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LAPLACE TRANSFORMS FOR CLASSES OF LIFE DISTRIBUTIONS[1]

BY HENRY W. BLOCK AND THOMAS H. SAVITS

University of Pittsburgh

Using Laplace transforms, necessary and sufficient conditions are given for classes of life distributions which are of interest in reliability. Using these conditions, the problem of class closure under convolution is discussed. Furthermore, converses to results of A-Hameed for gamma wear processes are given.

1. Introduction. Vinogradov (1973) has used the Laplace transform to give necessary and sufficient conditions for a distribution to have increasing failure rate. In this paper, we obtain similar results for other reliability classes.

Let F be a distribution function such that $F(0) = 0$ and let $\phi(s) = \int_0^\infty e^{-sx}dF(x)$, $s \geqslant 0$. We define

$$a_n(s) = \frac{(-1)^n}{n!}\frac{d^n}{ds^n}\left[\frac{1-\phi(s)}{s}\right], \qquad n \geqslant 0, s \geqslant 0$$

and set $\alpha_{n+1}(s) = s^{n+1}a_n(s)$ for $n \geqslant 0$, $s \geqslant 0$ and let $\alpha_0(s) = 1$ for all $s \geqslant 0$.

The reliability classes of distributions which are IFR, IFRA, NBU and NBUE have been widely discussed in the literature. Definitions can be found in Barlow and Proschan (1975). The class of DMRL distributions and the class with PF_2 densities can be found in Esary, Marshall and Proschan (1973) along with the discrete analogs of all six of these classes. The terms increasing and decreasing will mean nondecreasing and nonincreasing respectively. The main theorem can now be given.

(1.1) THEOREM. *For nonnegative distributions F:*

(1) *F is* IFR *if and only if $\{\alpha_n(s), n \geqslant 1\}$ is* log *concave in n for all $s > 0$;*

(2) *F is* DMRL *if and only if $\Sigma_{k=n}^\infty \alpha_k(s)/\alpha_n(s)$ is decreasing in $n = 1, 2, \cdots$ for all $s > 0$;*

(3) *F is* IFRA *if and only if $[\alpha_n(s)]^{1/n}$ is decreasing in $n = 1, 2 \cdots$ for all $s > 0$;*

(4) *F is* NBU *if and only if $\alpha_{n+m}(s) \leqslant \alpha_n(s)\alpha_m(s)$ for all $n, m > 0$, all $s > 0$;*

(5) *F is* NBUE *if and only if $\alpha_n(s)\Sigma_{k=0}^\infty \alpha_k(s) \geqslant \Sigma_{k=n}^\infty \alpha_k(s)$ for all $n > 0$ and all $s > 0$.*

The proof of this theorem will be given in Section 2. An interpretation of the conditions of this theorem in terms of a shock model is given in Section 3 along

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with a probabilistic interpretation of the α's. This leads to the expression

$$\alpha_n(s) = E\big(\bar{F}(T_n(s))\big), \tag{1.2}$$

where $\bar{F}(x) = 1 - F(x)$ and $T_n(s)$ is a random variable which has a gamma distribution with parameters n and s. Since $E(T_n(s)) = n/s$ and Var $T_n(s) = n/s^2$ it follows by choosing $s = n/x$ that

$$\alpha_n(s) \to \bar{F}(x) \quad \text{as} \quad n \to \infty. \tag{1.3}$$

A detailed proof of (1.3) is given in Lemma 2.3. Thus from (1.2) it can be seen that properties of $\bar{F}$ can be translated into properties of the α's and from (1.3) properties of the α's are inherited by $\bar{F}$. This is the idea of the proof of Theorem 1; the details are carried out in Section 2. It should be noted that for a specific Laplace transform it might be very difficult to check the various conditions of Theorem 1.1. In this sense it would still be very useful to have conditions which can be easily checked for the various classes of life distributions.

In Section 3, a quantity related to the α_n is defined and using it, other classes from the theory of total positivity are characterized. Convolutions of the distributions in the various classes are discussed in Section 4 and in Section 5 converses to results of A-Hameed (1975) for gamma wear processes are given.

