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ON THE LIMITING BEHAVIOUR OF THE MODE AND MEDIAN OF A SUM OF INDEPENDENT RANDOM VARIABLES

BY PETER HALL

Australian National University

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables, and let M_n and m_n denote respectively the mode and median of $\Sigma_1^n X_i$. Assuming that $E(X_1^2) < \infty$ we obtain a number of limit theorems which describe the behaviour of M_n and m_n as $n \to \infty$. When $E|X_1|^3 < \infty$ our results specialize to those of Haldane (1942), but under considerably more general conditions.

1. Introduction and summary. One of the most curious results in classical statistical theory is the empirical relationship which exists between the mean, the mode and the median of many continuous distributions:

$$\text{mean-mode} \sim 3\ (\text{mean-median}). \tag{1}$$

For a symmetric unimodal distribution the result is trivial, but it holds quite closely for many moderately skew distributions. For example, the mean, mode and median of a gamma (α) distribution are respectively equal to α, $\alpha - 1$ and $\alpha - \frac{1}{3} + 0(\alpha^{-1})$ (as $\alpha \to \infty$). The relationship (1) was discovered by Pearson (1895) who gave an empirical explanation in the case of his Type III distributions. By considering a distribution as a convolution of n independent and identically distributed (i.i.d.) components, Haldane (1942) was able to provide a more satisfactory explanation. Let $X_1, X_2, \cdots$ be i.i.d. random variables with $E(X) = 0$, $E(X^2) = 1$ and $E(X^3) = \tau$ (assumed to exist), and set $S_n = \Sigma_1^n X_j$, $M_n = \text{mode}\,(S_n)$ and $m_n = \text{median}\,(S_n)$, assuming that these quantities are uniquely defined. Haldane showed that $M_n \to -\frac{1}{2}\tau$ and $m_n \to -\frac{1}{6}\tau$ as $n \to \infty$. In his investigation he makes a number of stringent assumptions, the most severe being that S_n have a density which admits a convergent Edgeworth expansion. Haldane acknowledged that his formulae might be inappropriate when these conditions are violated.

Our aim in this paper is to examine the asymptotic behavior of M_n and m_n under general moment conditions (assuming only that $E(X^2) < \infty$), and without regard to the convergence of Edgeworth expansions. As a corollary we obtain Haldane's results under very weak assumptions, and show that the formula (1) cannot be expected to hold even approximately if the third moment is infinite. Our proofs are considerably different from Haldane's and are based on characteristic function techniques.

The behaviour of the mode is considered in Section 2, and the median in Section 3. We also examine the rate of convergence of M_n to $-\frac{1}{2}\tau$ and m_n to $-\frac{1}{6}\tau$.

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Throughout this paper, $X, X_1, X_2, \cdots$ denote i.i.d. variables with distribution function F and characteristic function f, and moments $E(X) = 0$ and $E(X^2) = 1$. We set $S_n = \Sigma_1^n X_j$. We will sometimes suppose that for all sufficiently large n, S_n is absolutely continuous with density function p_n. The mode and median of S_n are denoted by M_n and m_n, respectively, whenever they exist. If they are not uniquely defined, M_n and m_n stand for any of the possible values, with the provisor that M_n is a "largest mode" of S_n—

$$p_n(M_n) = \sup_x p_n(x).$$

2. The limit behaviour of the mode. Suppose that p_n exists and has an integrable derivative for all sufficiently large n—say $n \geqslant n_0$. Then $|f(t)|^{n_0} = o(|t|^{-1})$ as $|t| \to \infty$ (Lemma 4, page 514, Feller (1971)) and so

$$\int_{-\infty}^{\infty} |f(t)|^n \, dt < \infty$$

for $n \geqslant 2n_o$. Therefore the density of $S_n/n^{\frac{1}{2}}$ converges uniformly to the standard normal density (see Feller's Theorem 2, page 516 and his ensuing remarks), and so $M_n/n^{\frac{1}{2}} \to 0$ as $n \to \infty$.

Our first result describes the asymptotic behaviour of M_n when $E|X|^{2.5} < \infty$.

THEOREM 1. *Suppose that $E|X|^{2.5} < \infty$, and p_n' exists and is integrable for all $n \geqslant n_o$. Then*

$$M_n(1 + o(1)) = -(2\pi)^{-\frac{1}{2}} n \int_{-\infty}^{\infty} E\left[X\left(1 - \cos\left(tX/n^{\frac{1}{2}}\right)\right)\right] e^{-\frac{1}{2}t^2} \, dt + o(1)$$

as $n \to \infty$.

The problem of describing the behaviour of M_n under more general moment conditions is complicated by the possibility of symmetry. For example, if X has a unimodal symmetric distribution then $M_n = 0$ for all n, irrespective of whether or not moments are finite. The case of a skew distribution is more interesting, and is considered next.

THEOREM 2. *Suppose that p_n' exists and is integrable for all $n \geqslant n_o$. If*

$$x^3[1 - F(x) + F(-x)] \to \infty \text{ as } x \to \infty, \tag{2}$$

$$\liminf_{x \to \infty} |[1 - F(x)]/[1 - F(x) + F(-x)] - \tfrac{1}{2}| > 0 \text{ and} \tag{3}$$

$$[1 - F(x(1 + \varepsilon)) + F(-x(1 + \varepsilon))]/[1 - F(x) + F(-x)] \to 1 \text{ as } x \to \infty \tag{4}$$

for any function $\varepsilon = \varepsilon(x) \to 0$, then $|M_n| \to \infty$ and

$$M_n \sim -(2\pi)^{-\frac{1}{2}} n \int_{-\infty}^{\infty} E\left[X\left(1 - \cos\left(tX/n^{\frac{1}{2}}\right)\right)\right] e^{-\frac{1}{2}t^2} \, dt$$

$$= -(2\pi)^{-\frac{1}{2}} n^{\frac{3}{2}} \int_0^{\infty} \left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right] dx \int_{-\infty}^{\infty} t^2(1 - \cos tx) e^{-\frac{1}{2}t^2} \, dt$$

as $n \to \infty$.

