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AN EXTENSION TO A STRONG LAW RESULT OF MITTAL AND YLVISAKER FOR THE MAXIMA OF STATIONARY GAUSSIAN PROCESSES

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Mittal and Ylvisaker have shown a law of iterated logarithms for the maxima of certain stationary Gaussian processes. Their result is extended in this paper in that it is shown to be valid under weaker mixing conditions.

1. Introduction. Let $\{X_k, k \geqslant 0\}$ be a stationary sequence of standard normal random variables. Let $r_n = EX_oX_n$, $c_n = (2 \ln n)^{\frac{1}{2}}$ and set $M_n = \max_{0<k<n} X_k$, $\overline{X}_n = 1/(n+1)\sum_{k=0}^n X_k$ and Z_n to be the Markov estimate of the mean, that is the estimate of minimum variance among all estimates of the form $\sum_{k=0}^n c_k X_k$ with $\sum_{k=0}^n c_k = 1$.

Mittal and Ylvisaker [6] have shown for suitably smooth correlation functions r_n with $r_n = o(1)$, $(r_n \ln n)^{-1}$ monotone for large n and $o(1)$, that, with probability one,

$$(1.1)\qquad \begin{aligned} &\liminf \frac{2c_n\big(M_n - (1-r_n)^{\frac{1}{2}}c_n - Z_n\big)}{\ln\ln n} = -1 \quad \text{and} \\ &\limsup \frac{2c_n\big(M_n - (1-r_n)^{\frac{1}{2}}c_n - Z_n\big)}{\ln\ln n} = 1 \end{aligned}$$

provided $(r_n \ln n)/(\ln \ln n)^2 = o(1)$ and further, when $(r_n \ln n)/(\ln\ln n)^2 \to \infty$ as $n \to \infty$,

$$(1.2)\qquad \lim_{n\to\infty} r_n^{-\frac{1}{2}}\big(M_n - (1-r_n)^{\frac{1}{2}}c_n - \overline{X}_n\big) = o \text{ a.s.}$$

The main result of this paper establishes the validity of (1.1) with $\overline{X}_n$ in place of Z_n without the assumption $(r_n \ln n)/(\ln\ln n)^2 = o(1)$. (1.2) is then an immediate consequence of this result. It is also noted in [6] that $\overline{X}_n$ and Z_n are interchangeable in (1.1). A version of this result for continuous time Gaussian processes is also indicated.

2. Preliminaries. Let f be a probability density function vanishing off $[0, \infty)$ with the following properties. There exists $\beta > 0$ such that for $0 < x < \beta$, $f(x) > f(\beta)$ and f is nonincreasing on $[\beta, \infty)$. Set

$$(2.1)\qquad r(t) = \int_{-\infty}^{\infty} f(x) \wedge f(x+t)\, dx.$$

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Functions of the form (2.1) are by a result of Berman [2] correlation functions. Moreover, there exists a stochastic integral representation for a process having such a correlation function. Let $A_t = \{(x, y): -\infty < x < \infty, 0 \leqslant y < \infty \text{ and } f(x + t) > y\}$ and Z be Gaussian random measure on the plane with Lebesgue measure as its covariance kernel. Then

$$X_t = \int\int_{A_t} Z(dx \times dy) \tag{2.2}$$

is a stationary Gaussian process having correlation function r.

Following Mittal and Ylvisaker we use (2.2) to write for $0 \leqslant t \leqslant T$ and $T \geqslant \beta$

$$\begin{aligned} X_t &= \int\int_{A_t \cap (A_0 \cap A_T)^c} Z(dx \times dy) + \int\int_{A_0 \cap A_T} Z(dx \times dy) \\ &= (1 - r(T))^{\frac{1}{2}} Y_t^T + I_T. \end{aligned} \tag{2.3}$$

From this representation it is apparent that I_T is independent of $\{Y_t^T, 0 \leqslant t \leqslant T\}$ and the following properties follow easily. For $T \geqslant \beta$

$$\begin{aligned} &\text{(i) } EY_s^T Y_t^T = \frac{r(s - t) - r(T)}{1 - r(T)}, && 0 \leqslant s, t \leqslant T \\ &\text{(ii) } EX_t I_T = r(T), && 0 \leqslant t \leqslant T \\ &\quad\;\; EI_t X_T = EI_t I_T = r(T) && \beta \leqslant t \leqslant T. \end{aligned} \tag{2.4}$$

Let us now assume that we may take a version of the process at (2.2) having continuous sample paths and define $\bar{X}_t = \bar{X}_t^c = 1/t\int_0^t X_s\, ds$. The remainder of this section will be devoted to showing that the processes $\{\bar{X}_t, t > 0\}$ and $\{I_t, t > 0\}$ are appropriately close. This is accomplished by constructing an intermediary process through which we make our comparisons.

