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A RENEWAL MODEL WITH RANDOMLY SELECTED PARAMETERS

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Let $\{\mu_1, \mu_2, \cdots\}$ be chosen from a strictly stationary, ergodic sequence of random variables each with distribution concentrated on $(0, \infty)$. Let $S_n = T_1 + \cdots + T_n$ be a sum of independent random variables where T_j is exponential with mean μ_j. Limiting properties of S_n are considered. More limiting properties are derived under the assumption that $\{\mu_1, \mu_2, \cdots\}$ is strongly mixing and then under the assumption of independence.

1. The model. Let $T_1, T_2, \cdots$ be independent, exponential random variables with parameters (means) respectively $\mu_1, \mu_2, \cdots$. The sequence $\Lambda = \{\mu_1, \mu_2, \cdots\}$ constitutes the *parameter sequence* for the renewal process $\{S_n = T_1 + \cdots + T_n\}_{n=0}^{\infty}(S_0 \equiv 0)$. The μ_i's are chosen previous to the renewal process; they form a sample from a strictly stationary sequence of random variables each with distribution G concentrated on $(0, \infty)$. This paper is concerned with limit behaviors of the renewal process $\{S_n\}$ given a "typical" parameter sequence Λ.

NOTATION. Set $\lambda_i = \mu_i^{-1}$ for all i. Let F_i be the exponential distribution with mean μ_i and let f_i be the corresponding density. As usual count time 0 as renewal number 1. The convolution of distribution functions H_1 and H_2 is

$$H_1 * H_2(t) = \int_{-\infty}^{\infty} H_1(t-x)H_2(dx)$$

whereas the convolution of densities h_1 and h_2 is

$$h_1 * h_2(t) = \int_{-\infty}^{\infty} h_1(t-x)h_2(x)\,dx.$$

N_t denotes the number of renewals in $(0, t]$ so that

$$\begin{aligned} P(N_t = n) &= P(T_1 + \cdots + T_n \leqslant t, T_1 + \cdots + T_{n+1} > t) \\ &= F_1 * \cdots * F_n(t) - F_1 * \cdots * F_{n+1}(t) \\ &= \mu_{n+1} \cdot f_1 * \cdots * f_{n+1}(t) \end{aligned}$$

as can easily be verified for exponential distributions. Finally, $U(t)$ is the expected number of renewals in $[0, t]$

$$U(t) = \Sigma_{n=0}^{\infty} F_1 * \cdots * F_n(t)$$

—the addend for index $n = 0$ being the atom at the origin (evaluated at t).

The main results in this paper are contained in these two theorems:

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THEOREM 1. *Suppose* $\{\mu_1, \mu_2, \cdots\}$ *is chosen from a strictly stationary, ergodic sequence. Then for* a.e. *parameter sequence*

(a) $U(t) = 1 + E(N_t) < \infty$ *for all* t,

(b) $t^{-1}U(t) \to (E(\mu_1))^{-1}$ *as* $t \to \infty$.

If $\{\mu_1, \mu_2, \cdots\}$ *is strongly mixing, then for* a.e. *parameter sequence*

(c) $n^{-1}S_n \to E(\mu_1)$, $t^{-1}N_t \to (E(\mu_1))^{-1}$ a.e. *respectively as* $n, t \to \infty$.

THEOREM 2. *In addition if* $\{\mu_1, \mu_2, \cdots\}$ *are independent, identically distributed*

(d) $t^{-1} \cdot \int_0^t P(\mu(s) \leq x)\, ds \to (E(\mu_1))^{-1} \cdot \int_0^x yG(dy)$

where $\mu(t) = \mu_{N_t+1}$ = *parameter of the component in operation at time* t,

(e) $t^{-1} \cdot \int_0^t P(H_t > \xi)\, dt \to (E(\mu_1))^{-1} \cdot \int_0^\infty y \exp(-\xi y^{-1})G(dy)$

where H_t *is the residual waiting time* $S_{N_t+1} - t$, *the spent waiting time* $t - S_{N_t}$, *or the interarrival time containing* $t : S_{N_t+1} - S_{N_t}$.

2. Proofs. The proof of Theorem 1 is straightforward enough; the proof of Theorem 2 relies on Lemma 4 below.

Feller [2], page 452, shows $N_t \to \infty$ for all t (this is a pure birth process) if and only if $\Sigma_{n=1}^\infty \mu_i = \infty$. Since the ergodic theorem implies that $n^{-1}\Sigma_{i=1}^n \mu_i \to E(\mu_1)$ a.e., N_t is finite for all t for almost every parameter sequence.

