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ALMOST SURE LIMIT POINTS OF MAXIMA OF STATIONARY GAUSSIAN SEQUENCES

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Let $\{X_n, n \geqslant 1\}$ be a discrete-parameter stationary Gaussian process with $E(X_i) = 0$, $E(X_i^2) = 1$ for all i and $E(X_i X_{i+n}) = r(n)$. Let $M_n = \text{maximum}(X_1, X_2 \cdots X_n)$. Under the condition that either $(\log n)^{1+\gamma} r(n) = O(1)$ as $n \to \infty$, for some $\gamma > 0$ or $\sum_{j=1}^{\infty} r^2(j) < \infty$, the set of all almost sure limit points of the vector sequence

$$\left\{ \frac{M_{1,n} - b_n}{a_n}, \frac{M_{2,n} - b_n}{a_n}, \cdots \frac{M_{p,n} - b_n}{a_n} \right\}$$

is obtained, where $(M_{j,n})$, $j = 1, 2 \cdots p$ are independent copies of (M_n); $a_n = (\log\log n)(2 \log n)^{-\frac{1}{2}}$ and $b_n = (2 \log n)^{\frac{1}{2}}$.

1. Introduction. Let $\{X_n, n \geqslant 1\}$ be a discrete-parameter stationary Gaussian process with $E(X_i) = 0$, $E(X_i^2) = 1$ for all i and $E(X_i X_{i+n}) = r(n)$. Let $M_n =$ maximum $(X_1, X_2 \cdots X_n)$; $a_n = (\log\log n)(2 \log n)^{-\frac{1}{2}}$ and $b_n = (2 \log n)^{\frac{1}{2}}$. It can be observed that (cf. Pickands (1969), Deo ((1973), Corollary, page 407), Mittal (1974)) the set of all limit points of $(M_n - b_n)/a_n$, under almost sure convergence, coincides with the closed interval $S_1 = [-\frac{1}{2}, \frac{1}{2}]$ provided either $(\log n) r(n) = O(1)$ as $n \to \infty$ or $\sum_{j=1}^{\infty} r^2(j) < \infty$.

The object of this paper is to extend the above result to vector sequence. We prove the following.

THEOREM. *The almost sure limit points of*

$$\xi_n^{(p)} = \left\{ \frac{M_{1,n} - b_n}{a_n}, \frac{M_{2,n} - b_n}{a_n}, \cdots \frac{M_{p,n} - b_n}{a_n} \right\}, n \geqslant 1, \quad p \geqslant 1,$$

coincide with the set

$$S_p = \left\{ (x_1, x_2, \cdots x_p) : x_i \geqslant -\tfrac{1}{2}, i = 1, 2, \cdots p; \textstyle\sum_{i=1}^{p} x_i \leqslant 1 - p/2 \right\},$$

provided either there exists $\gamma > 0$ *such that* $(\log n)^{1+\gamma} r(n) = O(1)$ *as* $n \to \infty$ *or* $\sum_{j=1}^{\infty} r^2(j) < \infty$, *where* $(M_{j,n})$, $j = 1, 2 \cdots p$ *are independent copies of* (M_n).

It appears that the condition: $(\log n) r(n) = O(1)$ as $n \to \infty$ is not sufficient for the theorem to hold. Almost sure limit sets of random vectors with independent components have received attention in the recent literature (cf. LePage (1973), Pakshirajan and Vasudeva (1977) and Strassen (1964)). We present the proof of the above theorem for $p = 2$ only to avoid cumbersome notation. Consequently, in view of the result for the one-dimensional case ($p = 1$), it is clear that we need consider only the points (x_1, x_2) such that $|x_i| \leqslant \frac{1}{2}$, $i = 1, 2$.

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Let $\{Y_n\}$ be an independent copy of $\{X_n\}$ and let $M_{1,n}$ = maximum $(X_1, X_2, \cdots X_n)$ and $M_{2,n}$ = maximum$(Y_1, Y_2, \cdots Y_n)$. Write $U_n = (M_{1,n} - b_n)/a_n$ and $V_n = (M_{2,n} - b_n)/a_n$. For any positive number u, $[u]$ denotes the greatest integer $\leqslant u$. The abbreviation i.o. stands for infinitely often.