Define a probability density function

$$\begin{aligned} g(x) &= c, \quad 0 \leqslant x < \beta \\ &= f(x), \quad \beta \leqslant x < \infty \end{aligned}$$

where if $\beta > 0$, we define $c = 1/\beta\int_0^\beta f(x)\, dx$ and where f is the probability density function at (2.1). Our assumptions on f imply that g is nonincreasing and we may define a convex correlation function $\tilde{r}(t) = \int_t^\infty g(x)\, dx$. Note that $r(t) = \tilde{r}(t)$ for $t \geqslant \beta$. Next define

$$\tilde{X}_t = \int\int_{B_t} Z(dx \times dy)$$

where $B_t = \{(x, y): -\infty < x < \infty, 0 \leqslant y < \infty \text{ and } g(x + t) > y\}$. We may take a version of $\tilde{X}_t$ having continuous sample paths since, if $\beta > 0$, $1 - \tilde{r}(t) = ct$ for t in a neighborhood of zero and we may define $\bar{\tilde{X}} = 1/t\int_0^t \tilde{X}_s\, ds$, $t > 0$. Now by their representations as stochastic integrals, one may readily verify that the process $\{\zeta_t = X_t - \tilde{X}_t, t > 0\}$ is β-dependent, that is ζ_s and ζ_t are independent for $|s - t| \geqslant \beta$.

Next we establish for $0 < S \leqslant T$

$$\text{(2.5)}\qquad \text{(i) } E\left\{T^{\frac{1}{2}}\bar{\zeta}_T - S^{\frac{1}{2}}\bar{\zeta}_S\right\}^2 \leqslant 13\beta\left\{\left(\frac{T-S}{S}\right)^2 + \left(\frac{T-S}{S}\right)\right\}$$

$$\text{(ii) } \Gamma^2 = \Gamma^2_{S,T} = \sup_{S \leqslant t \leqslant T} E\left\{t^{\frac{1}{2}}\bar{\zeta}_t\right\}^2 \leqslant 4\beta$$

where $\bar{\zeta}_T = 1/T\int_0^T\zeta_t\,dt$. To show (2.5i) let $D = \{(s, t): |s - t| < \beta\}$. Then

$$\begin{aligned}
E\left\{T^{\frac{1}{2}}\bar{\zeta}_T - S^{\frac{1}{2}}\bar{\zeta}_S\right\}^2 &= (T^{\frac{1}{2}} - S^{\frac{1}{2}})^2 E\bar{\zeta}_T^2 + 2S^{\frac{1}{2}}(T^{\frac{1}{2}} - S^{\frac{1}{2}})E\bar{\zeta}_T(\bar{\zeta}_T - \bar{\zeta}_S) \\
&\quad + SE(\bar{\zeta}_T - \bar{\zeta}_S)^2 \\
&\leqslant \frac{(T-S)^2}{2ST^2}\int_0^T\int_0^T I_D(s, t)\,ds\,dt \\
&\quad + 2(T - S)\left\{\frac{1}{T^2}\int_0^T\int_S^T I_D(s, t)\,ds\,dt + \frac{1}{T}\left(\frac{1}{S} - \frac{1}{T}\right)\int_0^T\int_0^S I_D(s, t)\,ds\,dt\right\} \\
&\quad + 2S\left\{\frac{1}{T^2}\int_S^T\int_S^T I_D(s, t)\,ds\,dt + \frac{2}{T}\left(\frac{1}{S} - \frac{1}{T}\right)\int_S^T\int_0^S I_D(s, t)\,ds\,dt\right. \\
&\quad \left. + \left(\frac{1}{T} - \frac{1}{S}\right)^2\int_0^S\int_0^S I_D(s, t)\,ds\,dt\right\} \\
&\leqslant \beta\frac{(T-S)^2}{ST} + 8\beta\frac{(T-S)^2}{T^2} + 2S\left\{\frac{6\beta(T-S)}{T^2} + \frac{2\beta(T-S)^2}{ST^2}\right\} \\
&\leqslant 13\beta\left\{\left(\frac{T-S}{S}\right)^2 + \left(\frac{T-S}{S}\right)\right\}.
\end{aligned}$$

The proof of (2.5ii) is similar.

Now let $t_m = e^m$, $m = 1, 2, \cdots$. Then

$$P\left\{\max_{t_m \leqslant t < t_{m+1}} \frac{t^{\frac{1}{2}}}{\ln\ln t}\bar{\zeta}_t > \epsilon\right\} \leqslant P\left\{\max_{0 \leqslant t < 1}\tau^{\frac{1}{2}}\bar{\zeta}_\tau > \epsilon\ln m\right\}$$

$$\text{(2.6)}\qquad \leqslant c_1\Phi\left(-\frac{\epsilon}{c_2}\ln m\right)$$

where $\tau = \tau(t) = t_m + t(t_{m+1} - t_m)$ and c_1, c_2 are constants not depending on m. The last inequality follows by Fernique's inequality [3] since by (2.5i)

$$\begin{aligned}
E\left\{(\tau(t))^{\frac{1}{2}}\bar{\zeta}_{\tau(t)} - (\tau(s))^{\frac{1}{2}}\bar{\zeta}_{\tau(s)}\right\}^2 &\leqslant 13\beta\left\{\left(\frac{\tau(t) - \tau(s)}{\tau(s)}\right)^2 + \left(\frac{\tau(t) - \tau(s)}{\tau(s)}\right)\right\} \\
&\leqslant 13\beta\left\{(e - 1)^2(t - s)^2 + (e - 1)(t - s)\right\} \\
&= \psi(t - s)
\end{aligned}$$

and neither the function ψ nor the bound on the $E\{t^{\frac{1}{2}}\bar{\zeta}_t\}^2$ depends on m.Thus from (2.6) we conclude by a Borel-Cantelli argument that