2. Proof of theorem. Before proving the theorem some useful versions of a_n and α_{n+1} will be given. These easily verified forms are

$$a_n(s) = \int_0^\infty \frac{u^n}{n!} e^{-su}\bar{F}(u)\, du \text{ for } n \geqslant 0 \text{ and } s \geqslant 0, \tag{2.1}$$

$$\alpha_{n+1}(s) = s\int_0^\infty \frac{(su)^n}{n!} e^{-su}\bar{F}(u)\, du \text{ for } n \geqslant 0 \text{ and } s \geqslant 0. \tag{2.2}$$

The necessity of the conditions will be established first.

(1) This has been established in Vinogradov (1973), but can also be proven by an argument similar to the one which follows in (2).

(2) A standard sign change argument can be used to establish the condition. As on page 93 of Barlow and Porschan (1975) (or see Karlin (1968)), the kernel $\exp(-su)\cdot(su)^n/n!$ is PF_2 in $s \in (0, \infty)$ and $n \in \{0, 1, 2, \cdots\}$. To show the condition we use the identity, for $n \geqslant 0$, $s > 0$, and $c \geqslant 0$,

$$s\int_0^\infty \frac{e^{-st}(st)^n}{n!}\Big(\int_t^\infty \bar{F}(y)\, dy - c\bar{F}(t)\Big)\, dt = \frac{1}{s}\sum_{i=n+2}^\infty \alpha_i(s) - c\alpha_{n+1}(s).$$

Thus since F is DMRL, $\int_t^\infty \bar{F}(y)\, dy - c\bar{F}(t)$ changes sign at most once and in the order $+$, $-$ if a change occurs. The variation diminishing property of the above kernel yields that $\sum_{i=n+1}^\infty \alpha_i(s)/\alpha_n(s)$ is decreasing in $n = 1, 2, \cdots$ for all $s > 0$. Finally adding $1 = \alpha_n(s)/\alpha_n(s)$ to the above gives the condition.

(3) and (4). Fix $s \geqslant 0$. Then

$$a_n(s) = \int_0^\infty \frac{u^n}{n!}\bar{G}(u)\, du$$

where $\bar{G}(u) = e^{-su}\bar{F}(u)$. Clearly $\bar{G}(u)$ is IFRA or NBU if and only if $\bar{F}(u)$ has the corresponding property. Also

$$a_n(s) = \int_0^\infty \frac{u^n}{n!}\bar{G}(u)\,du = \frac{\rho_{n+1}}{(n+1)!} = \gamma_{n+1}$$

where ρ_r is the rth moment of G. By Corollary 6.5, page 112, of Barlow and Proschan (1975), if G is IFRA, then $(\gamma_r)^{1/r}$ is decreasing in r and so $\alpha_n^{1/n}(s)$ is decreasing in $n > 0$. Similarly if G is NBU, by problem 3, page 187, of Barlow and Proschan (1975), the necessity of the condition in (4) follows.

(5) Since F is NBUE we have

$$\bar{F}(u) \geqslant \frac{1}{\mu}\int_u^\infty \bar{F}(v)\,dv \qquad \text{for each} \quad u \geqslant 0,$$

where μ is the mean of F. Now, for $n \geqslant 0$, $s > 0$,

$$\begin{aligned} a_n(s) &= \int_0^\infty \frac{u^n}{n!}e^{-su}\bar{F}(u)\,du \geqslant \frac{1}{\mu}\int_0^\infty \frac{u^n}{n!}e^{-su}\int_u^\infty \bar{F}(v)\,dv\,du \\ &= \frac{1}{\mu}\int_0^\infty \bar{F}(v)\int_0^v \frac{u^n}{n!}e^{-su}\,du\,dv \\ &= \frac{1}{\mu}\int_0^\infty \bar{F}(v)\frac{1}{s^{n+1}}\left\{1 - \Sigma_{k=0}^n \frac{s^k v^k}{k!}e^{-sv}\right\}dv \\ &= \frac{1}{\mu}\frac{1}{s^{n+1}}\int_0^\infty \bar{F}(v)\Sigma_{k=n+1}^\infty \frac{s^k v^k}{k!}e^{-sv}\,dv \\ &= \frac{1}{\mu}\frac{1}{s^{n+1}}\Sigma_{k=n+1}^\infty s^k a_k(s). \end{aligned}$$