Condition (2) implies that $E|X|^3 = \infty$. Condition (3) asks that the tails of X not be symmetrically balanced, and is a little weaker than the more familiar balancing condition,

$$[1 - F(x)]/[1 - F(x) + F(-x)] \to p, \qquad 0 \leqslant p \leqslant 1, \quad p \neq \tfrac{1}{2}. \tag{5}$$

Condition (4) asserts that the tails of X are ultimately smooth, and is satisfied by many distributions such as those with regularly varying tails, which we consider next.

COROLLARY 1. *Suppose that p_n' exists and is integrable for all $n \geqslant n_0$, and (5) holds with $p \neq 0$, and that we can write $1 - F(x) = x^{-\alpha}L(x)$ where L is slowly varying at ∞ and $2 \leqslant \alpha \leqslant 3$. If $2 \leqslant \alpha < 3$ then*

$$M_n \sim C_\alpha(p^{-1} - 2)n^{\frac{3}{2}}\left[1 - F(n^{\frac{1}{2}})\right] \qquad \textit{where}$$

$$C_\alpha = (2\pi)^{-\frac{1}{2}}\textstyle\int_{-\infty}^{\infty} t^2 e^{-\frac{1}{2}t^2}\,dt \int_0^\infty x^{-\alpha}(1 - \cos tx)\,dx$$

$$= -(2^{\alpha+1}/\pi)^{\frac{1}{2}}\sin\left(\tfrac{1}{2}\alpha\pi\right)\Gamma(1 - \alpha)\Gamma\left(1 + \tfrac{1}{2}\alpha\right) > 0,$$

for $\alpha \neq 2$, and $C_2 = (2\pi)^{\frac{1}{2}}$. If $\alpha = 3$ and $E|X|^3 = \infty$ then

$$M_n \sim \tfrac{3}{2}(p^{-1} - 2)\textstyle\int_0^{n^{\frac{1}{2}}} x^2[1 - F(x)]\,dx.$$

PROOF. When $2 \leqslant \alpha < 3$ the conditions of Theorem 2 are satisfied, and by Theorems 2.6 and 2.7 of Seneta (1976),

$$\textstyle\int_0^\infty\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right]dx \int_{-\infty}^{\infty} t^2(1 - \cos tx)e^{-\frac{1}{2}t^2}\,dt$$

$$\sim (2 - p^{-1})\left(1 - F(n^{\frac{1}{2}})\right)\textstyle\int_0^\infty x^{-\alpha}\,dx \int_{-\infty}^{\infty} t^2(1 - \cos tx)e^{-\frac{1}{2}t^2}\,dt,$$

completing the proof. The case $\alpha = 3$ is handled using Theorem 1, as follows. For sequences $\{a_n\}$ and $\{b_n\}$ we have

$$n\textstyle\int_{-\infty}^{\infty} E\left[X\left(1 - \cos\left(tX/n^{\frac{1}{2}}\right)\right)\right]e^{-\frac{1}{2}t^2}\,dt$$

$$= n^{\frac{3}{2}}\left[\textstyle\int_0^1 + \int_1^\infty\right]\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right]dx \int_{-\infty}^{\infty} t^2(1 - \cos tx)e^{-\frac{1}{2}t^2}\,dt$$

$$= n\textstyle\int_0^{n^{\frac{1}{2}}}[1 - F(x) - F(-x)]\,dx \int_{-\infty}^{\infty} t^2\left(1 - \cos\left(tx/n^{\frac{1}{2}}\right)\right)e^{-\frac{1}{2}t^2}\,dt + a_n$$

$$= \tfrac{1}{2}\textstyle\int_0^{n^{\frac{1}{2}}} x^2[1 - F(x) - F(-x)]\,dx \int_{-\infty}^{\infty} t^4 e^{-\frac{1}{2}t^2}\,dt + a_n + b_n.$$

Here

$$(2\pi)^{-\frac{1}{2}}|a_n| \leqslant n^{\frac{3}{2}}\textstyle\int_1^\infty\left[1 - F(n^{\frac{1}{2}}x) + F(-n^{\frac{1}{2}}x)\right]dx \sim n^{\frac{3}{2}}\left[1 - F(n^{\frac{1}{2}})\right]p^{-1}\int_1^\infty x^{-3}\,dx$$

$$= o\left(\textstyle\int_0^{n^{\frac{1}{2}}} x^2[1 - F(x)]\,dx\right),$$

using Seneta's Theorems 2.6 and 2.7, and Theorem 1, page 281, of Feller (1971), and for a finite constant C,

$$|b_n| \leqslant (1/24n)\int_0^{n^{\frac{1}{2}}} x^4[1 - F(x) + F(-x)]\,dx \int_{-\infty}^{\infty} t^6 e^{-\frac{1}{2}t^2}\,dt$$
$$\leqslant Cn^{-1}\int_0^{n^{\frac{1}{2}}} x^4[1 - F(x)]\,dx \sim \tfrac{1}{2}Cn^{\frac{3}{2}}\left[1 - F(n^{\frac{1}{2}})\right].$$

Since $E|X|^3 = \infty$ then

$$\int_0^{n^{\frac{1}{2}}} x^2[1 - F(x) - F(-x)]\,dx \sim (2 - p^{-1})\int_0^{n^{\frac{1}{2}}} x^2[1 - F(x)]\,dx,$$

and the result now follows from Theorem 1.