LEMMA 2.1. *Let r be a correlation function of the form* (2.1) *and X_t be the process having the correlation function defined at* (2.2). *Then with $\tilde{X}_t$ defined as before we have*

$$\frac{T^{\frac{1}{2}}}{\ln \ln T}\left(\bar{X}_T - \bar{\tilde{X}}_T\right) \to 0 \text{ as } T \to \infty \text{ a.s.}$$

Now for $t \geqslant \beta$ set $\sigma_t^2 = E\{\bar{\tilde{X}}_t - I_t\}^2$ and for $s, t \geqslant \beta$ set

$$\rho_{s,t} = \frac{E\left(\bar{\tilde{X}}_s - I_s\right)\left(\bar{\tilde{X}}_t - I_t\right)}{\sigma_s \sigma_t}.$$

Also note that (2.4ii) holds with $\tilde{X}_t$ in place of X_t. This observation is crucial to the proof of the following lemma.

LEMMA 2.2. *Suppose $\tilde{r}(t) = E\tilde{X}_0\tilde{X}_t$ satisfies*

$$\text{(2.7)}\qquad \begin{aligned} &\text{(i) } \tilde{r}(t) = o(1) \\ &\text{(ii) } (\tilde{r}(t)\ln t)^{-1} \textit{ is monotone for large } t \textit{ and is } o(1). \end{aligned}$$

Then

$$\text{(i) } \sigma_T^2 = 0\left(\frac{\tilde{r}_T}{\ln T}\right)$$

$$\text{(2.8)}\qquad \text{(ii) } \frac{T^2\sigma_T^2}{S^2\sigma_S^2} \leqslant 1 + 2\Sigma_{k=1}^4\binom{4}{k}\left(\frac{T-S}{S}\right)^k = 2\left(\frac{T}{S}\right)^4 - 1$$

$$\textit{for all } \beta \leqslant S \leqslant T$$

$$\text{(iii) } \rho_{S,T} \geqslant \frac{S\sigma_S}{T\sigma_T} \textit{ for all } \beta \leqslant S \leqslant T.$$

PROOF. (i) This is simply a continuous version of Lemma 5.1 of [6].

(ii) Let $T > \beta$ be fixed and $\alpha \geqslant 1$ be arbitrary. Then

$$\frac{(\alpha T)^2\sigma_{\alpha T}^2}{T^2\sigma_T^2} = \frac{\int_0^{\alpha T}\int_0^{\alpha T}\tilde{r}(u-v)\,du\,dv - (\alpha T)^2\tilde{r}(\alpha T)}{\int_0^T\int_0^T\tilde{r}(u-v)\,du\,dv - T^2\tilde{r}(T)}$$

$$= \frac{\int_0^{\alpha T}(\alpha T - u)(\tilde{r}(u) - \tilde{r}(\alpha T))\,du}{\int_0^T(T-u)(\tilde{r}(u) - \tilde{r}(T))\,du} = \frac{\alpha^2\int_0^T(T-u)(\tilde{r}(\alpha u) - \tilde{r}(\alpha T))\,du}{\int_0^T(T-u)(\tilde{r}(u) - \tilde{r}(T))\,du}$$

$$= \frac{\alpha^2\int_0^T(T-u)\int_{\alpha u}^{\alpha T}g(z)\,dz\,du}{\int_0^T(T-u)(\tilde{r}(u) - \tilde{r}(T))\,du} = \frac{\alpha^3\int_0^T(T-u)\int_u^T g(\alpha z)\,dz\,du}{\int_0^T(T-u)(\tilde{r}(u) - \tilde{r}(T))\,du}$$

$$\leqslant \frac{\alpha^3\int_0^T(T-u)\int_u^T g(z)\,dz\,du}{\int_0^T(T-u)(\tilde{r}(u) - \tilde{r}(T))\,du} = \alpha^3 \leqslant 2\alpha^4 - 1.$$

(iii) For $\beta \leqslant S \leqslant T$ we have

$$\begin{aligned} STE\left(\bar{\tilde{X}}_S - I_S\right)\left(\bar{\tilde{X}}_T - I_T\right) &= E\textstyle\int_0^S(\tilde{X}_s - I_s)\,ds\int_0^T\tilde{X}_t\,dt \\ &= E\textstyle\int_0^S(\tilde{X}_s - I_s)\,ds\left[\int_0^S\tilde{X}_t\,dt + \int_S^T\tilde{X}_t\,dt\right] \\ &= S^2\sigma_S^2 + \textstyle\int_0^S\left[\int_S^T(\tilde{r}(s-t) - \tilde{r}(t))\,dt\right]ds \\ &\geqslant S^2\sigma_S^2. \end{aligned}$$

