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# ON MARTINGALES IN THE LIMIT

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The purpose of this note is to show that the set of $L^1$-bounded "martingales in the limit", unlike the set of $L^1$-bounded "amarts", is *not* a vector lattice.

**1. Introduction.** In recent years A. G. Mucci introduced the notion of "martingale in the limit" [2], [3] which considerably generalizes that of martingale. He also proved [3] that every (real) $L^1$-bounded martingale in the limit converges a.s., thus extending the Martingale Convergence Theorem.

Subsequently G. A. Edgar and L. Sucheston [1] showed that every (real) "amart" is a "martingale in the limit"; the notion of "amart" (short for asymptotic martingale) had been introduced and developed earlier also in an attempt to generalize the concept of martingale and to extend the Martingale Convergence Theorem. Edgar and Sucheston also showed in their paper that "several crucial properties possessed by amarts fail for martingales in the limit, namely: the maximal inequality, Riesz decomposition, optional stopping theorem, optional sampling theorem". The purpose of this note is to add one rather basic property to this "negative list." In fact, we shall show by way of example that the set of $L^1$-bounded martingales in the limit (unlike the set of $L^1$-bounded amarts) is *not* a vector lattice.

We now make precise our setting and terminology. Below $(\Omega, \mathfrak{F}, P)$ is a fixed *nonatomic* probability space. All the rv's considered in what follows are *real*. We denote as usual by $L^1 = L^1(\Omega, \mathfrak{F}, P)$ the space of all integrable rv's. For a rv $X \in L^1$ we write

$$\|X\|_1 = \int_\Omega |X(\omega)| dP(\omega).$$

We recall that a sequence $(X_n)_{n \in N}$ of rv's belonging to $L^1$ is said to be *$L^1$-bounded* if

$$\sup_{n \in N} \|X_n\|_1 < \infty.$$

If $\mathcal{G} \subset \mathfrak{F}$ is a sub-$\sigma$-field, $E^{\mathcal{G}}$ denotes the corresponding *conditional expectation operator* in $L^1$.

The notation $(X_n, \mathfrak{F}_n)_{n \in N}$ means that the sequence of rv's $(X_n)_{n \in N}$ is *adapted* to the increasing sequence of sub-$\sigma$-fields $(\mathfrak{F}_n)_{n \in N}$ (i.e., $\mathfrak{F}_m \subset \mathfrak{F}_n \subset \mathfrak{F}$ for $m \leqslant n$).

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Finally we recall the definition of the martingale in the limit [2], [3]:

DEFINITION. $(X_n, \mathcal{F}_n)_{n\in N}$ is a *martingale in the limit* if $X_n \in L^1$ for each $n \in N$ and if

$$\sup_{(n, m)p<n<m}|E^{\mathcal{F}_n}(X_m)(\omega) - X_n(\omega)| \to 0 \quad \text{a.s.} \qquad \text{as } p \to \infty$$

**2. The example.** *There exists a sequence $(X_n, \mathcal{F}_n)_{n\in N}$ such that: a) $(X_n)_{n\in N}$ converges to 0 a.s. and in $L^1$; b) $(X_n, \mathcal{F}_n)_{n\in N}$ is a martingale in the limit, but $(|X_n|, \mathcal{F}_n)_{n\in N}$ is not. Hence the set of $L^1$-bounded martingales in the limit is not a vector lattice.*

PROOF. We define by induction: a sequence $(\pi_p)$ of partitions of $\Omega$, a sequence $(G_p)$ of "successive generations" (the atoms of $G_p$ can be thought of as the "distinguished" atoms of $\pi_p$), and a sequence $(c_p)$ of constants as follows:

*Step* 1. We divide $\Omega$ into a partition of 4 sets

$$\pi_1 = \{A'_{(1)}, B'_{(1)}, A''_{(1)}, B''_{(1)}\}$$

where $P(A'_{(1)}) = P(A''_{(1)}) = \frac{1}{4}$ and $P(B'_{(1)}) = P(B''_{(1)})$. We let

$$G_1 = \{A'_{(1)}, A''_{(1)}\}$$
$$c_1 = 4.$$

We have

$$k_1 = |\pi_1| = 4$$
$$\Delta(\pi_1) = \sup\{P(C)|C \in \pi_1\} < \tfrac{1}{2}$$
$$1(G_1) = P(A'_{(1)}) + P(A''_{(1)}) = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