From Theorem 1 we easily obtain Haldane's result under considerably weaker conditions.

COROLLARY 2. *Suppose that $E|X|^3 < \infty$, $E(X^3) = \tau$ and p_n' exists and is integrable for all $n \geqslant n_o$. Then $M_n \to -\frac{1}{2}\tau$ as $n \to \infty$.*

PROOF. Since $1 - \cos x = \frac{1}{2}x^2 + r(x)$ where $|r(x)| \leqslant \min(x^2, x^4)$, then

$$E\left|X\left[1 - \cos(tX/n^{\frac{1}{2}}) - \tfrac{1}{2}(tX/n^{\frac{1}{2}})^2\right]\right| = E\left|Xr(tX/n^{\frac{1}{2}})\right|$$
$$\leqslant n^{-2}t^4E\left[|X|^5 I(|X| \leqslant n^{\frac{1}{4}})\right] + n^{-1}t^2E\left[|X|^3 I(|X| > n^{\frac{1}{4}})\right].$$

Therefore, for a constant C,

$$|M_n(1 + o(1)) + \tfrac{1}{2}\tau + o(1)| \leqslant C\left\{n^{-\frac{1}{2}}E|X|^3 + E\left[|X|^3 I(|X| > n^{\frac{1}{4}})\right]\right\} = o(1),$$

completing the proof.

As our final result in this section we obtain some information on the rate of convergence of M_n.

THEOREM 3. *Suppose that $E|X|^{3+\delta} < \infty$ for some $0 < \delta < 2$, $E(X^3) = \tau$, and p_n' exists and is integrable for $n \geqslant n_o$. Then $M_n = -\frac{1}{2}\tau + o(n^{-\delta/2})$. If $E|X|^5 < \infty$ then*

$$M_n = -\tfrac{1}{2}\tau + \tfrac{1}{n}\left(\tfrac{1}{8}\kappa_5 - \tfrac{5}{12}\tau\kappa_4 + \tfrac{1}{4}\tau^3\right) + o(n^{-1}), \tag{6}$$

where κ_4 and κ_5 are respectively the 4th *and* 5th *cumulants of X.*

The remainder of this section is devoted to the proofs of Theorems 1, 2 and 3.

PROOF OF THEOREM 1. By the Fourier inversion theorem,

$$p_n(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-itx}f(t)^n\,dt$$

for $n \geqslant 2n_o$. If $n \geqslant 3n_o$ it is permissible to differentiate under the integral sign, so that

$$p_n'(x) = -i(2\pi)^{-1}\int_{-\infty}^{\infty} te^{-itx}f(t)^n\,dt.$$

Write $f(t/n^{\frac{1}{2}})^n = A_n(t) + iB_n(t)$ where A_n and B_n are real valued functions. Setting $p_n'(x) = 0$ and taking real parts we deduce that

$$(7) \qquad \int_{-\infty}^{\infty} t \sin\left(tM_n/n^{\frac{1}{2}}\right)A_n(t)\, dt = \int_{-\infty}^{\infty} t \cos\left(tM_n/n^{\frac{1}{2}}\right)B_n(t)\, dt,$$

and so

$$(8) \quad \left(M_n/n^{\frac{1}{2}}\right)\int_{-\infty}^{\infty} t^2 A_n(t)\, dt + a_n\left(M_n/n^{\frac{1}{2}}\right)^3 \int_{-\infty}^{\infty} t^4 |A_n(t)|\, dt$$
$$= \int_{-\infty}^{\infty} tB_n(t)\, dt + b_n\left(M_n/n^{\frac{1}{2}}\right)^2 \int_{-\infty}^{\infty} |t^3 B_n(t)|\, dt,$$

where $|a_n|$ and $|b_n| \leqslant 1$. We can write $f(t) = \exp[-\frac{1}{2}t^2(1 + \gamma(t))]$ where $\gamma(t) \to 0$ as $t \to 0$, and

$$(9) \qquad \gamma(t) = -2t^{-2}\left[\log f(t) + \tfrac{1}{2}t^2\right] = -2t^{-2}\left[f(t) - 1 + \tfrac{1}{2}t^2\right] + 0(t^2)$$

as $t \to 0$. The Riemann-Lebesgue lemma implies that $|f(t)^{n_0}| \to 0$ as $|t| \to \infty$, and so we may choose a sequence c_n such that

$$(10) \qquad c_n \downarrow 0,\ c_n n^{\frac{1}{4}} \to \infty \text{ and } \sup_{|t|>c_n} |f(t)|^n = O(n^{-6}).$$

Let $d_n = c_n n^{\frac{1}{2}}$. Now,

$$\int_{\{|t|>d_n\}} t^4 |A_n(t)|\, dt \leqslant \int_{\{|t|>d_n\}} t^4 |f(t/n^{\frac{1}{2}})|^n\, dt$$
$$\leqslant \left(\sup_{|t|>c_n} |f(t)|\right)^{n-6n_0} n^{\frac{5}{2}} \int_{-\infty}^{\infty} t^4 |f(t)|^{6n_0}\, dt = O(n^{-2}), \text{ and}$$
$$\int_{-d_n}^{d_n} t^4 |A_n(t)|\, dt \leqslant \int_{-d_n}^{d_n} t^4 \left|\exp\left[-\tfrac{1}{2}t^2\left(1 + \gamma\left(t/n^{\frac{1}{2}}\right)\right)\right]\right|\, dt$$
$$\to \int_{-\infty}^{\infty} t^4 e^{-\frac{1}{2}t^2}\, dt.$$