2. Proof of the theorem. We need the following three lemmas for the proof of our theorem which, as stated in the previous section, will be established for $p = 2$.

LEMMA 2.1. *For all* $x_1, x_2 > -\frac{1}{2}$ *and for every* $\varepsilon > 0$,
(i) $P\{U_{n_k} > x_1 + \varepsilon,\ V_{n_k} > x_2 \text{ i.o.}\} = 0,$
(ii) $P\{U_{n_k} > x_1,\ V_{n_k} > x_2 + \varepsilon \text{ i.o.}\} = 0,$
where $n_k = [\exp(k^{(1+x_1+x_2)^{-1}})]$.

PROOF. Since U_{n_k} and V_{n_k} are independent

$$\begin{aligned} P\{U_{n_k} > x_1 + \varepsilon, V_{n_k} > x_2\} &= P\{U_{n_k} > x_1 + \varepsilon\}P\{V_{n_k} > x_2\} \\ &\leqslant n_k \cdot P(X_1 > (x_1 + \varepsilon)a_{n_k} + b_{n_k}) \\ &\quad \times n_k \cdot P(Y_1 > x_2 a_{n_k} + b_{n_k}) \\ &\sim \text{const. } k^{-(1+\varepsilon/(1+x_1+x_2))} \quad \text{as} \quad k \to \infty, \end{aligned}$$

using the known result $1 - \Phi(x) \sim (2\pi)^{-\frac{1}{2}}x^{-1}\exp(-x^2/2)$ as $x \to \infty$ for the standard normal distribution function Φ. Thus $\Sigma_k P(U_{n_k} > x_1 + \varepsilon, V_{n_k} > x_2) < \infty$. An application of the Borel–Cantelli lemma completes the proof of (i). Proof of (ii) is similar.

LEMMA 2.2. *For* $x_1, x_2 > -\frac{1}{2}$ *and* $x_1 + x_2 \leqslant 0$,

$$P(U_{n_k} > x_1, V_{n_k} > x_2 \text{ i.o.}) = 1,$$

provided either there exists $\gamma > 0$ *such that* $(\log n)^{1+\gamma}r(n) = O(1)$ *as* $n \to \infty$ *or* $\Sigma_{j=1}^{\infty} r^2(j) < \infty$.

PROOF. Define the integer sequence $m_k = [n_k(\log k)^{-\frac{1}{2}}.]$ Let $F_k = \{\max_{n_k - m_k + 1 \leqslant \nu \leqslant n_k} X_\nu > c_{n_k}\}$; $G_k = \{\max_{n_k - m_k + 1 \leqslant \nu \leqslant n_k} Y_\nu > d_{n_k}\}$ and $E_k = F_k \cap G_k$ where $c_{n_k} = x_1 a_{n_k} + b_{n_k}$ and $d_{n_k} = x_2 a_{n_k} + b_{n_k}$. When $r(n) \equiv 0$ the corresponding variables and events are indicated by an asterisk. Thus we have X_k^*, E_k^*, G_k^*, etc. Note that $E_k \subset (U_{n_k} > x_1, V_{n_k} > x_2)$. Hence the lemma will be established if we show

$$P(E_k \text{ i.o.}) = 1. \tag{2.1}$$

This, in turn, will follow when we show as $n \to \infty$, that

$$E(J_n) \to \infty \tag{2.2}$$

$$J_n/E(J_n) \to_p 1 \tag{2.3}$$

where $J_n = \Sigma_{k=N}^{n} I_k$ for sufficiently large N, I_k being the indicator function of E_k.

In order to establish (2.2) consider

$$\begin{aligned} P(E_k) - P(E_k^*) &= P(F_k)P(G_k) - P(F_k^*)P(G_k^*) \\ &= P(F_k)\{P(G_k) - P(G_k^*)\} + P(G_k^*)\{P(F_k) - P(F_k^*)\}. \end{aligned} \tag{2.4}$$

Therefore

$$|P(E_k) - P(E_k^*)| \leq P(F_k)|P(G_k) - P(G_k^*)| + P(G_k^*)|P(F_k) - P(F_k^*)| = A_k, \quad \text{say.} \tag{2.5}$$

Observe

(1) from the tail behaviour of $\Phi(x)$ that

(a) $P(F_k) \leq m_k P(X_1 > c_{n_k}) \sim$ const. $(\log k)^{-\frac{1}{2}}(\log n_k)^{-(x_1+\frac{1}{2})}$ as $k \to \infty$.