THEOREM 2.1. *Suppose* $r(t) = EX_0X_t$ *is of the form* (2.1) *and satisfies the conditions at* (2.7 i *and* ii). *Then*

$$\frac{(\ln t)^{\frac{1}{2}}}{\ln\ln t}\left(\bar{X}_t - I_t\right) \to 0 \text{ as } t \to \infty \text{ a.s.} \tag{2.9}$$

PROOF. By Lemma 2.1 it suffices to show

$$\frac{(\ln t)^{\frac{1}{2}}}{\ln\ln t}\left(\bar{\tilde{X}}_t - I_t\right) \to 0 \text{ as } t \to \infty \text{ a.s.} \tag{2.10}$$

Further since r and $\tilde{r}$ agree on $[\beta, \infty)$, conditions (2.7i and ii) hold for $\tilde{r}$ also. Let $\xi_t = \bar{\tilde{X}}_t - I_t/\sigma_t$. Then by (2.8 ii and iii) it follows that

$$E(\xi_t - \xi_s)^2 \leqslant 4\Sigma_{k=1}^4\binom{4}{k}\left(\frac{t-s}{s}\right)^k, \qquad \beta \leqslant s \leqslant t \tag{2.11}$$

for

$$\begin{aligned} E\{\xi_t - \xi_s\}^2 &= 2[1 - \rho_{s,t}] \\ &\leqslant 2\left[1 - \frac{s\sigma_s}{t\sigma_t}\right] \\ &\leqslant 2\left(1 - \left[1 + 2\Sigma_{k=1}^4\binom{4}{k}\left(\frac{t-s}{s}\right)^k\right]^{-\frac{1}{2}}\right) \\ &\leqslant 4\Sigma_{k=1}^4\binom{4}{k}\left(\frac{t-s}{s}\right)^k. \end{aligned}$$

Thus if $t_m = e^m$, $m = 1, 2, \cdots$ and $\epsilon > 0$ is arbitrary, we have

$$\begin{aligned} P\left\{\max_{t_m \leqslant t < t_{m+1}}\frac{(\ln t)^{\frac{1}{2}}}{\ln\ln t}\left(\bar{\tilde{X}}_t - I_t\right) > \epsilon\right\} &\leqslant P\left\{\max_{t_m \leqslant t < t_{m+1}}\xi_t > \epsilon\frac{\ln\ln t_{m+1}}{\sigma_{t_m}(\ln t_{m+1})^{\frac{1}{2}}}\right\} \\ &\leqslant P\left\{\max_{0 \leqslant \tau < 1}\xi_\tau > \epsilon\frac{\ln\ln t_{m+1}}{K\tilde{r}_{t_m}^{\frac{1}{2}}}\right\} \\ &\leqslant c\Phi(-\ln m) \end{aligned} \tag{2.12}$$

for m large enough where c is a constant not depending on m. The next to last inequality followed by (2.8i) where K is some constant and the change of time is given by $\tau = \tau(t) = t_m + t(t_{m+1} - t_m)$. The last inequality followed by Fernique's inequality by virtue of (2.11). (2.10) now follows from (2.12) by Borel-Cantelli.

Let us also note that Theorem 2.1 contains the corresponding result for discrete time, that is

$$\frac{(\ln n)^{\frac{1}{2}}}{\ln \ln n}\left(\bar{X}_n^d - I_n\right) \to 0 \text{ as } n \to \infty \text{ a.s.} \tag{2.13}$$

where $\bar{X}_n^d = 1/(n+1)\Sigma_{k=0}^n X_k$. This follows upon checking that $nE(\bar{X}_n^d - \bar{X}_n^c)^2 = 0(1)$.

3. Iterated logarithm law for $\mathbf{M_n - \bar{X}_n}$.

THEOREM 3.1. *Suppose r is of the form* (2.1) *and satisfies*

(3.1)
(i) $r_n = o(1)$
(ii) $(r_n \ln n)^{-1}$ *is monotone for large n and is* $o(1)$.

Then with probability one we have

$$\liminf_{n\to\infty} \frac{2c_n\left(M_n - \bar{X}_n - (1 - r_n)^{\frac{1}{2}} c_n\right)}{\ln \ln n} = -1 \quad \textit{and}$$

$$\limsup_{n\to\infty} \frac{2c_n\left(M_n - \bar{X}_n - (1 - r_n)^{\frac{1}{2}} c_n\right)}{\ln \ln n} = 1 \tag{3.2}$$

PROOF. Recall the representation at (2.3). For $n \geqslant \beta$ and $0 \leqslant k \leqslant n$,

$$X_k = (1 - r_n)^{\frac{1}{2}} Y_k^n + I_n$$

Let $M_n{}^* = \max_{0\leqslant k\leqslant n} Y_k^n$. Then by (2.13) to show (3.2) it suffices to show

$$\liminf \frac{2c_n(M_n{}^* - c_n)}{\ln \ln n} = -1 \quad \text{and} \tag{3.3}$$

$$\limsup \frac{2c_n(M_n^* - c_n)}{\ln \ln n} = 1.$$

Now the proof of lim inf $(2c_n(M_n^* - c_n))/(\ln \ln n) \leqslant -1$ given in [6] does not depend on mixing conditions and applies here as well.

