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JOINT LIMIT LAWS OF SAMPLE MOMENTS OF A SYMMETRIC DISTRIBUTION[1]

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Let $\{X_n\}$ be a sequence of i.i.d. random variables with a common symmetric distribution F. Let $\mathcal{L}(Z)$ denote the distribution of a random variable Z, and let $\mathfrak{D}(\alpha)$ denote the domain of attraction of a stable law of characteristic exponent α. It is assumed that $\mathcal{L}(X_1^k) \in \mathfrak{D}(\alpha)$ for some integer $k \geqslant 2$ and $\alpha \in (0, 2]$. Let $\mathbf{S}_n$ denote the k-dimensional random vector whose jth coordinate is $\sum_{i=1}^n X_i^j$, and let $m = \max\{j : k\alpha/j \geqslant 2\}$. Then there exist a sequence of $k \times k$ matrices $\{A_n\}$ and a sequence of vectors $\{\mathbf{b}_n\}$ in $\mathbf{R}^k$ such that $A_n\mathbf{S}_n + \mathbf{b}_n$ converges in law to a random vector $\mathbf{S}$. The first m coordinates of $\mathbf{S}$ are jointly normal and are independent of the remaining $k - m$ coordinates. No pair of these remaining $k - m$ coordinates are independent, but their joint distribution is operator-stable with two orbits.

1. Introduction and summary. The multivariate central limit theorem offers a number of interesting problems that do not occur in the univariate case. One such problem is that of independent limit coordinates. The general theorem on asymptotic independence does not contain much depth; see Theorem 2 in [8]. Asymptotic independence becomes more interesting in special examples, some of which are treated in [2], [5], [8], [9] and [13]. What shall be dealt with here is essentially another example which in a sense extends Theorem 3 in [8]. Namely, from a sequence of independent, identically distributed random variables $\{X_n\}$, the problems considered here are those of existence of a joint limit law and independence of subsets of coordinates of this limit law of the sequence of a fixed number of sample moments.

The notation that is special follows. The symbol $\mathfrak{D}(\alpha)$ will denote the domain of attraction of some stable distribution with characteristic exponent $\alpha \in (0, 2]$, and the domain of normal attraction will be denoted by $\mathfrak{D}_\eta(\alpha)$. If X is a random variable with distribution function F, the relation $X \in \mathfrak{D}(\alpha)$ (or $\mathfrak{D}_\eta(\alpha)$) will mean $F \in \mathfrak{D}(\alpha)$ (or $\mathfrak{D}_\eta(\alpha)$). Some distinction should be made between a *coordinate of a random vector* $\mathbf{Z}$ and a component of the joint distribution function F of $\mathbf{Z}$; namely, in the latter case, a distribution function G is a component of F if there exists a distribution function H such that $F = G * H$. Following M. Sharpe [11] we shall say that the common distribution function of a sequence of k-dimensional independent, identically distributed random vectors $\{\mathbf{Z}_n\}$ is in the domain of attraction of

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a distribution function F, called *operator-stable*, if there exist a sequence of $k \times k$ matrices $\{B_n\}$ (called normalizing matrices) and a sequence of (centering) vectors $\{\mathbf{b}_n\}$ in $\mathbb{R}^k$ such that the distribution of $B_n \Sigma_{j=1}^n \mathbf{Z}_j + \mathbf{b}_n$ converges completely to F.