PROOF OF THEOREM 1. (a)

$$\begin{aligned} U(t) &= 1 + \Sigma_{n=1}^\infty P(S_n \leq t) \\ &\leq 1 + \Sigma_{n=1}^\infty P(T_1 \leq t) \cdots P(T_n \leq t) \\ &= 1 + \Sigma_{n=1}^\infty (1 - \exp(-\lambda_1 t)) \cdots (1 - \exp(-\lambda_n t)). \end{aligned}$$

But for a.e. fixed parameter sequence

$$\begin{aligned} \Pi_{j=1}^n(1\text{-}\exp(-\lambda_j t)) &= \exp\left[\Sigma_{j=1}^n \log(1 - \exp(-\lambda_j t))\right] \\ &\leq \exp\left[n(E(\log(1 - \exp(-\lambda_1 t))) + \varepsilon)\right] \end{aligned}$$

for n sufficiently large by the ergodic theorem. Choosing ε so that $E(\log(1 - \exp(-\lambda_1 t))) + \varepsilon < 0$ implies the tail of the series $U(t)$ is bounded above by the tail of a convergent geometric series.

(b) Assume $n^{-1}(\mu_1 + \cdots + \mu_n) \to E(\mu_1)$. Taking Laplace transforms

$$\Phi(s) = \int_0^\infty e^{-st}U(dt) = \Sigma_{n=0}^\infty \phi_1(s) \cdots \phi_n(s)$$

by monotone convergence where the addend for $n = 0$ is 1 and

$$\phi_j(s) = \int_0^\infty e^{-st}F_j(dt) = (1 + s\mu_j)^{-1}$$

for exponential distribution F_j. Since $(1 + s\mu_j)^{-1} \geq e^{-s\mu_j}$,

$$\Phi(s) \geq \Sigma_{n=0}^\infty \exp[-s(\mu_1 + \cdots + \mu_n)].$$

Given $\varepsilon > 0$, choose $N = N(\varepsilon)$ so large that $n > N$ implies $\mu_1 + \cdots + \mu_n \leq n(E(\mu_1) + \varepsilon)$. Hence

$$\Phi(s) \geq \Sigma_{n=0}^{N-1} \exp[-s(\mu_1 + \cdots + \mu_n)] + \Sigma_{n=N}^\infty \exp[-sn(E(\mu_1) + \varepsilon)].$$

Thus

$$\liminf_{s\downarrow 0} s\Phi(s) \geq \lim_{s\downarrow 0}\left(s\exp[-sN(\varepsilon)(E(\mu_1)+\varepsilon)]\right)/\left(1-\exp[-s(E(\mu_1)+\varepsilon)]\right)$$
$$= (E(\mu_1)+\varepsilon)^{-1}.$$

So $\liminf s\Phi(s) \geq (E(\mu_1))^{-1}$. On the other hand, let $\tau_j = \mu_j$ if $\mu_j \leq A$ and $\tau_j = A$ if $\mu_j > A$. For $a < 1$, choose δ so that $0 < x < \delta$ implies $1 + x \geq ae^x$. Then for $s \leq \delta/A$, $1 + s\mu_j \geq 1 + s\tau_j \geq ae^{s\tau_j}$. Thus as before, given $\varepsilon > 0$ so that $\tau_1 + \cdots + \tau_n \geq n(E(\tau_1) - \varepsilon)$ for $n \geq N = N(\varepsilon)$,

$$\Phi(s) \leq \Sigma_{n=0}^{N-1}\Pi_{j=1}^{n}(1+s\mu_j)^{-1} + \Sigma_{n=N}^{\infty}a^{-n}\exp[-ns(E(\tau_1)-\varepsilon)]$$

and

$$\limsup_{s\downarrow 0} s\Phi(s) \leq \lim_{s\downarrow 0} sa^{-N(\varepsilon)}\left(\exp[-sN(\varepsilon)(E(\tau_1)-\varepsilon)]\right)$$
$$/\left(1-\exp[-s(E(\tau_1)-\varepsilon)]\right)$$
$$= a^{-N(\varepsilon)}/\left(E(\tau_1)-\varepsilon\right) \qquad \text{(at least for } A \text{ large enough).}$$

Now letting $a\uparrow 1$ (it is independent of ε), $\varepsilon\downarrow 0$ and $A\uparrow\infty$ implies $\lim s\Phi(s) = (E(\mu_1))^{-1}$ as $s\downarrow 0$. Thus a Tauberian theorem [3], page 421, implies $t^{-1}U(t) \to (E(\mu_1))^{-1}$ as $t\uparrow\infty$.

