mu_1(B_n) > 0$ and μ_2 is compound Poisson. We may do this since for μ_m defined by

$$\hat{\mu}_m(x) = \exp\{\textstyle\int_{B_m}(\chi(x) - 1)d\nu(x)\}$$

we have $\mu_m \Rightarrow \delta_0$, hence for m large $\mu_m(B_n) \geqslant 1 - \varepsilon$. Take $\mu_1 = \mu_m$ and μ_2 with characteristic function

$$\hat{\mu}_2(\chi) = \exp\{\textstyle\int_{B_m^c}(\chi(x) - 1)d\nu(x)\},$$

a compound Poisson measure.

Now μ_2 has an atom of size $d = \exp\{-\nu(B_m^c)\}$ at 0 so $\mu(B_n) = (\mu_1 * \mu_2)(B_n) = \int \mu_1(B - t)d\mu_2(t) \geqslant (1 - \varepsilon)d > 0$. Since $\mu \ll \lambda$ and $\mu(B_n) > 0$ we must have $\lambda(S(\mu) \cap B_n) = \lambda(\mathbb{S} \cap B_n) > 0$. By claim 1 $\mathbb{S}^o$ is angular, and hence by Theorem 2.1 $\mu \sim \lambda$ over $S(\mu)$.

COROLLARY 2.2. (Hudson and Mason (1975)). *If μ is infinitely divisible with infinite absolutely continuous Lévy measure ν, then $\mu \sim \lambda$ over $S(\mu)$.*

PROOF. If $\nu \ll \lambda$, then $\mu \ll \lambda$ and $\mathbb{S}$ is the closure of an angular semigroup (see Hudson and Mason (1975) or Yuan and Liang (1976)). The result follows from Theorem 2.1.

REMARK. We conjecture that it is always true that $\mu \ll \lambda$ implies $\mathbb{S}$ is the closure of an angular semigroup (and hence $\mu \sim \lambda$ over $S(\mu)$); however, we have been unable to prove this for $\nu \not\ll \lambda$ and $\int|\chi(x) - 1|d\nu(x) = \infty$.

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