(b) $P(F_k^*) = 1 - \{\Phi(c_{n_k})\}^{m_k} = 1 - \exp\big(m_k \log\{1 - (1 - \Phi(c_{n_k}))\}\big) = 1 - \exp\{-m_k(1 - \Phi(c_{n_k}))(1 + o(1))\} \sim$ const. $(\log k)^{-\frac{1}{2}}(\log n_k)^{-x_1-\frac{1}{2}}$ as $k \to \infty$, whenever $x_1 > -\frac{1}{2}$. Similarly, $P(G_k^*) \sim$ const. $(\log k)^{-\frac{1}{2}}(\log n_k)^{-x_2-\frac{1}{2}}$ as $k \to \infty$, whenever $x_2 > -\frac{1}{2}$.

(2) By Lemma 3.1 of Berman (1964),

$$|P(F_k) - P(F_k^*)| = |P(F_k^c) - P(F_k^{*c})|$$

$$\leq (2\pi)^{-1}\Sigma_{j=1}^{m_k-1}|r(j)|(m_k - j)(1 - r^2(j))^{-\frac{1}{2}} \exp\big(-c_{n_k}^2/(1 + |r(j)|)\big)$$

and similarly $|P(G_k) - P(G_k^*)|$ is

$$\leq (2\pi)^{-1}\Sigma_{j=1}^{m_k-1}|r(j)|(m_k - j)(1 - r^2(j))^{-\frac{1}{2}} \exp\big(-d_{n_k}^2/(1 + |r(j)|)\big).$$

Then it can be easily seen that $\lim_{n\to\infty} \Sigma_{k=N}^n A_k < \infty$, under the condition that either $(\log n)r(n) = O(1)$ as $n \to \infty$ or $\Sigma_{j=1}^{\infty} r^2(j) < \infty$ (cf. proof of Theorem 3.1 of Berman (1964)). Further, $\Sigma_{k=N}^n P(F_k^*)P(G_k^*) =$ const. $\Sigma_{k=N}^n (k \log k)^{-1} \to \infty$ as $n \to \infty$. The proof of (2.2) is thus complete.

By Chebycheff's inequality

$$P\{|(J_n|E(J_n)) - 1| > \varepsilon\} \leq V(J_n)/\big(\varepsilon^2(E(J_n))^2\big) = \frac{\Sigma_{k=N}^n V(I_k) + 2\Sigma\Sigma_{N\leq k<l\leq n} \operatorname{Cov}(I_k, I_l)}{\varepsilon^2(E(J_n))^2}. \tag{2.6}$$

Clearly $\Sigma_{k=N}^n V(I_k) \leq \Sigma_{k=N}^n E(I_k) = o(E(J_n))^2$ as $n \to \infty$. Hence (2.3) will be established when we show

$$\sup_n|\Sigma\Sigma_{N\leq k<l\leq n} \operatorname{Cov}(I_k, I_l)| < \infty. \tag{2.7}$$

This can be done as below. For $k < l$,

$$\begin{aligned} \operatorname{Cov}(I_k, I_l) &= E(I_k I_l) - E(I_k)E(I_l) \\ &= P(F_k \cap F_l)P(G_k \cap G_l) - P(F_k)P(F_l)P(G_k)P(G_l) \\ &= P(F_k \cap F_l)\{P(G_k \cap G_l) - P(G_k)P(G_l)\} \\ &\quad + P(G_k)P(G_l)\{P(F_k \cap F_l) - P(F_k)P(F_l)\} \\ &= \{P(F_k \cap F_l) - P(F_k)P(F_l)\} \\ &\quad \times (\{P(G_k \cap G_l) - P(G_k)P(G_l)\} + P(G_k)P(G_l)) \\ &\quad + P(F_k)P(F_l)\{P(G_k \cap G_l) - P(G_k)P(G_l)\}. \end{aligned} \tag{2.8}$$