The example investigated here deals with a sequence of independent, identically distributed random variables $\{X_n\}$ with common symmetric distribution function F. Letting $\mathbf{Z}_n$ denote the k-dimensional random vector whose jth coordinate is X_n^j, we are concerned about the existence and structure of the operator-stable limit law of $\Sigma_{j=1}^n \mathbf{Z}_j$ under suitable norming and centering. Letting $m = \max\{j : k\alpha/j \geq 2\}$, it will be shown that the limiting distribution G (the distribution function of, say, a random vector $\mathbf{Z}$ whose ith coordinate is Z_i) does exist if $X_1^k \in \mathfrak{D}(\alpha)$ for some $\alpha \in (0,2]$. The first m coordinates are jointly normal. The remaining $k - m$ coordinates have no normal components, there are no independent pairs among them, and each univariate marginal is stable. The two sets of random variables $\{Z_1, \cdots, Z_m\}$ and $\{Z_{m+1}, \cdots, Z_k\}$ are independent of each other. The distribution of the random variable X_1^m may be either in $\mathfrak{D}_\eta(2)$ or in $\mathfrak{D}(2) \setminus D_\eta(2)$. If $X_1^m \in \mathfrak{D}_\eta(2)$, then the three sets of random variables $\{Z_{2i}, 1 \leq i \leq [m/2]\}$, $\{Z_{2i-1}, 1 \leq i \leq [(m+1)/2]\}$ and $\{Z_{m+1}, \cdots, Z_k\}$ are independent. If $X_1^m \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$, then the following four sets of random variables are independent:

$$\{Z_{2i}, 1 \leq i \leq [(2m-1)/2]\}, \quad \{Z_{2i-1}, 1 \leq i \leq [m/2]\}, \quad \{Z_m\},$$

and

$$\{Z_{m+1}, \cdots, Z_k\}.$$

The general form of the multivariate central limit theorem, due to E. L. Rvacheva [10], will be used; it is stated here as Lemma 1 as a convenience for the reader.

LEMMA 1. *Let $\{\{\mathbf{X}_{nj}\}\}$ be an infinitesimal system of row-wise independent p-dimensional random vectors, and define the probability measure H_{nj} over the measurable space $(\mathbb{R}^p, \mathfrak{B}^{(p)})$ by $H_{nj}(A) = P[\mathbf{X}_{nj} \in A]$ for all $A \in \mathfrak{B}^{(p)}$. Then there exists a sequence $\{\mathbf{c}_n\} \subset \mathbb{R}^p$ such that the distribution of $\Sigma_{j=1}^{k_n} \mathbf{X}_{nj} + \mathbf{c}_n$ converges completely to a (necessarily infinitely divisible) distribution function F if and only if there exists a Lévy spectral measure N over the Borel subsets $\mathfrak{B}^{(p)}$ of $\mathbb{R}^p$ and a nonnegative definite quadratic form $Q(\mathbf{u})$ defined over $\mathbb{R}^p$ which satisfy the following*:

(i) *for Borel sets of the form $S = \{\mathbf{x} \in \mathbb{R}^p : |\mathbf{x}| > R, \omega_\mathbf{x} \in A\}$, where A is a Borel subset of the surface of the unit sphere, $\omega_\mathbf{x}$ denotes the point of intersection of the vector $\mathbf{x}$ with the surface of the unit sphere, and such that $N(bdry\ S) = 0$,*

$$\Sigma_{j=1}^{k_n} H_{nj}(S) \to N(S) \quad \text{as } n \to \infty, \quad \text{and}$$

(ii) $\lim_{\varepsilon \downarrow 0} \overline{\lim}_{n\to\infty} \Sigma_{j=1}^{k_n} \{ \int_{\|\mathbf{x}\|<\varepsilon} (\mathbf{u}'\mathbf{x})^2 H_{nj}(d\mathbf{x}) - (\int_{\|\mathbf{x}\|<\varepsilon} \mathbf{u}'\mathbf{x} H_{nj}(d\mathbf{x}))^2 \} = Q(\mathbf{u})$, *where $\mathbf{u}'\mathbf{x} = \Sigma_{i=1}^p u_i x_i$. The characteristic function $\hat{F}(\mathbf{u})$ of $F(\mathbf{x})$ is given by*

$$\hat{F}(\mathbf{u}) = \exp\left\{ i\gamma'\mathbf{u} - Q(\mathbf{u})/2 + \int_{\|\mathbf{x}\|>0} \left(e^{i\mathbf{u}'\mathbf{x}} - 1 - \frac{i\mathbf{u}'\mathbf{x}}{1 + \|\mathbf{x}\|^2} \right) N(d\mathbf{x}) \right\},$$

where $\gamma \in \mathbb{R}^p$ is constant.

2. Multivariate normal limit. Two special cases of the result announced in Section 1 will be given in this and the next section, and the general theorem will be assembled in Section 4.