From these and similar results we deduce that the integrals in (8) are bounded uniformly in $n \geqslant 6n_0$, and that (8) implies

$$(11) \qquad \left(M_n/n^{\frac{1}{2}}\right)(1 + o(1))(2\pi)^{\frac{1}{2}} = \int_{-d_n}^{d_n} tB_n(t)\, dt + 0(n^{-2}).$$

Since $|e^z - 1 - z| \leqslant |z|^2 e^{|z|}$ then

$$B_n(t) = \operatorname{Im} \exp\left[-\tfrac{1}{2}t^2\left(1 + \gamma\left(t/n^{\frac{1}{2}}\right)\right)\right]$$
$$= -\tfrac{1}{2}t^2 e^{-\frac{1}{2}t^2} \operatorname{Im} \gamma\left(t/n^{\frac{1}{2}}\right) + a_n(t) t^4 |\gamma\left(t/n^{\frac{1}{2}}\right)|^2 \exp\left[-\tfrac{1}{2}t^2\left(1 - |\gamma\left(t/n^{\frac{1}{2}}\right)|\right)\right]$$

where $|a_n(t)| \leqslant 1$ for all n and t. If $E|X|^{2.5} < \infty$ then $\gamma(t) = o(|t|^{\frac{1}{2}})$ as $t \to 0$, and so

$$\int_{-d_n}^{d_n} |t|^5 |\gamma\left(t/n^{\frac{1}{2}}\right)|^2 \exp\left[-\tfrac{1}{2}t^2\left(1 - |\gamma\left(t/n^{\frac{1}{2}}\right)|\right)\right] dt = o(n^{-\frac{1}{2}}).$$

The expansion (9) implies that

$$(12) \qquad \operatorname{Im} \gamma\left(t/n^{\frac{1}{2}}\right) = -2nt^{-2}E\left[\sin\left(tX/n^{\frac{1}{2}}\right)\right] + t^2 0(n^{-1})$$

uniformly in $|t| \leqslant d_n$, and substituting the results above into (11),

$$\left(M_n/n^{\frac{1}{2}}\right)(1+o(1))(2\pi)^{\frac{1}{2}} = n\int_{-d_n}^{d_n} tE\left[\sin\left(tX/n^{\frac{1}{2}}\right)\right]e^{-\frac{1}{2}t^2}\,dt + o\left(n^{-\frac{1}{2}}\right)$$
$$= n\int_{-\infty}^{\infty} tE\left[\sin\left(tX/n^{\frac{1}{2}}\right) - \left(tX/n^{\frac{1}{2}}\right)\right]e^{-\frac{1}{2}t^2}\,dt + o\left(n^{-\frac{1}{2}}\right).$$

Theorem 1 follows on integrating by parts.

Proof of Theorem 2. In the notation above let $\gamma(t) = \alpha(t) + i\beta(t)$ where α and β are real valued functions. Then

$$B_n(t) = \operatorname{Im}\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)+i\beta\left(t/n^{\frac{1}{2}}\right)\right)\right]$$
$$= -\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)\right)\right]\sin\left[\tfrac{1}{2}t^2\beta\left(t/n^{\frac{1}{2}}\right)\right]$$
$$= \left[-\tfrac{1}{2}t^2\beta\left(t/n^{\frac{1}{2}}\right) + a_n(t)t^6\beta\left(t/n^{\frac{1}{2}}\right)^3\right]\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)\right)\right]$$

where $|a_n(t)| \leqslant 1$ for all n and t. Now from (11) we see that

$$\left(M_n/n^{\frac{1}{2}}\right)(1+o(1))(2\pi)^{\frac{1}{2}} = -\tfrac{1}{2}\int_{-d_n}^{d_n} t^3\beta\left(t/n^{\frac{1}{2}}\right)\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)\right)\right]dt$$
$$+\Delta_n\int_{-d_n}^{d_n}\left|t^7\beta\left(t/n^{\frac{1}{2}}\right)\right|\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)\right)\right]dt + 0(n^{-2}),$$

where $\Delta_n \to 0$. Furthermore,

$$E\left[\sin\left(tX/n^{\frac{1}{2}}\right)\right] = E\left[\sin\left(tX/n^{\frac{1}{2}}\right) - \left(tX/n^{\frac{1}{2}}\right)\right]$$
$$= -\int_0^\infty\left[\sin\left(tx/n^{\frac{1}{2}}\right) - \left(tx/n^{\frac{1}{2}}\right)\right]d\left[1 - F(x) - F(-x)\right]$$
$$= -t\int_0^\infty(1-\cos tx)\left[1 - F\left(n^{\frac{1}{2}}x\right) - F\left(-n^{\frac{1}{2}}x\right)\right]dx.$$