*Step* 2. Rewrite $\pi_1 = \{C_{(1),1}, C_{(1),2}, C_{(1),3}, C_{(1),4}\}$. We divide each atom $C = C_{(1),i} \in \pi_1$ into 4 sets:

$$C = A'_{(2),i} \cup B'_{(2),i} \cup A''_{(2),i} \cup B''_{(2),i},$$

where

$$P(A'_{(2),i}) = P(A''_{(2),i}) = \frac{1}{2^3}P(C)$$
$$P(B'_{(2),i}) = P(B''_{(2),i}).$$

We let

$$\pi_2 = \{A'_{(2),i}, B'_{(2),i}, A''_{(2),i}, B''_{(2),i};\ 1 \leqslant i \leqslant k_1\}$$
$$G_2 = \{A'_{(2),i}, A''_{(2),i};\ 1 \leqslant i \leqslant k_1\}$$
$$c_2 = 2^3.$$

We have:

$$k_2 = |\pi_2| = 4k_1 = 4^2$$

$$\Delta(\pi_2) = \sup\{P(C)|C \in \pi_2\} \leqslant \tfrac{1}{2}\Delta(\pi_1) < \frac{1}{2^2}$$

$$1(G_2) = \Sigma_{i=1}^{k_1}(P(A'_{(2),i}) + P(A''_{(2),i}))$$

$$= \frac{2}{2^3}\Sigma_{i=1}^{k_1}P(C_{(1),i}) = \frac{1}{2^2}P(\Omega) = \frac{1}{2^2}.$$

Assume now that $\pi_j$, $G_j$ and $c_j$ have been constructed for $1 \leqslant j \leqslant p$.

*Step* $p+1$. Rewrite $\pi_p = \{C_{(p),i};\ 1 \leqslant i \leqslant k_p\}$. We divide each atom $C = C_{(p),i} \in \pi_p$ into 4 sets:

$$C = A'_{(p+1),i} \cup B'_{(p+1),i} \cup A''_{(p+1),i} \cup B''_{(p+1),i},$$

where

$$P(A'_{(p+1),i}) = P(A''_{(p+1),i}) = \frac{1}{2^{p+2}}P(C)$$

$$P(B'_{(p+1),i}) = P(B''_{(p+1),i})$$

We now let

$$\pi_{p+1} = \{A'_{(p+1),i}, B'_{(p+1),i}, A''_{(p+1),i}, B''_{(p+1),i};\ 1 \leqslant i \leqslant k_p\}$$

$$G_{p+1} = \{A'_{(p+1),i}, A''_{(p+1),i};\ 1 \leqslant i \leqslant k_p\}$$

$$c_{p+1} = 2^{p+2}$$

We have, by induction

$$k_{p+1} = |\pi_{p+1}| = 4k_p = \cdots = 4^{p+1}$$

$$\Delta(\pi_{p+1}) = \sup\{P(C)|C \in \pi_{p+1}\} \leqslant \tfrac{1}{2}\Delta(\pi_p) < \frac{1}{2^{p+1}}$$

$$1(G_{p+1}) = \Sigma_{i=1}^{k_p}(P(A'_{(p+1),i}) + P(A''_{(p+1),i})$$

$$= \frac{2}{2^{p+2}}\Sigma_{i=1}^{k_p}P(C_{(p),i}) = \frac{2}{2^{p+2}}P(\Omega) = \frac{1}{2^{p+1}}.$$

In particular we have:

(1) $\Delta(\pi_p) \to 0$ as $p \to \infty$;

(2) $\Sigma_{p=1}^{\infty}1(G_p) < +\infty$.