Also the proof of lim sup $(2c_n(M_n^* - c_n))/(\ln \ln n) \leqslant 1$ in [6] does not make essential use of the condition $(r_n \ln n)/(\ln \ln n)^2 = o(1)$ and carries over to our case without difficulty. However, the proofs of lim inf $(2c_n(M_n^* - c_n))/(\ln \ln n) \geqslant -1$ and lim sup $(2c_n(M_n^* - c_n))/(\ln \ln n) \geqslant 1$ require substantial modification.

The proof of the lim inf proceeds by showing first that it is sufficient to consider the lim inf along a particular subsequence and second that the lim inf along that subsequence is at least -1. In establishing the latter half, Mittal and Ylvisaker use their mixing condition so that a bound on a certain probability derived in a previous paper might be employed to bound a probability that occurs in the proof. By working directly with this probability. (3.4ii) below, one can avoid the condition $(r_n \ln n)/(\ln \ln n)^2 = o(1)$.

Obtaining that the lim sup is at least 1 in the present context, is the more difficult extension. Essentially the difference in the two proofs is in the choice of blocking the variables in preparation for Slepian's lemma. Mittal and Ylvisaker use a fixed length between blocks which allows a convenient bound on the maximum correlation between blocks but in order that the bound be useful for an application of Slepian's lemma, their condition on r must be imposed. Removing that condition necessitates a delicate blocking scheme of fewer blocks (the number of blocks is of smaller order than the log of the number used in [6]) and such that the distance between the initial and later blocks grows geometrically as does the size of the blocks. With this scheme one can show that the maximum correlation between blocks is appropriately small without using the condition $(r_n \ln n)/(\ln \ln n)^2 = o(1)$ and that sufficiently many variables have been retained so that effective use of Slepian's lemma is possible.

LEMMA 3.1. $\liminf (2c_n(M_n^* - c_n))/(\ln \ln n) \geq -1$.

PROOF. Let $n_m = [e^m]$ and $\epsilon > 0$ be arbitrary. Define sets

$$G_m = \left\{ \frac{2c_{n_m}(M_{n_m}{}^* - c_{n_m})}{\ln \ln n_m} > -(1+\epsilon) \right\}$$

$$H_m = \left\{ \frac{2c_n(M_n^* - c_n)}{\ln \ln n} < -(1+2\epsilon) \text{ for some } n_m \leq n < n_{m+1} \right\}$$

$$J_m = G_m \cap G_{m+1} \cap H_m.$$

The lemma follows by showing

$$\text{(3.4)} \qquad \text{(i) } P\{J_m, \text{ i.o.}\} = 0 \qquad \text{(ii) } P\{G_m^c, \text{ i.o.}\} = 0.$$

The proof of (3.4 i) is accomplished by the same arguments as presented in [6]. To show (3.4 ii) we will demonstrate that the sequence

$$P\left\{ M_{n_m}{}^* \leq c_{n_m} - \frac{(1+\epsilon)\ln \ln n_m}{2c_{n_m}} \right\}$$

is summable.

First observe that since the correlation function r agrees with a convex correlation function from some point on, we may by Berman's comparison lemma [1] assume that r is convex without loss of generality. Thus if we set $t_n = [\exp\{(1 - (10 \ln \ln n)/(\ln n)\ln n\}]$ and $\bar{\rho}_n(j) = \rho_n(j) \vee \rho_n(t_n)$, $o \leq j \leq n$, then $\bar{\rho}_n$ defines a correlation function for $0 \leq j \leq n$ by convexity. Now let $\{W_k^n, 0 \leq k \leq n\}$ be a sequence of standard normal variables having correlation function $\bar{\rho}_n$. By

Slepian's lemma [7] we have

$$P\left\{M_n^* \leqslant c_n - \frac{(1+\epsilon)\ln\ln n}{2c_n}\right\}$$

$$\leqslant P\left\{\max_{0\leqslant k\leqslant n} W_k^n \leqslant c_n - \frac{(1+\epsilon)\ln\ln n}{2c_n}\right\}$$

$$\leqslant P\left\{(1-\rho_n(t_n))^{\frac{1}{2}} M_n(0) + \rho_n^{\frac{1}{2}}(t_n) V \leqslant c_n - \frac{(1+\epsilon)\ln\ln n}{2c_n}\right\}$$

$$(3.5) \qquad + \left| P\left\{(1-\rho_n(t_n))^{\frac{1}{2}} M_n(0) + \rho_n^{\frac{1}{2}}(t_n) V \leqslant c_n - \frac{(1+\epsilon)\ln\ln n}{2c_n}\right\} - P\left\{\max_{0\leqslant k\leqslant n} W_k^n \leqslant c_n - \frac{(1+\epsilon)\ln\ln n}{2c_n}\right\}\right|$$

where $M_n(0)$ denotes the maximum of $n+1$ independent standard normal variables and V is a standard normal variable independent of $M_n(0)$.