(c) We embedded the process in the larger one consisting of the Cartesian product of the set of parameter sequences $R = (0, \infty)^N$ and the set of component lifetimes $T = (0, \infty)^N$ where N denotes the set of positive integers. To define a probability measure on $(R \times T, F)$ where F is the σ-field generated by the cylinder sets, begin by letting Q_Λ denote the product space measure on T where the ith slot has exponential distribution with mean μ_i. (Here $\Lambda = \{\mu_1, \mu_2, \cdots\}$.) On the parameter sequences $R = \{\Lambda\}$ let M be the measure so that $\{\mu_i\}_{i=1}^{\infty}$ is the required strictly stationary, strongly mixing sequence—each μ_i distributed with distribution G. Now for $A \subset$ parameter sequences R and $B \subset$ set of component lifetimes T, each measurable with respect to the σ-fields generated by the cylinder sets, let

$$P(A \times B) = \textstyle\int_A Q_\Lambda(B)M(d\Lambda).$$

As in [4] where a similar model is considered, it is routine to show that P is well defined and extends to a probability measure on $(R \times T, F)$. And the very definition implies

LEMMA 3. *Let B be measurable $\subset$ set of component lifetimes T. Then $Q_\Lambda(B) = 1$ for* a.e. *environment Λ if and only if $P(R \times B) = 1$.*

Returning to the proof of (c), let T_i^* be the random variable on $R \times T$ defined by $T_i^*(\Lambda, \omega) = T_i(\omega) = \omega_i$ (= ith component of ω). Hence

$$P(T_i^* \leq t) = \textstyle\int_R Q_\Lambda(T_i^* \leq t)M(d\Lambda)$$
$$= \textstyle\int_0^\infty(1-\exp(-ty^{-1}))G(dy).$$

So $E(T_i^*) = E(\mu_1)$. A straightforward verification shows that strict stationarity and the strong mixing of $\mu_1, \mu_2, \cdots$ imply these properties hold for $T_1^*, T_2^*, \cdots$. Thus the ergodic theorem implies as $n \to \infty$

$$n^{-1}S_n^* = n^{-1}(T_1^* + \cdots + T_n^*) \to E(\mu_1) \quad \text{a.e.}$$

But $\{(\Lambda, \omega) : n^{-1}S_n^*(\Lambda, \omega) \to E(\mu_1) \text{ as } n \to \infty\} = \{(\Lambda, \omega) : n^{-1}S_n(\omega) \to E(\mu_1) \text{ as } n \to \infty\} = R \times \{\omega : n^{-1}S_n(\omega) \to E(\mu_1) \text{ as } n \to \infty\}$. Therefore Lemma 3 implies as $n \to \infty n^{-1}S_n \to E(\mu_1)$ a.e. for a.e. fixed parameter sequence.

Since N_t increases with t, $\{N_t \to \infty \text{ as } t \to \infty\} = \cap_{n=1}^{\infty} \cup_{j=1}^{\infty}\{N_j \geq n\} = \cap_{n=1}^{\infty} \cup_{j=1}^{\infty}\{S_n < j\}$ which is a set of measure 1 since each S_n has a proper probability distribution for each parameter sequence. Thus $N_t \to \infty$ a.e. Now $S_{N_t} \leq t < S_{N_t+1}$. So

$$(N_t)^{-1}S_{N_t} \leq (N_t)^{-1}t < (N_t)^{-1}S_{N_t+1} = (N_t)^{-1}(S_{N_t} + T_{N_t+1}).$$

Hence it remains to show that $n^{-1}T_{n+1} \to 0$ a.e. in the case where $E(\mu_1) < \infty$. But

$$P(|n^{-1}T_{n+1}| > \varepsilon) = P(T_{n+1} > n\varepsilon) = \exp(-n\varepsilon\lambda_{n+1})$$