THEOREM 1. *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables whose common distribution F is symmetric, let $k \geqslant 2$, and let $\mathbf{Z}_n$ be a k-dimensional random vector whose jth coordinate is X_n^j. If $X_1^k \in \mathfrak{D}(2)$, then there exist a sequence of $k \times k$ matrices $\{B_n\}$ and a sequence of vectors $\{\mathbf{b}_n\}$ in $\mathbb{R}^k$ such that the distribution of $B_n \Sigma_{i=1}^n \mathbf{Z}_i + \mathbf{b}_n$ converges to that of a multivariate normal random vector $\mathbf{W} = (W_1 \cdots W_k)^t$, and $\{W_{2i}, 1 \leqslant i \leqslant k/2\}$ is independent of $\{W_{2i-1}, 1 \leqslant i \leqslant (k+1)/2\}$. Moreover, if $X_1^k \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$, then $\{W_{2i}, 1 \leqslant i \leqslant (k-1)/2\}$, $\{W_{2i-1}, 1 \leqslant i \leqslant (k/2)\}$ and $\{W_k\}$ are independent.*

PROOF. Since $X_1^k \in \mathfrak{D}(2)$, then by Theorem 3 in Section 35 of [4], $E(|X_1|^{\delta k}) < \infty$ for all $\delta \in [0, 2)$. Hence, for $1 \leqslant j \leqslant k-1$, $E(X_1^{2j}) < \infty$, which implies that $X_1^j \in \mathfrak{D}_\eta(2)$, and thus $\{n^{\frac{1}{2}}\}$ serve as normalizing coefficients for $\{X_n^j, n \geqslant 1\}$. If $X_1^k \in \mathfrak{D}_\eta(2)$, then these same normalizing coefficients serve $\{X_n^k, n \geqslant 1\}$. If $X_1^k \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$, then by Lemma 5 in [12], normalizing coefficients for $\{X_n^k\}$ must be of the form $\{(n)^{\frac{1}{2}} L(n)\}$, where $L(x)$ is a nondecreasing slowly varying function satisfying $L(x) \uparrow \infty$ as $x \to \infty$. Let us denote

$$L_k(n) = 1 \quad \text{if} \quad X_1^k \in \mathfrak{D}_\eta(2)$$
$$= L(n) \quad \text{if} \quad X_1^k \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2),$$

and

$$B_n = \operatorname{diag}\left(1/n^{\frac{1}{2}}, \cdots, 1/n^{\frac{1}{2}} L_k(n)\right).$$

Clearly, the system $\{\{B_n \mathbf{Z}_1, \cdots, B_n \mathbf{Z}_n\}\}$ is an infinitesimal system. By the univariate central limit theorem,

$$nP\left[|X_1^j| \geqslant n^{\frac{1}{2}} \varepsilon\right] \to 0 \qquad \text{as} \quad n \to \infty$$

for $1 \leqslant j \leqslant k-1$, and $nP[|X_1^k| \geqslant n^{\frac{1}{2}} L_k(n) \varepsilon] \to 0$ as $n \to \infty$. Thus (i) of Lemma 1 is satisfied with $N(\cdot) = 0$. We now show that (ii) of Lemma 1 is satisfied. Let us denote $L_i(n) = 1$ if $i < k$ and $= L_k(n)$ if $i = k$, and, for $1 \leqslant r \leqslant s \leqslant k$, define

$$I(r, s; n, \varepsilon) = n\left\{E\left(\frac{X_1^r}{n^{\frac{1}{2}} L_r(n)} \frac{X_1^s}{n^{\frac{1}{2}} L_s(n)} \cdot I_{[|X_1^r| < n^{\frac{1}{2}} L_r(n)\varepsilon][|X_1^s| < n^{\frac{1}{2}} L_s(n)\varepsilon]}\right)\right.$$
$$\left. - E\left(\frac{X_1^r}{n^{\frac{1}{2}} L_r(n)} I_{[|X_1^r| < n^{\frac{1}{2}} L_r(n)\varepsilon]}\right) E\left(\frac{X_1^s}{n^{\frac{1}{2}} L_s(n)} I_{[|X_1^s| < n^{\frac{1}{2}} L_s(n)\varepsilon]}\right)\right\}.$$

We consider four cases:

CASE (i). $r + s$ is odd. In this case, the symmetry of F implies $I(r, s; n, \varepsilon) = 0$.