Combining these results and the expansion (12),

(13)

$$M_n(1+o(1))(2\pi)^{\frac{1}{2}} = -n^{\frac{3}{2}}\int_0^\infty\left[1 - F\left(n^{\frac{1}{2}}x\right) - F\left(-n^{\frac{1}{2}}x\right)\right]dx$$
$$\times\int_{-d_n}^{d_n} t^2(1-\cos tx)\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)\right)\right]dt$$
$$+\varepsilon_n n^{\frac{3}{2}}\int_0^\infty\left|1 - F\left(n^{\frac{1}{2}}x\right) - F\left(-n^{\frac{1}{2}}x\right)\right|dx\int_{-d_n}^{d_n} t^6(1-\cos tx)$$
$$\times\exp\left[-\tfrac{1}{2}t^2\left(1+\alpha\left(t/n^{\frac{1}{2}}\right)\right)\right]dt + 0\left(n^{-\frac{1}{2}}\right)$$
$$= I_n + J_n + 0\left(n^{-\frac{1}{2}}\right),$$

say, where $\varepsilon_n \to 0$. At this point we use condition (3), and note that in view of (4) we may suppose for definiteness that

$$\liminf_{x\to\infty}\left[1 - F(x)\right]/\left[1 - F(x) + F(-x)\right] = \tfrac{1}{2} + \delta$$

where $\delta > 0$. Then there exists an x_0 such that for all $x > x_0$,

$$[1 - F(x)]/[1 - F(x) + F(-x)] > \tfrac{1}{2}(1 + \delta).$$

Therefore for large n,

$$\tfrac{1}{2}n^{\frac{3}{2}}\int_0^\infty\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right] dx\int_{-d_n}^{d_n} t^2(1 - \cos tx)$$
$$\times\exp\left[-\tfrac{1}{2}t^2\left(1 + \alpha\left(t/n^{\frac{1}{2}}\right)\right)\right] dt \leqslant n^{\frac{3}{2}}\delta(1 + \delta)^{-1}\int_{x_0/n^{\frac{1}{2}}}^\infty\left[1 - F(n^{\frac{1}{2}})\right] dx$$
$$\times \int_{-d_n}^{d_n} t^2(1 - \cos tx) \exp\left[-\tfrac{1}{2}t^2(1 + \alpha_n)\right] dt + \tfrac{1}{2}n\int_0^{x_0}[1 - F(x) - F(-x)] dx$$
$$\times \int_{-d_n}^{d_n} t^2\left(1 - \cos\left(tx/n^{\frac{1}{2}}\right)\right) \exp\left[-\tfrac{1}{2}t^2\left(1 + \alpha\left(t/n^{\frac{1}{2}}\right)\right)\right] dt$$
$$\geqslant n^{\frac{3}{2}}\delta(1 + \delta)^{-1}\left[1 - F(n^{\frac{1}{2}})\right]\int_{\frac{1}{2}}^1 dx \int_{-d_n}^{d_n} t^2(1 - \cos tx) \exp\left[-\tfrac{1}{2}t^2(1 + \alpha_n)\right] dt$$
$$- \int_0^{x_0}x^2\, dx \int_{-\infty}^\infty t^4 \exp\left[-\tfrac{1}{2}t^2(1 + \beta_n)\right] dt \geqslant Cn^{\frac{3}{2}}\left[1 - F(n^{\frac{1}{2}})\right],$$

where the constants α_n and $\beta_n \to 0$, and $C > 0$. Condition (2) now implies that $I_n \to \infty$. Using a similar argument we deduce that for $\alpha_n \to 0$.

$$- n^{-\frac{3}{2}}I_n = (1 + o(1))\int_{x_0/n^{\frac{1}{2}}}^\infty\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right] dx \int_{-d_n}^{d_n} t^2(1 - \cos tx)$$
$$\times\exp\left[-\tfrac{1}{2}t^2\left(1 + \alpha\left(t/n^{\frac{1}{2}}\right)\right)\right] dt$$
$$\leqslant (1 + o(1))\int_{x_0/n^{\frac{1}{2}}}^\infty\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right] dx \int_{-\infty}^\infty t^2(1 - \cos tx)e^{-\frac{1}{2}t^2(1-\alpha_n)}\, dt$$
$$= (1 + o(1))\int_{x_0(1-\alpha_n)^{-\frac{1}{2}}n^{-\frac{1}{2}}}^\infty\left[1 - F\left(n^{\frac{1}{2}}(1 - \alpha_n)^{\frac{1}{2}}y\right) - F\left(-n^{\frac{1}{2}}(1 - \alpha_n)^{\frac{1}{2}}y\right)\right] dy$$
$$\times \int_{-\infty}^\infty u^2(1 - \cos uy)e^{-\frac{1}{2}u^2}\, du$$
$$= (1 + o(1))\int_0^\infty\left[1 - F\left(n^{\frac{1}{2}}y\right) - F\left(-n^{\frac{1}{2}}y\right)\right] dy \int_{-\infty}^\infty u^2(1 - \cos uy)e^{-\frac{1}{2}u^2},\, du$$

using (4). The reverse inequality can be established similarly, and so

$$I_n \sim -n^{\frac{3}{2}}\int_0^\infty\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right] dx\int_{-\infty}^\infty t^2(1 - \cos tx)e^{-\frac{1}{2}t^2}\, dt.$$

To handle the term J_n in (13) we note that for any $\varepsilon > 0$, $\sup_t t^4e^{-\frac{1}{2}\varepsilon t^2} \geqslant 16\varepsilon^{-2}$. Let $\varepsilon = \varepsilon_n^{\frac{1}{3}}$, and $\beta_n = \sup_{|t|\leqslant c_n}|\alpha(t)| + \varepsilon_n^{\frac{1}{3}} \to 0$. Then

$$J_n \leqslant 16\varepsilon_n^{\frac{1}{3}}n^{\frac{3}{2}}\int_0^\infty|1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)|\, dx \int_{-\infty}^\infty t^2(1 - \cos tx)e^{-\frac{1}{2}t^2(1-\beta_n)}\, dt$$
$$= \varepsilon_n^{\frac{1}{3}}0(|I_n|).$$

Theorem 2 now follows from (13).