Thus

$$\mu s s^{n+1}a_n(s) \geqslant \Sigma_{k=n+1}^\infty s^{k+1}a_k(s) \text{ for } n \geqslant 0 \text{ and } s > 0$$

or

$$\alpha_{n+1}(s)\Sigma_{k=1}^\infty \alpha_k(s) \geqslant \Sigma_{k=n+2}^\infty \alpha_k(s)$$

since $\mu s = \Sigma_{k=1}^\infty \alpha_k(s)$. Adding $\alpha_{n+1}(s)$ to both sides of the inequality gives that

$$\alpha_n(s)\Sigma_{k=0}^\infty \alpha_k(s) \geqslant \Sigma_{k=n}^\infty \alpha_k(s) \qquad \text{for} \quad n \geqslant 1.$$

Sufficiency of the conditions. We repeatedly use the following lemma.

(2.3) LEMMA. *Let $x > 0$ be a continuity point of the* df F. *Let $s = s(n, x)$ and $\lim_{n\to\infty} n/s = x$. Then $\lim_{n\to\infty}\alpha_{n+1}(s) = \bar{F}(x)$.*

PROOF. We have

$$\alpha_{n+1}(s) = \int_0^\infty \bar{F}(u)\,dG_n(u)$$

where $dG_n(u) = (s^{n+1}/n!)u^n\exp(-su)\,du$. Since $\lim_{n\to\infty} n/s = x$ it follows that G_n converges weakly to G, where G is the one point distribution concentrated at x. By Theorem 5.2 of Billingsley (1968) it follows that $\lim_{n\to\infty}\alpha_{n+1}(s) = \bar{F}(x)$.

Sufficiency of condition in (1). This was proven by Vinogradov (1973).

Sufficiency of condition in (2). The condition can be written for all $s > 0$ as

$$\alpha_{n+k+1}(s)\Sigma_{i=n+1}^{\infty}\alpha_i(s) \geqslant \alpha_{n+1}(s)\Sigma_{i=n+k+1}^{\infty}\alpha_i(s) \quad \text{for all } k,\ n \geqslant 0.$$

It is easily checked that

$$\Sigma_{i=n+1}^{\infty}\alpha_i(s) = \int_0^{\infty} sP\{Y_n \leqslant u\}\bar{F}(u)\, du$$

where Y_n has the df G_n given in the proof of the lemma. As in the lemma if $\lim_{n\to\infty} n/s = x > 0$, G_n converges weakly to the one point distribution at x and

$$\lim_{n\to\infty}\int_0^{\infty} P\{Y_n \leqslant u\}\bar{F}(u)\, du = \int_x^{\infty}\bar{F}(u)\, du.$$

Let x and $x + y$ be continuity points of F with $x, y > 0$. Let $n = 1, 2, \cdots$ and define $s = n/x$ and $k = [\![ny/x]\!]$. It follows that $x = n/s$ and $\lim_{n\to\infty}(n + k)/s = x + y$. The inequality above can be written as

$$\alpha_{n+k+1}(s)\int_0^{\infty} P\{Y_n \leqslant u\}\bar{F}(u)\, du \leqslant \alpha_{n+1}(s)\int_0^{\infty} P\{Y_{n+k} \leqslant u\}\bar{F}(u)\, du$$

and letting $n \to \infty$ gives

$$\bar{F}(x + y)\int_x^{\infty}\bar{F}(u)\, du \leqslant \bar{F}(x)\int_{x+y}^{\infty}\bar{F}(u)\, du.$$

Since the continuity points of F are dense the inequality holds for all $x, y \geqslant 0$ which gives that F is DMRL.