CASE (ii). $r + s$ is even and $X_1^k \in \mathfrak{D}_\eta(2)$, or $r + s$ is even, $s < k$ and $X_1^k \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$.
In this case it is easy to verify that

$$\lim_{\varepsilon \downarrow 0} \overline{\underline{\lim}}_{n \to \infty} I(r, s; n, \varepsilon) = E(X_1^{r+s}) - E(X_1^r)E(X_1^s).$$

CASE (iii). $r = s = k$ and $X_1^k \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$. In this case, the univariate central limit theorem implies that $\lim_{\varepsilon \downarrow 0} \overline{\underline{\lim}}_{n \to \infty} I(k, k; n, \varepsilon)$ exists and is positive. Deferring case (iv) momentarily, we see that in cases (i), (ii) and (iii) a multivariate normal limit distribution of $B_n \Sigma_{i=1}^n \mathbf{Z}_i +$ (some) $\mathbf{b}_n$ exists according to Lemma 1, and the first independence conclusion of the theorem is obtained.

CASE (iv). $r + s$ is even, $r < s = k$ and $X_1^k \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$. Since $E|X_1^{r+s}| < \infty$ and $L(n) \uparrow \infty$, it follows that $\lim_{n \to \infty} I(r, s; n, \varepsilon) = 0$. This case establishes the moreover part of the theorem. □

3. Multivariate stable limits. In his development of the theory of operator-stable distributions, M. Sharpe [11] defines an index of such a probability measure or distribution μ over $\mathbb{R}^k$ as an automorphism B on $\mathbb{R}^k$ satisfying $\mu^{*t} = t^B\mu * \delta(b(t))$ for all $t > 0$, where μ^{*t} is defined by its characteristic function $\hat{\mu}^t$ (which in turn is well defined because of the infinite divisibility of μ), where $t^B\mu$ is defined by $t^B\mu(A) = \mu((t^B)^{-1}(A))$ for Borel sets $A \subset \mathbb{R}^k$, and where $\delta(b(t))$ is the unit point mass at some $b(t) \in \mathbb{R}^k$. (Note: in case $k = 1$, Sharpe's index is unique but is the reciprocal of what is usually referred to as the index or characteristic exponent of a stable distribution.) He shows that the support of the Lévy spectral measure M associated with μ is the union of its orbits, i.e., of sets of the form $\{t^B\mathbf{x} : t > 0\}$ for $\mathbf{x} \in \mathbb{R}^k$.

THEOREM 2. *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables whose common distribution F is symmetric, let $k \geq 2$, and let $\mathbf{Z}_n$ be a k-dimensional random vector whose jth coordinate is X_n^j. If $X_1^k \in \mathfrak{D}(\alpha)$, where $\alpha < 2/k$, then there exist a sequence of $k \times k$ matrices $\{B_n\}$ and a sequence of vectors $\{\mathbf{b}_n\}$ in $\mathbb{R}^k$ such that the distribution of $B_n \Sigma_{i=1}^n \mathbf{Z}_i + \mathbf{b}_n$ converges to that of a random vector $\mathbf{Z}$, where $\mathbf{Z}$ is operator-stable with stable marginals with characteristic exponents $k\alpha, k\alpha/2, \cdots, \alpha$. No two coordinates of $\mathbf{Z}$ are independent, and $\mathbf{Z}$ has two orbits.*