The first term appearing at (3.5) is at most

$$(3.6) \quad \Phi\left(-2(\ln\ln n)^{\frac{1}{2}}\right) + P\left\{M_n(0) \leqslant c_n - \frac{(1+\epsilon/2)\ln\ln n}{2c_n}\right\}$$

$$\leqslant (\ln\ln n)^{-\frac{1}{2}} e^{-2\ln\ln n} + \exp\{-e^{\epsilon/4\ln\ln n}\}.$$

Evaluated at the point $n_m = [e^m]$, the sum at (3.6) is clearly summable on m.

For the second term at (3.5), Berman's lemma yields the upper bound

$$(3.7) \qquad n\Sigma_{j=1}^{t_n} \rho_n(k) \exp\left\{-\frac{\left(c_n - \frac{(1+\epsilon)\ln\ln n}{2c_n}\right)^2}{1+\rho_n(k)}\right\}.$$

Now define the following quantities

$$t_o = t_o(n) = n^\alpha \quad , o < \alpha < \frac{1-r_1}{1+r_1} \leqslant \frac{1-\rho_n(1)}{1+\rho_n(1)}$$

$$t_i = t_i(n) = \exp\{(1 - r_n^{i/2})\ln n\}, \qquad 1 \leqslant i \leqslant q_n$$

$$t_{q_n+1} = t_n$$

where q_n is defined to be that integer for which

$$r_n^{q_n/2} > \frac{10\ln\ln n}{\ln n} \geqslant r_n^{(q_n+1)/2}.$$

Note $q_n \geqslant 1$ satisfying the above will exist for n sufficiently large since $(r_n \ln n)^{-1} = o(1)$. Also note that $q_n < \ln \ln n$ and that

$$\rho_n(t_o) \leqslant 2\left(\frac{1-\alpha}{\alpha}\right) r_n$$

$$\rho_n(t_i) \leqslant 2 r_n r_n^{i/2}, \qquad 1 \leqslant i \leqslant q_n.$$

Now observe that we may bound (3.7) above by

$$n^{1+\alpha} \exp\left\{ -\frac{\left(c_n - (1+\epsilon)\frac{\ln \ln n}{2c_n}\right)^2}{1+\rho_n(1)} \right\} \tag{3.8}$$
$$+ n\Sigma_{i=o}^{q_n} t_{i+1} \exp\left\{ -\frac{\left(c_n - (1+\epsilon)\frac{\ln \ln n}{2c_n}\right)^2}{1+\rho_n(t_i)} \right\}.$$

The first term at (3.8) can easily be shown to be $o(n^{-\lambda})$ for some $\lambda > o$ not depending on n. Moreover, following the procedure given in [5] to bound the sum appearing at (2.17) in that paper, one can easily verify that the second term at (3.8) is at most $e^{-2 \ln \ln n}$. Since $e^{-2 \ln \ln n_m} \leqslant 2/m^2$ is summable, the proof of Lemma 3.1 is complete.

LEMMA 3.2. $\lim \sup(2c_n(M_n^* - c_n))/(\ln \ln n) \geqslant 1$.

PROOF. Let $\Theta_n = \Theta_n(\epsilon) = c_n + ((1-\epsilon) \ln \ln n)/2c_n$ We will establish the lemma by showing that for any fixed $\epsilon > 0$ and K,

$$\lim_{N\to\infty} P\{ Y_j^n \leqslant \Theta_n, 0 \leqslant j \leqslant n; K \leqslant n \leqslant N\} = 0. \tag{3.9}$$

Define the following quantities

$$L = L(N) = \left[N \exp\left\{ -\frac{\ln N}{(\ln \ln N)^2} \right\} \right]$$

$$q = q_N(\delta) = \left[\exp\{ r_N^{-\delta} \} \right]$$

for some $0 < \delta < 1$ to be specified later. $m = m_N(\delta)$ to be that integer for which $Lq^m \leqslant N < Lq^{m+1}$. $a_\gamma = Lq^\gamma$, $1 \leqslant \gamma \leqslant m$. Note

$$m \leqslant (r_N \ln N)^\delta \frac{(\ln N)^{1-\delta}}{(\ln \ln N)^2} \leqslant m + 1$$

so that

$$m \leqslant \ln N \tag{3.10}$$

and $m = m_N \to \infty$ as $N \to \infty$. Now (3.9) is bounded above by

$$P\{Y_j^n \leqslant \Theta_n, 0 \leqslant j \leqslant n, L \leqslant n \leqslant N\}$$

$$\leqslant P\{\cap_{\gamma=1}^m [Y_j^n \leqslant \Theta_n, a_{\gamma-1} \leqslant j \leqslant n, a_{\gamma-1} \leqslant n \leqslant a_\gamma]\}$$

$$\leqslant P\left\{\cap_{\gamma=1}^m \left[Y_j^{a_\gamma} \leqslant \Theta_{a_\gamma}, a_{\gamma-1} \leqslant j < a_{\gamma-1}\left(\frac{1+q}{2}\right)\right]\right\}. \tag{3.11}$$