(recalling that T_n has exponential distribution with mean μ_n when the parameter sequence is fixed). Thus the first Borel-Cantelli lemma [1], page 69, implies $n^{-1}T_{n+1} \to 0$ a.e. if $\Sigma_{n=0}^{\infty} \exp(-n\varepsilon\lambda_{n+1}) < \infty$. But this is true for a.e. parameter sequence since this series has a finite expectation: By monotone convergence

$$E(\Sigma_{n=0}^{\infty} \exp(-n\varepsilon\lambda_{n+1})) = \Sigma_{n=0}^{\infty}E(\exp(-n\varepsilon\lambda_{n+1})) = \Sigma_{n=0}^{\infty}\int_0^{\infty} \exp(-n\varepsilon y^{-1})G(dy).$$

This converges by the integral test since

$$\int_0^{\infty}\int_0^{\infty} \exp(-t\varepsilon y^{-1})G(dy)\, dt = \varepsilon^{-1}E(\mu_1).$$

Thus $t(N_t)^{-1} \to (E(\mu_1))^{-1}$ a.e. which completes the proof of Theorem 1.

The proof of Theorem 2 depends on

LEMMA 4. *Let $X_1, X_2, \cdots$ be independent, identically distributed each with finite expectation m and finite variance σ^2. Set*

$$Y(a) = \Sigma_{j=0}^{\infty}a(1 - a)^jX_{j+1}.$$

Then $\lim_{a\downarrow 0}Y(a) = m$ a.e.

PROOF. Setting $X_j = X_j^+ + X_j^-$ where $X_j^{\pm} = \max[\pm X_j, 0]$, and

$$Y(a)^{\pm} = a\Sigma_0^{\infty}(1 - a)^jX_{j+1}^{\pm}$$

shows that a proof for nonnegative random variables X_j suffices. Hence assume throughout that each X_j is nonnegative.

By monotone convergence $E(Y(a)) = m$; hence for each *fixed* $0 < a < 1$, $Y(a)$ converges a.e. Also by direct calculation $E(Y(a)^2) = m^2 + \sigma^2a/(2 - a)$. So $\sigma^2(Y(a)) = \sigma^2a/(2 - a)$. Hence $P(|Y(a) - m| > \varepsilon) \leq \sigma^2a/(\varepsilon^2(2 - a))$ by Chebyshev's inequality. Now the first Borel-Cantelli lemma implies that the sequence $\{Y(n^{-2})\}_{n=1}^{\infty}$ converges to m everywhere on a set Ω of probability 1. The claim is that the full set $\{Y(a)\}$ converges to m a.e. as $a\downarrow 0$. To see this suppose that $(n+1)^{-2} \leqslant a \leqslant n^{-2}$. Now $h(x) = x(1-x)^j$ is increasing on $[0, 1/(j+1)]$ and decreasing on $[1/(j+1)]$. Thus

$$a(1-a)^j \leqslant n^{-2}(1-n^{-2})^j \qquad \text{for } n^{-2} \leqslant 1/(j+1) \qquad \text{or } j \leqslant n^2 - 1$$

$$\leqslant (n+1)^{-2}(1-(n+1)^{-2})^j \qquad \text{for } (n+1)^{-2} \geqslant 1/(j+1) \qquad \text{or } j \geqslant n^2 + 2n.$$

When $n^2 \leqslant j \leqslant n^2 + 2n$ a bound for $h(a) = a(1-a)^j$ is obtained in this way: h is concave on $(n+1)^{-2} \leqslant a \leqslant n^{-2}$ (for $n \geqslant 3$); so

$$h(a) \leqslant h\big((n+1)^{-2}\big) + h'\big((n+1)^{-2}\big)\big(a - (n+1)^{-2}\big)$$
$$\leqslant h\big((n+1)^{-2}\big) + h'\big((n+1)^{-2}\big)\big(n^{-2} - (n+1)^{-2}\big).$$

But a routine calculation shows the second term in the last right-hand side is less than $h(n^{-2})$ for n large. Hence for $n^2 \leqslant j \leqslant n^2 + 2n$, n large

$$a(1-a)^j \leqslant n^{-2}(1-n^{-2}) + (n+1)^{-2}\big(1-(n+1)^{-2}\big).$$

So, for a close enough to 0

$$Y(a) \leqslant \Sigma_{j=0}^{n^2+2n} n^{-2}(1-n^{-2})^j X_{j+1} + \Sigma_{j=n^2}^{\infty}(n+1)^{-2}\big(1-(n+1)^{-2}\big)^j X_{j+1}$$
$$= Y\big((n+1)^{-2}\big) + Z_1 + Z_2$$

where

$$Z_1 = \Sigma_{j=0}^{n^2-1}\big[n^{-2}(1-n^{-2})^j - (n+1)^{-2}\big(1-(n+1)^{-2}\big)^j\big]X_{j+1}$$
$$Z_2 = \Sigma_{j=n^2}^{n^2+2n} n^{-2}(1-n^{-2})^j X_{j+1}.$$