Therefore

$$(2.9)\quad |\mathrm{Cov}(I_k, I_l)| \leqslant (|P(F_k \cap F_l) - P(F_k)P(F_l)|)(|P(G_k \cap G_l) - P(G_k)P(G_l)|) + P(G_k)P(G_l)\{|P(F_k \cap F_l) - P(F_k)P(F_l)|\} + P(F_k)P(F_l)\{|P(G_k \cap G_l) - P(G_k)P(G_l)|\}.$$

By Lemma 1.5 of Qualls and Watanabe (1971) we get

$$(2.10)\quad |P(F_k \cap F_l) - P(F_k)P(F_l)| = |P(F_k^c \cap F_l^c) - P(F_k^c)P(F_l^c)| \leqslant \Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}|r|\textstyle\int_0^1\phi(c_{n_k}, c_{n_l}; \lambda r)\, d\lambda,$$

where $\phi(u, v; \rho)$ is the standard bivariate normal density with correlation coefficient ρ and $r = r(n_l - m_l + \mu - n_k + m_k - \nu)$. Now consider the following cases.

CASE (i). Assume that for some $\gamma > 0$, $(\log n)^{1+\gamma}r(n) = O(1)$ as $n \to \infty$.

Then, clearly $(\log n)^{1+\gamma}\delta(n) = O(1)$ as $n \to \infty$ where $\delta(n) = \sup_{j \geqslant n}|r(j)|$. The stationarity of X_n's and the condition on $\{r(n)\}$ ensure that $\delta(1) < 1$. Observe that $0 < n_k/n_{k+1} \leqslant e^{-1}$, whenever $x_1 + x_2 \leqslant 0$ and hence $n_l - m_l + \mu - n_k + m_k - \nu \geqslant n_l - m_l - n_k \geqslant n_l\ (1 - (\log l)^{-\frac{1}{2}} - e^{-1})$. Therefore $|r(n_l - m_l + \mu - n_k + m_k - \nu)| \leqslant \delta(\text{const. } n_l) \leqslant \text{const. } (\log n_l)^{-1-\gamma}$ for all k and l such that $l > k \geqslant N$, N being a sufficiently large positive integer and when $x_1 + x_2 \leqslant 0$. Since

$$(2.11)\quad \phi(c_{n_k}, c_{n_l}; \lambda r) \leqslant (2\pi)^{-1}(1 - \delta^2(1))^{-\frac{1}{2}} \exp\left(-\left(c_{n_k}^2 - 2|r|c_{n_k}c_{n_l} + c_{n_l}^2\right)/2\right) \leqslant \text{const. } \exp\left(-\left(c_{n_k}^2 + (1 - 2|r|)c_{n_l}^2\right)/2\right)$$

because the c_{n_j}'s are monotonic increasing in j, (2.10) can be majorised by

$$(2.12)\quad \text{const. } m_k \cdot m_l\{\delta(\text{const. } n_l)\}n_k^{-1}(\log n_k)^{-x_1} \times \left(n_l(\log n_l)^{x_1}\right)^{-(1-2\delta(\text{const. } n_l))} \leqslant \text{const. } (\log k)^{-\frac{1}{2}}(\log l)^{-\frac{1}{2}}(\log n_k)^{-x_1}(\log n_l)^{-(1+\gamma+x_1)}$$

since $n_l^{2\delta(\text{const. } n_l)} = \exp\{2(\delta(\text{const. } n_l)) \log n_l\}$ is bounded. Similarly $|P(G_k \cap G_l) - P(G_k)P(G_l)|$ can be majorised by an expression which is obtained from (2.12) by replacing x_1's by x_2's. Hence the first term on the right-hand side of (2.9) is

$$\leqslant \text{const. } (\log k)^{-1}(\log l)^{-1}(\log n_k)^{-(x_1+x_2)}(\log n_l)^{-(2+2\gamma+x_1+x_2)}$$
$$\leqslant \text{const. } (\log k)^{-1}(\log n_k)^{-(1+\gamma+x_1+x_2)}(\log l)^{-1}(\log n_l)^{-(1+\gamma+x_1+x_2)}.$$