Proof of Theorem 3. The proof is conducted using the techniques above, and expansions of $f(t)$. For example, if $E|X|^5 < \infty$ then

$$f(t) = \exp\left[-\tfrac{1}{2}t^2 - \tfrac{1}{6}i\tau t^3 + \tfrac{1}{24}\kappa_4t^4 + \tfrac{1}{120}i\kappa_5t^5 + o(|t|^5)\right]$$

as $t \to 0$, and so

$$A_n(t) = e^{-\frac{1}{2}t^2}\left[1 + n^{-1}\left(\tfrac{1}{24}\kappa_4 t^4 - \tfrac{1}{72}\tau^2 t^6\right)\right] + n^{-1}(1 + t^{10})\alpha_n(t)e^{-\frac{1}{2}t^2(1-\varepsilon_n)}$$

and

$$B_n(t) = e^{-\frac{1}{2}t^2}\left[-\tfrac{1}{6}\tau n^{-\frac{1}{2}}t^3 + n^{-\frac{3}{2}}\left(\tfrac{1}{120}\kappa_5 t^5 - \tfrac{1}{144}\tau\kappa_4 t^7 + \tfrac{1}{1296}\tau^3 t^9\right)\right]$$
$$+ n^{-\frac{3}{2}}(1 + |t|^{15})\beta_n(t)e^{-\frac{1}{2}t^2(1-\varepsilon_n)},$$

where the functions α_n and $\beta_n \to 0$ uniformly in $|t| \leqslant d_n$, and the constants $\varepsilon_n \to 0$. Substituting into (7) we see that

$$M_n \int_{-\infty}^{\infty} t^2\left(1 - \tfrac{1}{24}n^{-1}\tau^2 t^2\right)\left[1 + n^{-1}\left(\tfrac{1}{24}\kappa_4 t^4 - \tfrac{1}{72}\tau^2 t^6\right)\right]e^{-\frac{1}{2}t^2}\,dt$$
$$= \int_{-\infty}^{\infty} t\left(1 - \tfrac{1}{8}n^{-1}\tau^2 t^2\right)\left[-\tfrac{1}{6}\tau t^3 + n^{-1}\left(\tfrac{1}{120}\kappa_5 t^5 - \tfrac{1}{144}\tau\kappa_4 t^7 + \tfrac{1}{1296}\tau^3 t^9\right)\right]$$
$$\times e^{-\frac{1}{2}t^2}\,dt + o(n^{-1}).$$

Therefore

$$M_n\left[1 - n^{-1}\left(\tfrac{19}{12}\tau^2 - \tfrac{5}{8}\kappa_4\right)\right] = -\tfrac{1}{2}\tau + n^{-1}\left(\tfrac{1}{8}\kappa_5 - \tfrac{35}{48}\tau\kappa_4 + \tfrac{25}{24}\tau^3\right) + o(n^{-1}),$$

and from this follows (6).

3. The limit behaviour of the median. The results and some of the proofs of this section are similar to those of Section 2, and we will keep our discussion brief. It is clear from the central limit theorem that $m_n/n^{\frac{1}{2}} \to 0$ as $n \to \infty$. Our first result generalizes Haldane's in the case of a finite third moment. We will have occasion to use Cramér's continuity condition,

$$\text{(C)} \qquad \limsup_{|t|\to\infty}|f(t)| < 1.$$

THEOREM 4. *If X is nonlattice with $E|X|^3 < \infty$ and $E(X)^3 = \tau$, then $m_n \to -\frac{1}{6}\tau$. If condition* (C) *is satisfied and $E|X|^{3+\delta} < \infty$, $0 < \delta < 2$, then $m_n = -\frac{1}{6}\tau + o(n^{-\frac{1}{2}\delta})$, and if $E|X|^5 < \infty$ then*

$$\text{(14)} \qquad m_n = -\tfrac{1}{6}\tau + n^{-1}\left(\tfrac{1}{40}\kappa_5 - \tfrac{1}{12}\tau\kappa_4 + \tfrac{17}{324}\tau^3\right) + o(n^{-1}).$$

PROOF. If X is nonlattice then from a result of Stone (1965) we see that $\sup_x P(S_n = x) = o(n^{-\frac{1}{2}})$, and a theorem of Esseen (see Gnedenko and Kolmogorov (1954), Theorem 2, page 210) implies that

$$P(S_n \leqslant m_n) - \Phi\left(m_n/n^{\frac{1}{2}}\right) = (2\pi n)^{-\frac{1}{2}}\tfrac{1}{6}\tau + o\left(n^{-\frac{1}{2}}\right),$$

where Φ is the standard normal distribution function. It follows that $m_n \to -\frac{1}{6}\tau$. The case $E|X|^{3+\delta} < \infty$ is handled using a more extensive expansion such as that in Theorem 2, page 168, of Petrov (1975). For example, if $E|X|^5 < \infty$ then writing

