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LIMIT POINTS ASSOCIATED WITH GENERALIZED ITERATED LOGARITHM LAWS

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The set of limit points associated with a generalized iterated logarithm law for sums of asymmetric i.i.d. random variables is shown to be the entire closed interval bounded by the lim inf and the lim sup.

1. Introduction. Klass and Teicher [3] proved one-sided iterated logarithm laws of the form lim sup $\Sigma_1^n X_i/b_n = 1$ a.s. and lim sup $\Sigma_1^n X_i/b_n = -1$ a.s. for asymmetric i.i.d. random variables, X_n, $n \geqslant 1$, with lim sup $= 1$ when $EX_1 = 0$ and lim sup $= -1$ when $E|X_1| = \infty$. In both cases lim inf $\Sigma_1^n X_i/b_n = -\infty$ a.s. In this note the set of limit points of $\Sigma_1^n X_i/b_n$ is found. Kesten [2] showed that for any sequence $b_n \to \infty$, the set of limit points of $\Sigma_1^n X_i/b_n$ is nonrandom with probability one. Here we prove that for the sequence b_n defined in [3], this nonrandom set is the entire interval between the lim inf and the lim sup.

First let us give some definitions, following Klass and Teicher [3]. For any unbounded random variable X, let

$$(1)\qquad \begin{array}{lll} \text{(i)} & \bar{\mu}(x) = \int_x^\infty P\{|X| > y\}\, dy & \text{if } E|X| < \infty \\ \text{(ii)} & \mu(x) = \int_0^x P\{|X| > y\}\, dy & \text{if } E|X| = \infty. \end{array}$$

Both $x/\bar{\mu}(x)$ and $x/\mu(x)$ are increasing in x, so we can define

$$(2)\qquad \begin{array}{lll} \text{(i)} & b_x = (x/\bar{\mu}(x))^{-1} & \text{when } E|X| < \infty \\ \text{(ii)} & b_x = (x/\mu(x))^{-1} & \text{when } E|X| = \infty \end{array}$$

where the superscript indicates the inverse function. The function b_x is defined for all large x and

$$(3)\qquad \begin{array}{ll} \text{(i)} & b_x/x = \bar{\mu}(b_x) \text{ if } E|X| < \infty \text{ and } b_x/x \downarrow 0 \text{ since } \bar{\mu} \text{ is decreasing,} \\ \text{(ii)} & b_x/x = \mu(b_x) \text{ if } E|X| = \infty \text{ and } b_x/x \uparrow \infty \text{ since } \mu \text{ is increasing.} \end{array}$$

2. Limit points. Let us first prove the following

LEMMA. *Let X be an unbounded random variable and let $\bar{\mu}$ or μ be slowly varying at infinity (definition, Feller* [1], *page* 269), *according as $E|X|$ is finite or infinite. If*
(i) $E[X^+/\bar{\mu}(X^+)] < \infty$ *when* $E|X| < \infty$, *or*
(ii) $E[X^+/\mu(X^+)] < \infty$ *when* $E|X| = \infty$,
then

$$(4)\qquad \Sigma_{n=1}^\infty P\{X^+ > \varepsilon b_n\} < \infty \qquad \textit{for all} \quad \varepsilon > 0.$$

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PROOF. It suffices to consider $\varepsilon \in (0, 1)$. We shall prove that $E[(X^+/\varepsilon)/\mu(X^+/\varepsilon)] < \infty$, which is equivalent to (4).

Suppose $E|X| = \infty$ and let $a(x) = x/\mu(x)$. Then $a(x)$ is increasing and

$$\begin{aligned} a(x) &\leq a(x/\varepsilon) \\ &= (1/\varepsilon)a(x)\mu(x)/\mu(x/\varepsilon). \end{aligned}$$

Since μ is slowly varying at infinity,

$$a(x) \leq a(x/\varepsilon) \leq (2/\varepsilon)a(x)$$

for large x. Therefore $Ea(X^+) < \infty$ if and only if $Ea(X^+/\varepsilon) < \infty$ for all $\varepsilon \in (0, 1)$, so (4) holds. The proof for $E|X| < \infty$ is analogous. □

THEOREM. *Let $X, X_n, n \geq 1$, be unbounded* i.i.d. *random variables. If*

(i) $EX = 0$, $E[X^+/\bar{\mu}(X^+)] < \infty$, *and* $\bar{\mu}(x) \sim \bar{\mu}(x \log_2 x)$ *as* $x \to \infty$, *or*

(ii) $E|X| = \infty$, $E[X^+/\mu(X^+)] < \infty$, *and* $\mu(x) \sim \mu(x \log_2 x)$ *as* $x \to \infty$,

then the set of limit points of S_n/b_n,

$$\cap_{m=1}^{\infty} \overline{\{S_n/b_n : n \geq m\}} = [\liminf S_n/b_n, \limsup S_n/b_n] \quad \text{a.s.},$$

where $S_n = \sum_{i=1}^n X_i$, $\log_2 x = \log\log x$, *and* $\bar{\mu}$, μ *and* b_n *are defined in Section* 1.

PROOF. Let $t \in (\liminf S_n/b_n, \limsup S_n/b_n)$. Then in both cases we have

$$P\{S_n/b_n \leq t \leq S_{n+1}/b_{n+1}, \text{ i.o.}\} = 1.$$

Hence, for infinitely many n,

$$\begin{aligned} |t - S_n/b_n| &= t - S_n/b_n \\ &\leq S_{n+1}/b_{n+1} - S_n/b_n \\ &= S_{n+1}/b_{n+1}(1 - b_{n+1}/b_n) + (X_{n+1}^+/b_{n+1})(b_{n+1}/b_n) \end{aligned}$$

with probability one.

By Theorems 3 and 4 of Klass and Teicher [3], in both cases $S_{n+1}/b_{n+1} < 2$ a.s. for all large n. Also by the preceding lemma, $X_{n+1}^+/b_{n+1} \to 0$ a.s., so in order to prove that

$$(5) \qquad P\{|t - S_n/b_n| \leq \varepsilon, \text{ i.o.}\} = 1 \qquad \text{for all} \quad \varepsilon > 0$$

it suffices to show that $b_n/b_{n+1} \to 1$ as $n \to \infty$.

CASE (i).

$$\begin{aligned} 1 &\leq b_{n+1}/b_n \\ &= [(b_{n+1}/(n+1))/(b_n/n)](n+1)/n \\ &\leq (n+1)/n \end{aligned}$$

since $b_n/n \downarrow 0$ by (3) (i). Thus $b_{n+1}/b_n \to 1$.

CASE (ii).

$$\begin{aligned}1 &\leq b_{n+1}/b_n \\ &= \left[(b_{n+1}/(n+1))/(b_n/n)\right](n+1)/n \\ &\leq \left[(b_{2n}/2n)/(b_n/n)\right](n+1)/n\end{aligned}$$

since $b_n/n\uparrow\infty$ by (3) (ii). But $\mu(x) \sim \mu(x \log_2 x)$ implies that b_n/n is slowly varying at infinity (see [3]), so $b_{n+1}/b_n \to 1$.

Therefore (5) holds, and since the set of limit points of S_n/b_n is nonrandom (see Kesten [2]), this set must be [lim inf S_n/b_n, lim sup S_n/b_n] because t was chosen arbitrarily. □

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