The second term in (2.9) is

$$\leqslant m_kP(Y_1 > d_{n_k}) \cdot m_lP(Y_1 > d_{n_l})\{|P(F_k \cap F_l) - P(F_k)P(F_l)|\}$$
$$\leqslant \text{const. } (\log k)^{-1}(\log n_k)^{-\left(x_1+x_2+\frac{1}{2}\right)}(\log l)^{-1}(\log n_l)^{-\left(x_1+x_2+\gamma+\frac{3}{2}\right)},$$

for $l > k \geqslant N$, with N sufficiently large;

$$\leqslant \text{const. } (\log k)^{-1}(\log n_k)^{-(1+x_1+x_2+\gamma/2)}(\log l)^{-1}(\log n_l)^{-(1+x_1+x_2+\gamma/2)}.$$

The third term in (2.9) is bounded by the same expression as the second.

From these bounds (2.7) follows immediately.

CASE (ii). Assume that $\Sigma_{j=1}^{\infty} r^2(j) < \infty$. From (2.8) observe that

$$|\mathrm{Cov}(I_k, I_l)| \leqslant |P(G_k \cap G_l) - P(G_k)P(G_l)| + |P(F_k \cap F_l) - P(F_k)P(F_l)|$$
$$\leqslant \Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}|r|\left(\textstyle\int_0^1\{\phi(d_{n_k}, d_{n_l}; \lambda r) + \phi(c_{n_k}, c_{n_l}; \lambda r)\}\, d\lambda\right)$$

by Lemma 1.5 of Qualls and Watanabe (1971), where 'r' is the same as defined at (2.10).

The above expression can be majorised by

$$\text{(2.13)} \quad \Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}|r|\left(\exp\left\{-\left(d_{n_k}^2 + (1-2|r|)d_{n_l}^2\right)/2\right\}\right.$$
$$\left. + \exp\left\{-\left(c_{n_k}^2 + (1-2|r|)c_{n_l}^2\right)/2\right\}\right), \qquad \text{(cf. (2.11))}.$$

Under the assumption on $r(n)$, it follows that $|r| = |r(n_l - m_l + \mu - n_k + m_k - \nu)| < \varepsilon$ for all $l > k \geqslant N$, where ε is a sufficiently small positive number and when $x_1 + x_2 \leqslant 0$. Then (2.13) can be majorised by

$$\text{(2.14)} \quad \left\{\left(\exp\left\{-\left(d_{n_k}^2 + (1-2\varepsilon)d_{n_l}^2\right)/2\right\}\right) + \left(\exp\left\{-\left(c_{n_k}^2 + (1-2\varepsilon)c_{n_l}^2\right)/2\right\}\right)\right\}$$
$$\Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}|r(n_l - m_l + \mu - n_k + m_k - \nu)|.$$

By the Cauchy–Schwarz inequality,

$$\Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}|r(n_l - m_l + \mu - n_k + m_k - \nu)|$$
$$\leqslant m_l^{\frac{1}{2}} \cdot \left(\Sigma_{\mu=1}^{m_l}(\Sigma_{\nu=1}^{m_k}|r(n_l - m_l + \mu - n_k + m_k - \nu)|)^2\right)^{\frac{1}{2}}$$
$$\leqslant m_l^{\frac{1}{2}} \cdot m_k^{\frac{1}{2}} \cdot \left(\Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}\{r(n_l - m_l + \mu - n_k + m_k - \nu)\}^2\right)^{\frac{1}{2}}$$
$$\leqslant m_l^{\frac{1}{2}} \cdot m_k \cdot \left(\Sigma_{j=1}^{\infty} r^2(j)\right)^{\frac{1}{2}}$$

since

$$\Sigma_{\mu=1}^{m_l}\Sigma_{\nu=1}^{m_k}\{r(n_l - m_l + \mu - n_k + m_k - \nu)\}^2$$
$$= \Sigma_{\mu=1}^{m_l}\{r(n_l - m_l + \mu - n_k + m_k - 1)\}^2$$
$$+ \Sigma_{\mu=1}^{m_l}\{r(n_l - m_l + \mu - n_k + m_k - 2)\}^2$$
$$+ \cdots + \Sigma_{\mu=1}^{m_l}\{r(n_l - m_l + \mu - n_k)\}^2$$
$$\leqslant m_k \cdot \Sigma_{j=1}^{\infty} r^2(j).$$