$r_n = m_n/n^{\frac{1}{2}}$ we deduce that

$$P(S_n \leq m_n) - \Phi(r_n) + (2\pi n)^{-\frac{1}{2}}e^{-\frac{1}{2}r_n^2}\Big[\tfrac{1}{6}\tau H_2(r_n) + \tfrac{1}{24}n^{-\frac{1}{2}}\kappa_4 H_3(r_n) + \tfrac{1}{120}n^{-1}\kappa_5 H_4(r_n) + \tfrac{1}{72}n^{-\frac{1}{2}}\tau^2 H_5(r_n) + \tfrac{1}{144}n^{-1}\tau\kappa_4 H_6(r_n) + \tfrac{1}{1296}n^{-1}\tau^3 H_8(r_n)\Big] = o(n^{-\frac{3}{2}}),$$

where H_k, $k \geq 0$, are the Hermite polynomials. Now

$$\text{(15)} \qquad \sup_x P(S_n = x) \leq \Sigma_x P(S_n = x) = \lim_{T\to\infty} \tfrac{1}{2}T^{-1}\textstyle\int_{-T}^{T}|f(t)|^{2n}\,dt \leq \left(\limsup_{|t|\to\infty}|f(t)|^{2n}\right) = 0(e^{-\varepsilon n})$$

for some $\varepsilon > 0$. Therefore $P(S_n \leq m_n) = \frac{1}{2} + 0(e^{-\varepsilon n})$. Since

$$\Phi(x) = \tfrac{1}{2} + (2\pi)^{-\frac{1}{2}}\left(x - \tfrac{1}{6}x^3 + 0(x^5)\right)$$

as $x \to 0$, then

$$m_n\left(1 - \tfrac{1}{6}r_n^2\right) = \left(1 - \tfrac{1}{2}r_n^2\right)\Big[\tfrac{1}{6}\tau(r_n^2 - 1) - \tfrac{1}{8}n^{-\frac{1}{2}}\kappa_4 r_n + \tfrac{1}{40}n^{-1}\kappa_5 + \tfrac{5}{24}n^{-\frac{1}{2}}\tau^2 r_n - \tfrac{5}{48}n^{-1}\tau\kappa_4 + \tfrac{35}{432}n^{-1}\tau^3\Big] + o(n^{-1}),$$

and since $r_n = -\frac{1}{6}n^{-\frac{1}{2}}\tau + o(n^{-\frac{1}{2}})$, this expansion implies (14).

In the remainder of this section we consider the behaviour of m_n under less restrictive moment conditions.

THEOREM 5. *If* (C) *holds and* $E|X|^{2.5} < \infty$ *then*

$$m_n(1 + o(1)) = (2\pi)^{-\frac{1}{2}}n^{\frac{3}{2}}\textstyle\int_{-\infty}^{\infty}t^{-1}E\left[\sin\left(tX/n^{\frac{1}{2}}\right)\right]e^{-\frac{1}{2}t^2}\,dt + o(1).$$

THEOREM 6. *If conditions* (2), (3), (4) *and* (C) *hold, then* $|m_n| \to \infty$ *and*

$$m_n \sim (2\pi)^{-\frac{1}{2}}n^{\frac{3}{2}}\textstyle\int_{-\infty}^{\infty}t^{-1}E\left[\sin\left(tX/n^{\frac{1}{2}}\right)\right]e^{-\frac{1}{2}t^2}\,dt$$

$$= -(2\pi)^{-\frac{1}{2}}n^{\frac{3}{2}}\textstyle\int_0^{\infty}\left[1 - F(n^{\frac{1}{2}}x) - F(-n^{\frac{1}{2}}x)\right]dx\,\int_{-\infty}^{\infty}(1 - \cos tx)e^{-\frac{1}{2}t^2}\,dt.$$

COROLLARY 3. *Suppose that* (C) *holds,*

$$[1 - F(x)]/[1 - F(x) + F(-x)] \to p, \quad 0 < p \leq 1, p \neq \tfrac{1}{2},$$

as $x \to \infty$, *and* $1 - F(x) = x^{-\alpha}L(x)$ *where* L *is slowly varying at* ∞ *and* $2 \leq \alpha \leq 3$.

If $2 \leqslant \alpha < 3$ *then*

$$m_n \sim D_\alpha(p^{-1} - 2)n^{\frac{3}{2}}\left[1 - F(n^{\frac{1}{2}})\right]$$

where

$$D_\alpha = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\, dt \int_0^\infty x^{-\alpha}(1 - \cos tx)\, dx$$

$$= -(2^{\alpha-1}/\pi)^{\frac{1}{2}} \sin(\tfrac{1}{2}\alpha\pi)\Gamma(1-\alpha)\Gamma(\tfrac{1}{2}\alpha) > 0,$$

for $a \neq 2$, *and* $D_2 = (\pi/2)^{\frac{1}{2}}$. *If* $\alpha = 3$ *and* $E|X|^3 = \infty$ *then*

$$m_n \sim \tfrac{1}{2}(p^{-1} - 2)\int_0^n x^2[1 - F(x)]\, dx.$$

REMARKS. Comparing Corollaries 1 and 3 we see that (1) holds asymptotically if $\alpha = 3$, but not if $2 \leqslant \alpha < 3$. As a general rule, (1) cannot be expected to hold if $E|X|^3 = \infty$. Since $C_\alpha / D_\alpha = \alpha$ then for a distribution with regularly varying tails of exponent α, $2 \leqslant \alpha \leqslant 3$, we have instead of (1) that

$$\text{mean-mode} \sim \alpha(\text{mean-median}).$$

PROOF OF THEOREMS. In view of (15) we have from Gil-Pelaez' (1951) inversion theorem that

$$P(S_n \leqslant x) = \tfrac{1}{2} + \int_0^\infty (2\pi i t)^{-1}\left[e^{itx}f(-t)^n - e^{-itx}f(t)^n\right] dt + 0(e^{-\varepsilon n})$$

uniformly in x, where the integral exists in the Riemann sense. With Q denoting the concentration function we deduce from Lemma 3, page 38, of Petrov (1976) that for any $\varepsilon > 0$,