PROOF. Since $X_1^k \in \mathfrak{D}(\alpha)$, then there exists a slowly varying function $Q(x)$ such that $\{n^{1/\alpha}Q(n)\}$ serve as normalizing coefficients for the sequence $\{X_n^k\}$. The function $Q(x)$ can be selected to satisfy

$$P\left[|X_1^k| \geq n^{1/\alpha}Q(n)\right] \sim 1/n.$$

Also, the hypothesis $X_1^k \in \mathfrak{D}(\alpha)$ implies that there exists a slowly varying function $R(x)$ such that

$$P\left[|X_1^k| \geq x\right] = x^{-\alpha}R(x).$$

From this it follows that for $1 \leqslant j \leqslant k-1$,

$$P[|X_1^j| \geqslant x] = x^{-k\alpha/j}S(x),$$

where $S(x) = R(x^{k/j})$ is also slowly varying. All the above implies that $X_1^j \in \mathfrak{D}(k\alpha/j)$ for $1 \leqslant j \leqslant k-1$, and $\{n^{1/(k\alpha/j)}Q^{j/k}(n)\}$ serves as a sequence of normalizing coefficients for $\{X_n^j\}$, $1 \leqslant j \leqslant k$. Denote

$$B_n = \operatorname{diag}(n^{-1/k\alpha}Q^{-1/k}(n), \cdots, n^{-1/\alpha}Q^{-1}(n)).$$

We now prove that there is a sequence $\{\mathbf{b}_n\}$ in $\mathbb{R}^k$ and a distribution of some k-dimensional random vector $\mathbf{Z}$ such that

$$B_n \Sigma_{j=1}^n \mathbf{Z}_j + \mathbf{b}_n \to_{\mathfrak{L}} \mathbf{Z}.$$

Let $\Gamma(r,n) = n^{1/(k\alpha/r)}Q^{r/k}(n)$, and denote

$$f(r,s;n,\varepsilon) = n\left\{E\left(\frac{X_1^r}{\Gamma(r,n)}\frac{X_1^s}{\Gamma(s,n)}I_{[|X_1^r|<\Gamma(r,n)\varepsilon][|X_1^s|<\Gamma(s,n)\varepsilon]}\right)\right.$$
$$\left. - E\left(\frac{X_1^r}{\Gamma(r,n)}I_{[|X_1^r|<\Gamma(r,n)\varepsilon]}\right)E\left(\frac{X_1^s}{\Gamma(s,n)}I_{[|X_1^s|<\Gamma(s,n)\varepsilon]}\right)\right\}.$$

Since $X_1^j \in \mathfrak{D}(k\alpha/j)$ and since $k\alpha/j < 2$, we obtain from the univariate central limit theorem that

$$\lim_{\varepsilon\downarrow 0} \overline{\lim}_{n\to\infty} f(j,j;n,\varepsilon) = 0 \qquad \text{for} \quad 1 \leqslant j \leqslant k.$$

By Schwarz's inequality, for $1 \leqslant r \leqslant s \leqslant k$, $f^2(r,s;n,\varepsilon) \leqslant f(r,r;n,\varepsilon)f(s,s;n,\varepsilon)$, and thus (ii) of Lemma 1 is satisfied with $Q(\mathbf{u}) \equiv 0$. We now verify (i) in Lemma 1. Consider $\mathbf{x} \in \mathbb{R}^k$, all of whose coordinates $x_1, \cdots, x_k$ are positive. Then

$$nP \cap_{j=1}^k [X_1^j > x_j\Gamma(j,n)] = nP \cap_{j=1}^k [X_1^k > x_j^{k/j}n^{1/\alpha}Q(n)]$$
$$= nP[X_1^k > \max\{x_i^{k/i} : 1 \leqslant i \leqslant k\}n^{1/\alpha}Q(n)].$$

Now $X_1^k \in \mathfrak{D}(\alpha)$ means there exists a constant $c_1 > 0$ such that

$$nP[X_1^k/n^{1/\alpha}Q(n) \geqslant t] \to c_1/t^\alpha \qquad \text{as} \quad n \to \infty$$

for all $t > 0$. Thus we have

$$\lim nP \cap_{j=1}^k [X_1^j > x_j\Gamma(j,n)]$$
$$= c_1/\left(\max\{x_j^{k/j} : 1 \leqslant j \leqslant k\}\right)^\alpha.$$