Now $\max_{0<x<1}|h'(x)| = 1$. Thus the mean value theorem implies that the term multiplying X_{j+1} in the series defining Z_1 is $\leqslant$ in absolute value

$$|n^{-2} - (n+1)^{-2}| = (2n+1)/\big(n^2(n+1)^2\big)$$
$$\leqslant 2/n^3.$$

Hence

$$|Z_1| \leqslant \Sigma_{j=0}^{n^2-1}(2/n^3)X_{j+1}$$
$$= (2/n)n^{-2}\cdot \Sigma_{j=0}^{n^2-1}X_{j+1}$$
$$\to 0 \text{ a.e.}$$

by the law of large numbers. Also,

$$|Z_2| \leqslant n^{-2}\Sigma_{j=n^2}^{n^2+2n}X_{j+1}$$
$$= (2n + 1/n^2)\cdot(2n+1)^{-1}\Sigma_{j=n^2}^{n^2+2n}X_{j+1}$$
$$\to 0 \text{ a.e.}$$

using the first Borel-Cantelli lemma and Chebyshev's inequality. (Note: for

$$Z' = (2n+1)^{-1}\Sigma_{j=n^2}^{n^2+2n}X_{j+1},$$

$E(Z') = m$, $\sigma^2(Z') = \sigma^2/(2n+1)$). Combining this with the analogous reverse inequality yields $Y(n^{-2}) + W_n \leq Y(a) \leq Y((n+1)^{-2}) + W_n'$ where W_n and $W_n' \to 0$ as $n\uparrow\infty$ a.e., say on set Ω' of measure 1. Therefore on $\Omega \cap \Omega'$, $Y(a) \to m$ as $a\downarrow 0$ a.e.

PROOF OF THEOREM 2. (d)

$$P(\mu(t) \leq x) = \Sigma_{n=1}^{\infty} P(\mu(t) \leq x | N_t = n-1) \cdot P(N_t = n-1)$$
$$= \Sigma_{n=1}^{\infty} \varepsilon_n \mu_n f_1 * \cdots * f_n(t)$$

where $\varepsilon_n = 1, 0$ respectively if $\mu_n \leq, > x$. So

$$\Theta(s) = \int_0^\infty e^{-st} P(\mu(t) \leq x)\, dt$$
$$= \Sigma_{n=1}^{\infty} \varepsilon_n \mu_n \phi_1(s) \cdots \phi_n(s)$$
$$= \Sigma_{n=1}^{\infty} \varepsilon_n \mu_n (1 + s\mu_1)^{-1} \cdots (1 + s\mu_n)^{-1}.$$

In the same way as in (b),

$$\lim_{s\downarrow 0} s\Theta(s) = \lim_{s\downarrow 0} s\Sigma_{n=1}^{\infty} \varepsilon_n \mu_n e^{-snE(\mu_1)}$$
$$= \lim_{s\downarrow 0} s/(1 - e^{-sE(\mu_1)}) \cdot (1 - e^{-sE(\mu_1)}) \Sigma_{n=1}^{\infty} \varepsilon_n \mu_n e^{-snE(\mu_1)}.$$

Lemma 4 now applies with the result

$$\lim_{s\downarrow 0} s\Theta(s) = (E(\mu_1))^{-1} E(\varepsilon_1 \mu_1) = (E(\mu_1))^{-1} \int_0^x y G(dy).$$

Application of the same Tauberian theorem yields result (d).

(e) Details are similar in all three cases and much the same as in (d); so only the outline for the case H_t = residual waiting time $S_{N_t+1} - t$ is here presented. Now

$$P(H_t > \xi) = \Sigma_{n=1}^{\infty} P(H_t > \xi | N_t = n-1) \cdot P(N_t = n-1)$$
$$= \Sigma_{n=0}^{\infty} P(T_n > \xi) \cdot \mu_n f_1 * \cdots * f_n(t)$$

by the "memoryless" property of exponential random variables. Let $\rho(s)$ be

$$\int_0^\infty e^{-st} P(H_t > \xi)\, dt = \Sigma_{n=0}^{\infty} \mu_n e^{-\lambda_n \xi} \phi_1(s) \cdots \phi_n(s).$$

As in (d)

$$\lim_{s\downarrow 0} s\rho(s) = \lim_{s\downarrow 0} s\Sigma_{n=0}^{\infty} \mu_n e^{-\lambda_n \xi} e^{-snE(\mu_1)}$$
$$= (E(\mu_1))^{-1} E(\mu_1 e^{-\lambda_1 \xi}).$$

Application of the same Tauberian theorem completes the proof. (Note that the proofs apply with the usual modifications when $E(\mu_1) = \infty$.).