$$Q(S_n; e^{-\varepsilon n}) \leqslant Ce^{-\varepsilon n}\int_{-e^{\varepsilon n}}^{e^{\varepsilon n}} |f(t)|^n\, dt \tag{16}$$

$$\leqslant 2C\left\{e^{-\varepsilon n} + \left[\sup_{|t|>1}|f(t)|^n\right]\right\} = 0(e^{-\varepsilon' n})$$

where $0 < \varepsilon' \leqslant \varepsilon$. Suppose $0 < \delta_n \leqslant 1$. We can write

$$P(S_n \leqslant x) = \tfrac{1}{2} + \int_0^{\delta_n} (2\pi i t)^{-1}\left[e^{itx}f(-t)^n - e^{-itx}f(t)^n\right] dt$$

$$+ r_n(x) + 0(e^{-\varepsilon n})$$

where

$$|r_n(x)| \leqslant \int_{\delta_n}^{2^n} t^{-1}|f(t)|^n\, dt + \int_{-\infty}^{\infty}\left|\int_{2^n}^{\infty} \frac{\sin(x-y)t}{t}\, dt\right| dP(S_n \leqslant y). \tag{17}$$

The first term on the right is dominated by

$$\left[\sup_{|t|>\delta_n}|f(t)|\right]^n \log(2^n/\delta_n).$$

Choose $\delta_n \downarrow 0$ so slowly that $[\sup_{|t|>\delta_n}|f(t)|]^n = O(n^{-k})$ for all $k > 0$, $n^{\frac{1}{2}}\delta_n \to \infty$ and $\delta_n > 2^{-n}$. Then the first term on the right in (17) is $O(n^{-k})$ for all $k > 0$. Since

$$\left|\int_a^\infty \frac{\sin tz}{t}\,dt\right| \leqslant \min(1, 1/|az|),$$

then the second is dominated by

$$\begin{aligned} &CP(|S_n - x| \leqslant 2^{-n} \text{ or } > 2^n) \\ &\quad + \left[\Sigma_1^{2^n-1}\int_{j2^{-n}<|y-x|\leqslant\{(j+1)2^{-n}\}} + \Sigma_1^{2^n}\int_{\{j<|y-x|\leqslant j+1\}}\right] \\ &\quad \left|\int_{2^n}^\infty \frac{\sin(x-y)t}{t}\,dt\right| dP(S_n \leqslant y) \leqslant CP(|S_n - x| \leqslant 2^{-n} \text{ or } > 2^n) \\ &\quad + C\Sigma_1^{2^n-1}P(j2^{-n} < |S_n - x| \leqslant (j+1)2^{-n})/2^n \cdot j2^{-n} + C\Sigma_1^{2^n}1/2^n \cdot j \\ &\quad \leqslant 2C(\log 2^n)Q(|S_n - x|;\, 2^{-n}) + C(\log 2^n)2^{-n} + CP(|S_n - x| > 2^n). \end{aligned}$$

The inequality (16) and the fact that $m_n = o(n^{\frac{1}{2}})$ as $n \to \infty$ now imply that $r_n(m_n) = O(n^{-k})$ for all $k > 0$. Therefore for a certain sequence $\delta_n \to 0$,

$$\begin{aligned} P(S_n \leqslant m_n) &= \tfrac{1}{2} + \int_0^{n^{\frac{1}{2}}\delta_n}(2\pi i t)^{-1} \\ &\quad \left[e^{itm_n/n^{\frac{1}{2}}}f\left(-t/n^{\frac{1}{2}}\right)^n - e^{-itm_n/n^{\frac{1}{2}}}f\left(t/n^{\frac{1}{2}}\right)^n\right] dt + O(n^{-k}) \end{aligned}$$

for all $k > 0$. Taking real parts we see that

$$\int_0^{n^{\frac{1}{2}}\delta_n} t^{-1}\sin\left(tm_n/n^{\frac{1}{2}}\right)A_n(t)\,dt = \int_0^{n^{\frac{1}{2}}\delta_n} t^{-1}\cos\left(tm_n/n^{\frac{1}{2}}\right)B_n(t)\,dt + O(n^{-k}).$$

Let $d_n = n^{\frac{1}{2}}\delta_n$. Now,

(18)

$$\int_0^{d_n} t^{-1}\sin\left(tm_n/n^{\frac{1}{2}}\right)A_n(t)\,dt = \left(m_n/n^{\frac{1}{2}}\right)\int_0^{d_n}A_n(t)\,dt + a_n\left(m_n/n^{\frac{1}{2}}\right)^3\int_0^{d_n}t^2|A_n(t)|\,dt$$

where $|a_n| \leqslant 1$, and

$$\int_0^{d_n}A_n(t)\,dt \to \int_0^\infty e^{-\frac{1}{2}t^2}\,dt \quad\text{and}\quad \int_0^{d_n}t^2|A_n(t)|\,dt \to \int_0^\infty t^2e^{-\frac{1}{2}t^2}\,dt.$$

Instead of (1) we now deduce from (18) and a similar result that

$$\left(m_n/n^{\frac{1}{2}}\right)(1 + o(1))(\pi/2)^{\frac{1}{2}} = \int_0^{d_n}t^{-1}B_n(t)\,dt + O(n^{-k}) \tag{19}$$

for all $k > 0$, from which follows Theorem 5.

Theorem 6 and Corollary 3 are proved as before, using (19) in place of (11).

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Department of Statistics, S.G.S.
The Australian National University
P.O. Box 4
Canberra, A.C.T.
Australia