Sufficiency of condition in (3). Let $\alpha = p/q$ where $0 < p < q$ are integers and let $x > 0$ and αx be continuity points of F. The condition can be written for all $s > 0$ as

$$\alpha_{n+1}(s) \geqslant [\alpha_{n+k+1}(s)]^{(n+1)/(n+k+1)} \quad \text{for all } n,\ k \geqslant 0.$$

For integers $m = 1, 2, \cdots$ set $n = mp$ and $k = m(q - p)$ and let $s = n/(\alpha x)$. Then as $m \to \infty$ (or equivalently as $n \to \infty$ and $k \to \infty$) we have that $n/s \to \alpha x$, $(n + k)/s \to x$ and $(n + 1)/(n + k + 1) \to \alpha$. Thus by Lemma 2.3

$$\bar{F}(\alpha x) \geqslant \bar{F}^{\alpha}(x).$$

Since the set of points x of the type considered above are dense, the inequality holds for all $x \geqslant 0$, $0 < \alpha \leqslant 1$ so that F is IFRA.

Sufficiency of condition in (4). The condition can be written for all $s > 0$ as

$$\alpha_{n+1}(s)\alpha_{m+1}(s) \geqslant \alpha_{n+m+2}(s) \quad \text{for all } n,\ m \geqslant 0.$$

For continuity points $x, y > 0$ of F and $n = 1, 2, \cdots$ define $m = [\![ny/x]\!]$ and take $s = n/x$. Then $\lim_{n\to\infty} m/s = y$ and $\lim_{n\to\infty}(n + m + 1)/s = x + y$ and so by the lemma

$$\bar{F}(x)\bar{F}(y) \geqslant \bar{F}(x + y).$$

As in the previous proofs it follows that F is NBU.

Sufficiency of condition in (5). The condition can be written for all $s > 0$ as

$$\alpha_{n+1}(s)\int_0^\infty s\bar{F}(u)\,du \geq \sum_{k=n+2}^\infty \alpha_k(s)$$

or as

$$\alpha_{n+1}(s)\int_0^\infty s\bar{F}(u)\,du \geq \int_0^\infty sP\{Y_{n+1} \leq u\}\bar{F}(u)\,du$$

using the notation in the proof of sufficiency of the condition in (2). For a continuity point $x > 0$ of F, define $s = n/x$. Then $\lim_{n\to\infty} n/s = \lim_{n\to\infty}(n+1)/s = x$ and so

$$\bar{F}(x)\mu \geq \int_x^\infty \bar{F}(u)\,du$$

where μ is the mean of F. As in the previous proofs it follows that F is NBUE.

(2.4) REMARK. A similar theorem holds for DFR, IMRL, DFRA, NWU and NWUE where the necessary and sufficient conditions are obtained by making the obvious modifications on the conditions in Theorem 1.1.

(2.5) REMARK. It is clear that the theorem is still true if the phrase "all $s > 0$" is replaced by the phrase "all s sufficiently large".

3. Total positivity and its role. In the previous section necessary and sufficient conditions were obtained in terms of the Laplace transform for characterizing the various classes of life distributions in reliability. These were obtained in the spirit of Vinogradov (1973). In this section we show how the tools of total positivity can be applied to obtain results similar to those of Theorem 1.1, but for other classes of life distributions. Also it is shown that Theorem 1.1 is actually a statement about a type of shock model. The implications of this observation will also be discussed.

In Theorem 1.1 it should be noticed that the conditions on the $\{\alpha_n(s)\}$ are the exact analogs of the life properties for discrete distributions and that Theorem 1.1 in fact closely resembles Theorem 3.1 of Esary, Marshall, and Proschan (1973). An important difference is that the conditions in this latter theorem are not necessary and sufficient. However, the $\{\alpha_n(s)\}$ can be viewed as probabilities for a certain type of shock process. This will now be described.