Next for any $\gamma < \tilde{\gamma}$ and any j, k with

$$a_{\gamma-1} \leqslant j < a_{\gamma-1}\left(\frac{1+q}{2}\right) < a_{\tilde{\gamma}-1} \leqslant k < a_{\tilde{\gamma}-1}\left(\frac{1+q}{2}\right)$$

we have

$$EY_j^{a_\gamma} Y_k^{a_{\tilde{\gamma}}} \leqslant 2[r(j-k) - r(a_{\tilde{\gamma}})].$$

But $k - j \geqslant a_{\tilde{\gamma}}((q-1)/2)$ so that

$$r(k-j) - r(a_{\tilde{\gamma}}) \leqslant 2r_N\left[\frac{\ln a_{\tilde{\gamma}}}{\ln a_{\tilde{\gamma}-2}\left(\frac{q-1}{2}\right)} - 1\right] \leqslant 3\frac{r_N^{1-\delta}}{\ln N}.$$

Thus if we define $\sigma_N = 6\,(r_N^{1-\delta}/\ln N)$, then

$$EY_j^{a_\gamma} Y_k^{a_{\tilde{\gamma}}} \leqslant \sigma_N.$$

Now put

$$\rho_\gamma(j-k) = EY_j^{a_\gamma} Y_k^{a_\gamma} = \frac{r(j-k) - r(a_\gamma)}{1 - r(a_\gamma)}$$

and define random variables

$$Z_{\gamma,n} = (1-\sigma_N)^{\frac{1}{2}} U_{\gamma,n} + \sigma_N^{\frac{1}{2}} V$$

where $0 \leqslant n < a_{\gamma-1}((q-1)/2)$ and $1 \leqslant \gamma \leqslant m$. The $U_{\gamma,n}$ and V are standard normal random variables, V is independent of the $U_{\gamma,n}$ and

$$\begin{aligned} EU_{\gamma,n} \quad U_{\tilde{\gamma},\tilde{n}} &= \rho_\gamma(n-\tilde{n}) \qquad \text{if} \quad \gamma = \tilde{\gamma} \\ &= 0 \qquad \text{if} \quad \gamma \neq \tilde{\gamma}. \end{aligned}$$

Then we clearly have

$$EY_{a_{\gamma-1}+n}^{a_\gamma} Y_{a_{\tilde{\gamma}-1}+\tilde{n}}^{a_{\tilde{\gamma}}} \leqslant EZ_{\gamma,n} Z_{\tilde{\gamma},\tilde{n}}.$$

Thus by Slepian's lemma [7] the probability at (3.11) is at most

$$P\left\{\cap_{\gamma=1}^m \left[Z_{\gamma,n} \leqslant \Theta_{a_\gamma}, 0 \leqslant n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right]\right\}$$

$$= P\left\{\cap_{\gamma=1}^m \left[(1-\sigma_N)^{\frac{1}{2}} U_{\gamma,n} + \sigma_N^{\frac{1}{2}} V \leqslant \Theta_{a_\gamma}, 0 \leqslant n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right]\right\}$$

$$\leq \Phi(-\ln\ln N)$$

$$+P\left\{\cap_{\gamma=1}^{m}\left[U_{\gamma,n} \leq \Theta_{a_\gamma}(\epsilon/2), 0 \leq n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right]\right\}$$

$$= o(1) + \Pi_{\gamma=1}^{m} P\left\{U_{\gamma,n} \leq \Theta_{a_\gamma}, 0 \leq n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right\} \tag{3.12}$$

where $\Theta_{a_\gamma} = \Theta_{a_\gamma}(\epsilon)$ by redefining ϵ.

Now by Berman's lemma [1]

$$|P\left\{U_{\gamma,n} \leq \Theta_{a_\gamma}, 0 \leq n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right\} - \exp\left\{a_{\gamma-1}\left(\frac{q-1}{2}\right)\ln\Phi(\Theta_{a_\gamma})\right\}$$

$$\leq a_\gamma \Sigma_{k=1}^{a_\gamma} \rho_\gamma(k) \exp\left\{-\frac{\Theta_{a_\gamma}^2}{1+\rho_\gamma(k)}\right\}. \tag{3.13}$$

Now if $0 < \alpha < (1-\rho_0(1))/(1+\rho_0(1)) \leq (1-\rho_\gamma(1))/(1+\rho_\gamma(1))$, $1 \leq \gamma \leq m$, we can bound (3.13) above by

$$\exp\left\{-\tfrac{1}{2}\left(\frac{1-\rho_0(1)}{1+\rho_0(1)} - \alpha\right)\ln a_\gamma\right\} + c a_\gamma \Sigma_{(a_\gamma)^\alpha+1}^{a_\gamma} \rho_\gamma(k) \exp\left\{-\frac{b_{a_\gamma}^2 + (2-\epsilon)\ln\ln a_\gamma}{1+\rho_\gamma(k)}\right\}$$