3. Randomizing the parameter sequence. The above process may be compared with the process in which the μ_j's are random independent, identically distributed rather than preselected and fixed. The probabilistic setting for this new process has been defined at the beginning of the proof of Theorem 1: $T_1, T_2, \cdots$ are independent, identically distributed each with density

$$f(t) = \int_0^\infty y e^{-ty} G(dy)$$

for $t \geqslant 0$. So the model reduces to the standard renewal model of [3], chapter 11. Still it may be of interest to calculate the distribution of $\mu(t) = \mu_{N_t+1}$ = parameter of the component in operation at time t.

THEOREM 5. *In the renewal model in which* $\{\mu_i\}_{i=1}^{\infty}$ *is a sequence of independent, identically distributed random variables with distribution* G *concentrated on* $(0, \infty)$, (a) $\{\mu(t)\}_{t>0}$ *is a Markov process*; (b) $\mu(t)$ *approaches in distribution* $(E(\mu_1))^{-1} \cdot yG(dy)$ *when* $E(\mu_1) < \infty$.

PROOF. It is clear that $\{\mu_t\}$ is Markovian since each T_j is exponentially distributed. Now $\{\mu(t)\}_0^{\infty}$ constitutes a jump process. Given $\mu(t) = x$, the waiting time till the next jump is exponential with mean $1/x$ at which time the process jumps to another state according to distribution G independent of x. Hence with $Q_t(x, \Omega) = P(\mu(t) \in \Omega | \mu(0) = x)$ Kolmogorov's backward equations are

$$\frac{\partial Q_t(x, \Omega)}{\partial t} = x^{-1} Q_t(x, \Omega) + x^{-1} \int_0^{\infty} Q_t(y, \Omega) G(dy)$$

[3], page 317. The infinitesimal generator associated with Q_t is thus

$$Uu(x) = -x^{-1}\left[u(x) - \int_0^{\infty} u(y) G(dy)\right].$$

So the resolvent operator is

$$R_\tau w(x) = (1 + \tau x)^{-1}\left[xw(x) + C\right]$$

where

$$C = \left(\int_0^{\infty} \tau y(1 + \tau y)^{-1} G(dy)\right)^{-1} \cdot \int_0^{\infty} y w(y)(1 + \tau y)^{-1} G(dy)$$

since R_τ is the inverse of $\tau - U$. Or

$$R_\tau w(x) = (1 + \tau x)^{-1} x w(x) + L(h_1 * h_2 * U)(\tau)$$

where L indicates the Laplace transform of the function $h_1 * h_2 * U$ and

$$h_1(s) = x^{-1} e^{-s/x}, \qquad s \geqslant 0$$

$$h_2(s) = \int_0^{\infty} w(y) e^{-s/y} G(dy), \qquad s \geqslant 0$$

$$U(t) = \Sigma_0^{\infty} F^{*n}(t),\ F(t) = \int_0^{\infty} (1 - e^{-t/y}) G(dy) \qquad \text{for } t \geqslant 0.$$

Now $P(\mu(t) \leqslant x | \mu(0) = \mu_0) = \int_0^{\infty} w(y) Q_t(\mu_0, dy)$ where $w(y) = 1$ if $0 \leqslant y \leqslant x$ and 0 otherwise. Since the Laplace transform of this function (as a function of t) is $R_\tau w(x)$, taking inverse transforms implies

$$P(\mu(t) \leqslant x | \mu(0) = \mu_0) = e^{-t/x} w(x) + h_1 * h_2 * U(t).$$

Since $h_1 * h_2$ is directly Riemann integrable, the renewal theorem of [3], page 349, implies as $t \to \infty$

$$P(\mu(t) \leqslant x | \mu(0) = \mu_0) = (E(\mu_1))^{-1} \int_0^{\infty} h_1 * h_2(t)\, dt$$

$$= (E(\mu_1))^{-1} \int_0^x y G(dy).$$

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