Let $\{N_s(t), t \geq 0\}$ be a Poisson process with rate $s > 0$. Then, if X is a rv with survival function $\bar{F}(u)$,

$$\begin{aligned}\alpha_{n+1}(s) &= s\int_0^\infty \frac{e^{-su}(su)^n}{n!}\bar{F}(u)\,du \\ &= s\int_0^\infty P\{N_s(u) = n\}\bar{F}(u)\,du \\ &= \int_0^\infty P\{N_s(u) > n\}\,dF(u) = P\{N_s(X) > n\}.\end{aligned} \tag{3.1}$$

Furthermore if $Y_1, Y_2, \cdots$ are the (exponential) arrival times for the process, then

$$\alpha_{n+1}(s) = P\{\textstyle\sum_{i=1}^{n+1} Y_i \leq X\} = \int_0^\infty G^{(n+1)}(u)\,dF(u) \tag{3.2}$$

where $\bar{G}(u) = \exp(-su)$, $u \geq 0$. Thus (3.2) shows that the $\{\alpha_n(s), n \geq 1\}$ are the discrete survival probabilities for a special case of the random threshold cumulative damage model of EMP (1973). This observation leads to a slightly strengthened version of Theorem 5.2(b) of these authors' paper. That is, (3) of Theorem 1.1 and the previous interpretation give that for $\bar{P}_k = \alpha_k(s)$, $\bar{P}_k^{1/k}$ decreases in $k = 1, 2, \cdots$ whenever F is IFRA. Furthermore, using the above interpretation, Theorem 5.3 of EMP (1973) provides an alternate proof of the necessity of (4).

Using (3.1) define

$$\beta_n(s) = P\{N_s(X) = n\} \quad \text{for } n \geq 0, \quad s > 0. \tag{3.3}$$

Also let $\beta_n(0) = 0$ for $n > 0$ and $\beta_0(0) = 1$. It is clear that for $n \geq 0$, $s \geq 0$

$$\beta_n(s) = \alpha_n(s) - \alpha_{n+1}(s) = \int_0^\infty \frac{e^{-su}(su)^n}{n!}\, dF(u) \tag{3.4}$$

and also if ϕ is the Laplace transform of F

$$\beta_n(s) = \frac{(-s)^n}{n!} \frac{d^n}{ds^n}\phi(s) \quad \text{for } n \geq 0, s \geq 0. \tag{3.5}$$

Using the total positivity of the kernel in (3.4) we now establish criteria for the density f to have certain properties in terms of properties of the $\beta_n(s)$. For the definition of SC_r, and PF_r in the following see Karlin (1968).

(3.6) THEOREM. *Let f be a continuous density on* $[0, \infty)$. *Then $f(x + y)$ is SC_r for $x, y > 0$ if and only if for all $s > 0$, $\beta_{n+m}(s)$ is SC_r for n, $m \geq 0$.*

PROOF. Let $\beta_n'(s) = s\beta_n(s) = (1/n!)\int_0^\infty v^n e^{-v} f(v/s)\, dv$. Then $\beta_{n+m}'(s)$ is SC_r if and only if $\beta_{n+m}(s)$ is SC_r. The necessity of the condition follows from Theorem 5.4, Chapter 3 of Karlin (1968) since the function $g(x + y) = e^{-(x+y)} f((x + y)/s)$ is also SC_r for each $s > 0$ if $f(x + y)$ is SC_r. For the sufficiency let $0 < u_i, v_j$ for $i, j = 1, \cdots, r$ and set $x_i = \sum_{k=1}^i u_k$, $y_j = \sum_{k=1}^j v_k$. Then for any nonnegative integers p_i, $i = 1, \cdots, r$, $q_j, j = 1, \cdots, r$, we have that for each $s > 0$

$$\varepsilon_r \det\left(\beta_{n_i+m_j}'(s)\right) \geq 0$$

where ε_r is the signature of β' and $n_i = \sum_{k=1}^i p_k$, $m_j = \sum_{k=1}^j q_k$. Choosing $s = [\![n/x_1]\!]$, $p_i = [\![nu_i/x_1]\!]$, $q_j = [\![nv_j/x_1]\!]$ and appropriately modifying Lemma 2.3 we obtain that for $i, j = 1, \cdots, r$

$$\beta_{n_i+m_j}'(s) \to f(x_i + y_j) \text{ as } n \to \infty$$

since $(n_i + m_j)/s \to x_i + y_j$ and we are done.

Using Theorem 1.1 b of Chapter 3 of Karlin (1968), the proof of the following corollary is immediate.