Thus (2.14) is bounded by

$$\text{const. } m_k \cdot m_l^{\frac{1}{2}} n_k^{-1} n_l^{-(1-2\varepsilon)}\left((\log n_k)^{-x_2}(\log n_l)^{-(1-2\varepsilon)x_2}\right.$$
$$\left. + (\log n_k)^{-x_1}(\log n_l)^{-(1-2\varepsilon)x_1}\right).$$

Then (2.7) follows immediately.

Thus, via (2.6), the proof of the lemma is now complete.

LEMMA 2.3. *For all* $x_1, x_2 \geqslant -\frac{1}{2}$ *with* $x_1 + x_2 \geqslant 0$ *and for every* $\varepsilon > 0$,

$$P(U_n > x_1 + \varepsilon, V_n > x_2 + \varepsilon \text{ i.o.}) = 0.$$

PROOF. Let $\beta(k) = [\exp(k)]$. Notice that, for x fixed, $xa_r + b_r$ is ultimately monotonic increasing in r. Therefore, it is seen that $\{M_{1,n} > (x_1 + \varepsilon)a_n + b_n, M_{2,n} > (x_2 + \varepsilon)a_n + b_n$ for infinitely many $n\}$ is $\subset \{M_{1,\beta(k+1)} > (x_1 + \varepsilon)a_{\beta(k)} + b_{\beta(k)}, M_{2,\beta(k+1)} > (x_2 + \varepsilon)a_{\beta(k)} + b_{\beta(k)}$ for infinitely many $k\}$. Since, as in the proof of Lemma 2.1,

$$P\{M_{1,\beta(k+1)} > (x_1 + \varepsilon)a_{\beta(k)} + b_{\beta(k)}, M_{2,\beta(k+1)} > (x_2 + \varepsilon)a_{\beta(k)} + b_{\beta(k)}\}$$

$$\leqslant \text{const.} \left(\frac{\beta(k+1)}{\beta(k)}\right)^2 (\log \beta(k))^{-(1+x_1+x_2+2\varepsilon)}$$

$$= \text{const.}\ k^{-(1+x_1+x_2+2\varepsilon)} \quad \text{as} \quad k \to \infty,$$

the proof of the lemma is completed by observing that $\Sigma_k k^{-(1+x_1+x_2+2\varepsilon)} < \infty$ when $x_1 + x_2 \geqslant 0$.

With these lemmas we can furnish the proof of our theorem.

PROOF OF THE THEOREM. From the result for one-dimensional case it is clear that the limit set of $(\xi_n^{(2)})$ must be within the square $\{(x_1, x_2): |x_i| \leqslant \frac{1}{2}, i = 1, 2\}$. It follows from Lemma 2.3 that the limit set is contained in S_2. We conclude from Lemmas 2.1 and 2.2 that every point of S_2 except $(-\frac{1}{2}, -\frac{1}{2})$ is a limit point. That the point $(-\frac{1}{2}, -\frac{1}{2})$ is also a limit point follows from continuity consideration. This completes the proof of the theorem.

REMARK. Suppose $(X_{j,n})$, $j = 1, 2, \cdots p$ are p independent sequences of stationary Gaussian variables with means zero, variances one and the covariance functions $r_j(n), j = 1, 2, \cdots p$. Define $M_{j,n} = \max_{1 \leqslant k \leqslant n} X_{j,k}$. Let us say that the sequence $\{r_j(n)\}$ satisfies the condition α_j if $\Sigma_{k=1}^{\infty} r_j^2(k) < \infty$ and the condition α_j' if $(\log n)^{1+\gamma_j} r_j(n) = O(1)$ as $n \to \infty$, for some $\gamma_j > 0, j = 1, 2, \cdots p$. Then it is not difficult to show that the set of all almost sure limit points of $\{\xi_n^{(p)}\}$ is S_p provided for each $j, j = 1, 2, \cdots p$ either α_j holds or α_j' holds.

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