We now define the following sequence of rv's:

$$Y_{(1)} = c_1 1_{A'(1)} - c_1 1_{A''(1)}$$

$$Y_{(2),i} = c_2 1_{A'(2),i} - c_2 1_{A''(2),i} \text{ for } 1 \leqslant i \leqslant k_1$$

and in general for $p > 1$

$$Y_{(p),i} = c_p 1_{A'_{(p),i}} - c_p 1_{A''_{(p),i}} \text{ for } 1 \leqslant i \leqslant k_{p-1}.$$

Note that if we write

(3) $\Omega_{(p)} = \bigcup_{1 \leqslant i \leqslant k_{p-1}} \text{Supp } Y_{(p),i}$,

then

(3′) $P(\Omega_{(p)}) = l(G_p)$.

We now define correspondingly the following sequence of $\sigma$-fields:

$$\mathcal{G}_{(1)} = \sigma(\pi_1)$$

$$\mathcal{G}_{(2),i} = \sigma(\pi_1 \cup \{A'_{(2),j}, B'_{(2),j}, A''_{(2),j}, B''_{(2),j};\ 1 \leqslant j \leqslant i\})$$

and each $1 \leqslant i \leqslant k_1$; and in general for $p > 1$

$$\mathcal{G}_{(p),i} = \sigma(\pi_{p-1} \cup \{A'_{(p),j}, B'_{(p),j}, A''_{(p),j}, B''_{(p),j};\ 1 \leqslant j \leqslant i\})$$

for each $1 \leqslant i \leqslant k_{p-1}$. It is clear that

$$\mathcal{G}_{(1)} \subset \mathcal{G}_{(2),i} \qquad \text{for} \quad 1 \leqslant i \leqslant k_1$$

and that for each $p > 1$;

$$\mathcal{G}_{(p),s} \subset \mathcal{G}_{(p),t} \qquad \text{if} \quad 1 \leqslant s \leqslant t \leqslant k_{p-1}$$

$$\mathcal{G}_{(p),i} \subset \mathcal{G}_{(p+1),j} \qquad \text{for any} \quad 1 \leqslant i \leqslant k_{p-1}, 1 \leqslant j \leqslant k_p.$$

Thus $(\mathcal{G}_{(p),i})$ arranged "lexicographically" is increasing and clearly $Y_{(p),i}$ is $\mathcal{G}_{(p),i}$-measurable. By (2), (3), (3′), $P(\limsup_p \Omega_{(p)}) = 0$ and thus $Y_{(p),i} \to 0$ a.s. Also

$$\int |Y_{(p),i}| dP = 2c_p P(A'_{(p),i}) = 2P(C_{(p-1),i}) \leqslant 2\Delta(\pi_{p-1})$$

and hence by (1), $Y_{(p),i} \to 0$ in $L^1$.

Finally note that if $p > 1$

$$E^{\mathcal{G}_{(p),i-1}}(Y_{(p),i}) = 0 \qquad \text{for} \quad 1 < i \leqslant k_{p-1}$$

$$E^{\mathcal{G}_{(p-1),k_{p-2}}}(Y_{(p),i}) = 0 \qquad \text{for} \quad i = 1.$$

Thus $(Y_{(p),i}, \mathcal{G}_{(p),i})$ is a martingale in the limit.

On the other hand, if $p > 1$, then for $1 < i \leqslant k_{p-1}$ we have

$$E^{\mathcal{G}_{(p),i-1}}(|Y_{(p),i}|)(\omega) = \frac{2c_p P(A'_{(p),i})}{P(C_{(p-1),i})} = 2 \qquad \text{for} \quad \omega \in C_{(p-1),i}$$

and for $i = 1$

$$E^{\mathcal{G}_{(p-1),k_{p-2}}}(|Y_{(p),1}|)(\omega) = \frac{2c_p P(A'_{(p),1})}{P(C_{(p-1),1})} = 2 \qquad \text{for} \quad \omega \in C_{(p-1),1}$$

and the sets $\{C_{(p-1),i};\ 1 \leqslant i \leqslant k_{p-1}\}$ form the partition $\pi_{p-1}$ of $\Omega$. Thus $(|Y_{(p),i}|, {}_{(p),i})$ is not a martingale in the limit.

The desired sequence $(X_n, \mathcal{F}_n)_{n \in N}$ is obtained by arranging the sequence $(Y_{(p),i}, \mathcal{G}_{(p),i})$ "lexicographically" and relabelling it. This completes the proof.

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