(3.7) COROLLARY. *Let f be a continuous density on* $[0, \infty)$. *Then $f(x)$ is PF_r in $x > 0$ if and only if $\beta_n(s)$ is PF_r in $n \geq 0$ for any $s > 0$.*

(3.8) REMARK. Karlin, Proschan and Barlow (1961), obtain the result that if f is a probability density on $[0, \infty)$, then f is PF_∞ if and only if the normalized moments $\gamma_n = (1/n!)\int_0^\infty x^n f(x)\, dx$, $n \geq 0$, form a (one-sided) PF_∞ sequence. Note that $\gamma_n = \lim_{s\to 0}(\beta_n(s)/s^n)$. Consequently Corollary 3.7 is a partial answer to their unresolved question in that paper.

4. Convolutions. In Vinogradov (1975), it is shown that (1) of Theorem 1.1 can be used to prove that convolutions of IFR distributions are IFR. In this section we show how this idea can be extended to other life classes. We first establish a convolution formula for the $\alpha_{n+1}(s)$. Let

$$\alpha_{n+1}^{(X)}(s) = s\int_0^\infty \frac{(su)^n}{n!} e^{-su} P\{X > u\}\, du \quad \text{for } s > 0, n \geqslant 0,$$

$$\alpha_{n+1}^{(Y)}(s) = s\int_0^\infty \frac{(su)^n}{n!} e^{-su} P\{Y > u\}\, du \quad \text{for } s > 0, n \geqslant 0,$$

and $\alpha_0^{(X)}(s) = \alpha_0^{(Y)}(s) = 1$ where X and Y are independent nonnegative random variables. Also let

$$\beta_n^{(X)}(s) = \alpha_n^{(X)}(s) - \alpha_{n+1}^{(X)}(s), \quad \beta_n^{(Y)}(s) = \alpha_n^{(Y)} - \alpha_{n+1}^{(Y)}(s).$$

Using properties of the Poisson process, it is clear from (3.1) and (3.3) that

$$(4.1) \qquad \beta_n^{(X+Y)}(s) = \Sigma_{k=0}^n \beta_k^{(X)}(s)\beta_{n-k}^{(Y)}(s) \quad \text{for } n \geqslant 0, s > 0$$

and

$$(4.2) \qquad \alpha_{n+1}^{(X+Y)}(s) = \alpha_{n+1}^{(Y)}(s) + \Sigma_{k=0}^n \alpha_{k+1}^{(X)}(s)\left(\alpha_{n-k}^{(Y)}(s) - \alpha_{n-k+1}^{(Y)}(s)\right).$$

This gives that convolution of the continuous random variables X and Y gives rise to convolution (of independent versions) of the discrete random variables $N_s(X)$ and $N_s(Y)$. Thus Vinogradov's proof essentially shows that if X and Y are IFR then $\alpha_n^{(X+Y)}(s)$ is discrete IFR and so $X + Y$ is IFR. Similar results can be demonstrated for any of the classes considered in Theorem 1.1. For example, assume X and Y are NBU. Then by Theorem 1.1 $\alpha_n^{(X)}(s)$ and $\alpha_n^{(Y)}$ satisfy condition (4) of Theorem 1.1. As mentioned in Section 3, this is the discrete NBU condition. Since convolutions of discrete NBU distributions are discrete NBU, it follows that $\alpha_n^{(X+Y)}$ satisfies the condition in (4) of Theorem 1.1. Thus $X + Y$ is NBU by Theorem 1.1. Unfortunately the proof of the discrete convolutions is usually no easier than the proof of the continuous analog, so that in most cases not much is gained by this technique.