$$\leq a_\gamma^{-\lambda} + \frac{c}{(\ln a_\gamma)^{\frac{3}{2}}} a_\gamma \Sigma_{(a_\gamma)^\alpha+1}^{a_\gamma} \rho_\gamma(k) \exp\left\{-\frac{b_{a_\gamma}^2}{1+\rho_\gamma(k)}\right\} \tag{3.14}$$

where $\lambda > 0$ is some constant not depending on γ. The expression $a_\gamma \Sigma_{(a_\gamma)^\alpha+1}^{a_\gamma} \rho_\gamma (k)\exp\{-\frac{b_{a_\gamma}^2}{1+\rho_\gamma(k)}\}$ may be seen to be $o(1)$ by following the procedure given in [4] to show the sum appearing at (2.13) in that paper was $o(1)$. Therefore (3.14) is at most $(\ln a_\gamma)^{-\frac{3}{2}}$ for all N sufficiently large. Hence

$$P\left\{U_{\gamma,n} \leq \Theta_{a_\gamma}, 0 \leq n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right\}$$

$$\leq (\ln a_\gamma)^{-\frac{3}{2}} + \exp\left\{a_{\gamma-1}\left(\frac{q-1}{2}\right)\ln\Phi(\Theta_{a_\gamma})\right\}$$

$$\leq (\ln a_\gamma)^{-\frac{3}{2}} + \exp\left\{-c\frac{a_{\gamma-1}\left(\frac{q-1}{2}\right)}{\Theta_{a_\gamma}} e^{-\frac{1}{2}\Theta_{a_\gamma}^2}\right\}$$

$$\leq (\ln a_\gamma)^{-\frac{3}{2}} + \exp\left\{-\frac{c}{(\ln a_\gamma)^{1-\epsilon/2}}\right\}$$

for some constant c. Thus

$$\ln \Pi_{\gamma=1}^{m} P\left\{ U_{\gamma,n} \leqslant \Theta_{a_\gamma}, 0 \leqslant n < a_{\gamma-1}\left(\frac{q-1}{2}\right)\right\}$$

$$\leqslant \Sigma_{\gamma=1}^{m} \ln\left[(\ln a_\gamma)^{-\frac{3}{2}} + \exp\left\{-\frac{c}{(\ln a_\gamma)^{1-\epsilon/2}}\right\}\right]$$

$$\leqslant \Sigma_{\gamma=1}^{m}\left\{\frac{-c}{(\ln a_\gamma)^{1-\epsilon/2}} + 2(\ln a_\gamma)^{-\frac{3}{2}}\right\}$$

$$\leqslant o(1) - c\int_1^{m+1} \frac{dx}{(\ln L + x \ln q)^{1-\epsilon/2}}$$

$$\leqslant o(1) - cr_N^{\delta}\int_{\ln qL}^{\ln N} \frac{dx}{x^{1-\epsilon/2}}$$

$$= o(1) - \frac{2c}{\epsilon} r_N^{\delta}\left[(\ln N)^{\epsilon/2} - (\ln qL)^{\epsilon/2}\right]$$

$$\leqslant o(1) - \frac{2c}{\epsilon} r_N^{\delta}(\ln N)^{\epsilon/4}$$

$\rightarrow -\infty$ as $N \rightarrow \infty$ for δ small enough. Thus (3.12) is $o(1)$ and hence Lemma 3.2 holds proving Theorem 3.1

Let $\{X_t, t \geqslant 0\}$ be a separable stationary Gaussian process with $EX_0 = 0$, $EX_0^2 = 1$ and $r(t) = EX_0X_t$ of the form (2.1). Further suppose for some constant $c > 0$ that $1 - r(t) = c|t|^{\alpha} + o(|t|^{\alpha})$ for t in a neighborhood of zero so that the sample paths are continuous and we may define $M_T = \max_{0 \leqslant t \leqslant T} X_t$. By the representation at (2.3) we may write $M_T = (1 - r_T)^{\frac{1}{2}} M_T^* + I_T$ where $M_T^* = \max_{0 \leqslant t \leqslant T} Y_t^T$. Now assume the hypothesis of Theorem 2.1 holds so that by Theorem 2.1 the process $(\ln T)^{\frac{1}{2}}/(\ln \ln T)\ (I_T - \bar{X}_T)$ is a.s. asymptotically negligible and therefore, in order to prove a continuous version of Theorem 3.1, we need only modify the discrete time arguments of Section 3 to a continuous time setting. These modifications are essentially standard and will not be given. We state the result below.

THEOREM 3.2. *Assume the hypothesis of Theorem* 2.1 *holds and* $1 - r(t) = c|t|^{\alpha} + o(|t|^{\alpha})$ *as* $t \rightarrow 0$. *Then*

$$\liminf \frac{c_T\left(M_T - \bar{X}_T - (1 - r(T))^{\frac{1}{2}} c_T\right)}{\ln \ln T} = \frac{1}{\alpha} - \frac{1}{2}$$

$$\limsup \frac{c_T\left(M_T - \bar{X}_T - (1 - r(T))^{\frac{1}{2}} c_T\right)}{\ln \ln T} = \frac{1}{\alpha} + \frac{1}{2}.$$

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