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ON A THEOREM OF MARUYAMA

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It is shown that a wide-sense stationary process is strictly stationary (i) *only if* it is rotation invariant, (ii) *if* it is rotation invariant and has a (two dimensional) random spectral measure with independent increments.

1. Introduction. A real, wide-sense stationary process is not necessarily strictly stationary. However, if the process is strictly stationary and L^2, then it is stationary in the wide sense. The question therefore arises: under what conditions does a stationary in the wide sense process become strictly stationary? In [2] Maruyama gives necessary and sufficient conditions for this when the random spectral measure has independent increments, but these conditions are not quite correct. In what follows this theorem is established without the restrictive assumption on the random spectral measure.

2. Strict and wide-sense stationarity. Let $X(t)$ be a real stationary in the wide sense process which is continuous in mean square, with spectral representation

$$X(t) = \int_0^\infty \cos t\lambda \, d\xi(\lambda) + \int_0^\infty \sin t\lambda \, d\eta(\lambda). \tag{1}$$

For each τ we define the processes $\xi_\tau(\lambda)$ and $\eta_\tau(\lambda)$ by

$$\xi_\tau(\lambda) = \int_0^\lambda \cos \tau u \, d\xi(u) + \int_0^\lambda \sin \tau u \, d\eta(u)$$

and

$$\eta_\tau(\lambda) = -\int_0^\lambda \sin \tau u \, d\xi(u) + \int_0^\lambda \cos \tau u \, d\eta(u).$$

It follows that

$$X(t+\tau) = \int_0^\infty \cos t\lambda \, d\xi_\tau(\lambda) + \int_0^\infty \sin t\lambda \, d\eta_\tau(\lambda). \tag{2}$$

We shall say that $(d\xi(\lambda), d\eta(\lambda))$ is rotation invariant for $0 \leqslant \lambda_1 < \lambda_2$ the distribution of $(\xi_\tau(\lambda_2) - \xi_\tau(\lambda_1), \eta_\tau(\lambda_2) - \eta_\tau(\lambda_1))$ is independent of τ, and let $d\xi_\tau(\lambda)$ denote the process of the λ-increments of $\xi_\tau(\lambda)$. We have the following:

THEOREM. *Let $X(t)$ be as in* (1). *Then*

(i) *$X(t)$ is strictly stationary if and only if the distribution of $(d\xi_\tau(\lambda), d\eta_\tau(\lambda))$ does not depend on τ.*

(ii) *If $X(t)$ is strictly stationary, then $(d\xi(\lambda), d\eta(\lambda))$ is rotation invariant.*

(iii) *If for all $n = 2, 3, \cdots$ and $0 \leqslant \lambda_1 < \lambda_2 \cdots < \lambda_n$ the increments $(\xi(\lambda_{k+1}) - \xi(\lambda_k), \eta(\lambda_{k+1}) - \eta(\lambda_k))$, $k = 1, \cdots, n-1$ are independent, and if $(d\xi(\lambda), d\eta(\lambda))$ is rotation invariant then X is strictly stationary.*

PROOF. The proof of (i) follows by letting $X_\tau(t) = X(t+\tau)$ and noting that X is strictly stationary if and only if X and X_τ have the same distribution for all τ,

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which holds if and only if the distribution of $(d\xi_\tau(\lambda), d\eta_\tau(\lambda))$ does not depend on τ. The last remark follows from (2) and the inversion formulae, which express the increments $\xi_\tau(\lambda_2) - \xi_\tau(\lambda_1)$ at points of continuity λ_1, λ_2 and the jumps $\xi_\tau(\lambda) - \xi_\tau(\lambda -)$ in terms of X (see [1], page 527). Then (ii) is an immediate consequence of (i).

The proof of (iii) follows by writing, for each τ, and $N = 1, 2, \cdots$

$$X_{N,\tau}(t) = \Sigma_{k=1}^{N2^N}\left\{\cos\frac{tk}{2^N}\left(\xi_\tau\left(\frac{k}{2^N}\right) - \xi_\tau\left(\frac{k-1}{2^N}\right)\right) + \times \sin\frac{tk}{2^N}\left(\eta_\tau\left(\frac{k}{2^N}\right) - \eta_\tau\left(\frac{k-1}{2^N}\right)\right)\right\},$$

noting that the finite-dimensional distributions of the $X_{N,\tau}$ process are independent of τ for fixed N, and that $X_\tau(t) = L^2 - \lim_{N\to\infty} X_{N,\tau}(t)$ for each τ. This completes the proof of the theorem.

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REFERENCES

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