5. Necessary and sufficient conditions for a gamma wear process. A-Hameed (1975) has discussed preservation of certain life classes under a gamma wear process. In this section we give converses to these results. We first recall that the survival function for a gamma wear process is given by $\bar{\alpha}_s(0) = 0$ and

$$(5.1) \qquad \bar{\alpha}_s(t) = s\int_0^\infty \frac{1}{\Gamma(\Lambda(t))}(su)^{\Lambda(t)-1} e^{-su}\bar{F}(u)\, du \quad \text{for } t > 0$$

where $s > 0$, $\bar{F}$ is a survival function and $\Lambda(t)$ is a hazard function, i.e., $\Lambda(t)$ is increasing and right continuous on $[0, \infty)$, $\Lambda(0) = 0$ and $\lim_{t\to\infty}\Lambda(t) = \infty$. It is not hard to check that if $\Lambda(t)$ satisfies these conditions, then $\bar{\alpha}_s(t)$ is a survival function.

Notice that (5.1) reduces to (2.1) if $\Lambda(t) = t$ for nonnegative integers t. Notice also that $\bar{\alpha}_s$ and $\bar{F}$ play the roles of $\bar{F}$ and $\bar{G}$ in A-Hameed's paper. The converse result can now be stated. A function $\Lambda(t)$ is *antistarshaped* if $\Lambda(t)/t$ is decreasing in $t > 0$.

(5.2) THEOREM. *Let $\bar{\alpha}_s(t)$ be given by* (5.1). *The following hold*:

(a) *If $\bar{\alpha}_s(t)$ is* IFR *for each $s > 0$ and $\Lambda(t)$ is concave, then $\bar{F}(t)$ is* IFR.

(b) *If $\bar{\alpha}_s(t)$ is* IFRA *for each $s > 0$ and $\Lambda(t)$ is antistarshaped, then $\bar{F}(t)$ is* IFRA.

(c) *If $\bar{\alpha}_s(t)$ is* NBU *for each $s > 0$ and $\Lambda(t)$ is subadditive, then $\bar{F}(t)$ is* NBU.

PROOF. (a) Assume $\Lambda(t)$ is concave and $\bar{\alpha}_s(t)$ is IFR. Since Λ is increasing it follows that Λ^{-1} is defined and is convex. Now let $\bar{G}_s(0) = 1$ and

$$\bar{G}_s(v) = s\int_0^\infty \frac{1}{\Gamma(v)}(su)^{v-1}e^{-su}\bar{F}(u)\,du \quad \text{for } v > 0. \tag{5.3}$$

Since $\bar{\alpha}_s(t) = \bar{G}_s(\Lambda(t))$ is assumed to be IFR for each $s > 0$, for $t = \Lambda^{-1}(v)$ it follows that $\bar{G}_s(v)$ is IFR for each $s > 0$. For $0 \leqslant u_1 < u_2$ and $0 < v$

$$\bar{G}_s(u_2)\bar{G}_s(u_1 + v) \geqslant \bar{G}_s(u_2 + v)\bar{G}_s(u_1).$$

Now let $x_1, x_2, x_1 + y, x_2 + y$ be continuity points of F with $x_1 < x_2$. Fix $u_1 \geqslant 0$ and let $s = u_1/x_1$, $u_2 = sx_2$ and $v = sy$. By a modification of Lemma 2.3, as $u_1 \to \infty$

$$\lim_{s\to\infty}\bar{G}_s(u_1) = \bar{F}(x_1),\ \lim_{s\to\infty}\bar{G}_s(u_1 + v) = \bar{F}(x_1 + y)$$

and

$$\lim_{s\to\infty}\bar{G}_s(u_2 + v) = \bar{F}(x_2 + y)$$

so that

$$\bar{F}(x_2)\bar{F}(x_1 + y) \geqslant \bar{F}(x_2 + y)\bar{F}(x_1).$$

Thus F is IFR as in previous proofs of this type.

(b) The condition that $\bar{\alpha}_s$ is IFRA gives that

$$\bar{\alpha}_s(\alpha' t) \geqslant \bar{\alpha}_s^{\alpha'}(t) \quad \text{for } 0 < \alpha' \leqslant 1,\ 0 \leqslant t.$$

Using (5.3), this can be written as

$$\bar{G}_s(\Lambda(\alpha' t)) \geqslant \bar{G}_s^{\alpha'}(\Lambda(t)).$$

Since $\Lambda(t)$ is antistarshaped we can write

$$\bar{G}_s(\alpha'\Lambda(t)) \geqslant \bar{G}_s^{\alpha'}(\Lambda(t)).$$

Let x and $\alpha' x$ be continuity points of F and set $s = \Lambda(t)/x$. As in Lemma 2.3, letting $t \to \infty$ gives

$$\bar{F}(\alpha' x) \geqslant \bar{F}^{\alpha'}(x).$$

Thus F is IFRA.

(c) If $\bar{\alpha}_s$ is NBU, then

$$\bar{\alpha}_s(u)\bar{\alpha}_s(v) \geqslant \bar{\alpha}_s(u+v) \quad \text{for } u,\ v \geqslant 0$$

and using (5.3) this can be written as

$$\bar{G}_s(\Lambda(u))\bar{G}_s(\Lambda(v)) \geqslant \bar{G}_s(\Lambda(u+v)).$$

Using the subadditivity of Λ gives

$$\bar{G}_s(\Lambda(u))\bar{G}_s(\Lambda(v)) \geqslant \bar{G}_s(\Lambda(u)+\Lambda(v)).$$

Now let x and y be continuity points of F. Let $s = \Lambda(u)/x$ and $v = \inf\{t : \Lambda(t) \geqslant sy\}$. Then $\lim_{u\to\infty}\Lambda(v)/s = y$, so that as in Lemma 2.3 as $u \to \infty$ (i.e., $s \to \infty$)

$$\bar{F}(x)\bar{F}(y) \geqslant \bar{F}(x+y).$$

Thus F is NBU.

(5.4) COROLLARY. *Let* $\bar{\alpha}_s(t)$ *be given by* (5.1) *with* $\Lambda(t) = t$. *Then the following hold*:

(a) $\bar{\alpha}_s(t)$ *is* IFR *for each* $s > 0$ *if and only if* $\bar{F}(t)$ *is* IFR.

(b) $\bar{\alpha}_s(t)$ *is* IFRA *for each* $s > 0$ *if and only if* $\bar{F}(t)$ *is* IFRA.

(c) $\bar{\alpha}_s(t)$ *is* NBU *for each* $s > 0$ *if and only if* $\bar{F}(t)$ *is NBU.*

PROOF. Apply the previous theorem and Theorem 1 of A-Hameed (1975).

(5.5) REMARK. Corollary 5.4 can also be obtained by modifying the proofs of (1), (3) and (4) of Theorem 1.1.

(5.6) REMARK. Obvious modifications of the theorem and corollary exist for DFR, DFRA and NWU.

NOTE. As this paper was being written the paper by Vinogradov (1976) came to our attention. It turns out that there is a little overlap with the present paper. Vinogradov's paper contains four theorems. Theorem 1 is a restatement of a result of Karlin, Proschan and Barlow. Theorem 2 and 3 are properly contained in our Theorem 3.6 and our Corollary 3.7 and Theorem 4 is contained in (4) of our Theorem 1.1.

REFERENCES

A-HAMEED, M. (1975). A gamma wear process. *IEEE Transactions on Reliability* **R-24** 152–153.

BARLOW, R. E. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models.* Holt, Rinehart and Winston, New York.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley & Sons, New York.

ESARY, J. D., MARSHALL, A. W. and PROSCHAN, F. (1973). Shock models and wear processes. *Ann Probability* **1** 627–649.

KARLIN, S. (1968). *Total Positivity, Vol. 1.* Stanford Univ. Press.

KARLIN, S., PROSCHAN, F. and BARLOW, R. E. (1961). Moment inequalities of Polya frequency functions. *Pacific J. Math.* **11** 1023–1033.

VINOGRADOV, O. P. (1973). The definition of distribution functions with increasing hazard rate in terms of the Laplace transform. *Theor. Probability Appl.* **18** 811–814.

VINOGRADOV, O. P. (1975). A property of logarithmically concave sequences. *Math. Notes* **18** 865–868.

VINOGRADOV, O. P. (1976). On applications of an inversion formula for the Laplace transform (Russian). *Teor. Veroyatnost. i Primenen.* **21** 857–860.

DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF ARTS AND SCIENCES
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260