Consider the following curve in the orthant where all coordinates are positive: for all $t > 0$,

$$x_1 = t^{1/k}$$
$$x_2 = t^{2/k}$$
$$\vdots$$
$$x_k = t.$$

If, after fixing the value of $x_k = t$, we were to decrease the value of x_j from $t^{j/k}$, then there would be no change in the above limit relation. This shows that the support of the Lévy spectral measure N in this orthant is the curve $x_i = t^{i/k}$, $1 \leqslant i \leqslant k$, for all $t > 0$. Note that the Lévy spectral measure is

$$N\{(t^{1/k}, t^{2/k}, \cdots, t) : t \geqslant s\} = c_1/s^{\alpha}.$$

The only other orthant in which N has nonzero mass is that in which the odd-coordinates are negative and the even ones are positive. Then, by symmetry and the above argument, we have: $(-1)^j x_j > 0$, $1 \leqslant j \leqslant k$, and

$$\lim_{n\to\infty} nP \cap_{j=1}^{k}\left[X_1^j > (-1)^j x_j \Gamma(j,n)\right]$$
$$= c_1/\left(\max\{(-1)^j x_j^{k/j} : 1 \leqslant j \leqslant k\}\right)^{\alpha}.$$

By the same reasoning as above, it is easy to see that in this orthant the Lévy spectral measure N has as its support the curve

$$x_i = (-1)^i t^{i/k}, \qquad 1 \leqslant i \leqslant k,$$

for all $t > 0$. By Theorem 2 of [8], it follows that no pair of coordinates of $\mathbf{Z}$ are independent. □

REMARK. Theorem 2 will be applied in the proof of Theorem 3 in slightly altered form; the proof of this altered form of Theorem 2 remains the same. Instead of the hypothesis $\alpha < 2/k$, one has only $0 < \alpha < 2$. The definition of $\mathbf{Z}_n$ is altered. Letting $m = \max\{j : k\alpha/j \geqslant 2\}$, $\mathbf{Z}_n$ will be assumed to be a $(k-m)$-dimensional random vector whose ith coordinate is X_n^{m+i}. The same conclusion holds except that the characteristic exponents of the univariate stable marginals are now $k\alpha/(k-m+1), k\alpha/(k-m+2), \cdots, \alpha$.

4. The general case. In this case the joint asymptotic distribution of the first k sample moments of $\{X_n\}$ is investigated when $X_1^k \in \mathfrak{D}(\alpha)$, $0 < \alpha < 2$. In this case some powers X_1^j are in $\mathfrak{D}(2)$ and others are in $\mathfrak{D}(\beta)$ for $0 < \beta < 2$. The following lemma is given in special cases in [5] and [11] with two different proofs. Yet another proof of it is sketched here.

LEMMA 2. *If $\{U_1, \cdots, U_r, V_1, \cdots, V_s\}$ are $r + s$ random variables with a joint infinitely divisible distribution, if each U_i is normally distributed, and if each V_j has no Gaussian component, then the two sets of random variables $\{U_1, \cdots, U_r\}$ and $\{V_1, \cdots, V_s\}$ are independent.*

PROOF. Let $\mathbf{X}$ be the $(r+s)$-dimensional random vector whose ith coordinate is U_i, $1 \leqslant i \leqslant r$, and whose $(r+j)$th coordinate is V_j, $1 \leqslant j \leqslant s$. Since each V_i has no Gaussian component, it follows that the last s rows (columns) of the covariance matrix of the Gaussian component of $\mathbf{X}$ consist entirely of zeros. Since each U_i is Gaussian, it follows that the support of the Lévy spectral measure for $\mathbf{X}$ is a subset of the subspace $u_1 = 0, \cdots, u_r = 0$. One then obtains the marginal joint characteristic functions for $\{U_1, \cdots, U_r\}$ and for $\{V_1, \cdots, V_s\}$ and then notes that their joint characteristic function factors into the product of these two marginals. □

The principal theorem covering all cases follows.

THEOREM 3. *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables whose common distribution F is symmetric, let $k \geq 2$, and let $\mathbf{Z}_n$ be a k-dimensional random vector whose jth coordinate is X_n^j. Assume that $m = \max\{j : k\alpha/j \geq 2\} \geq 1$. If $X_1^k \in \mathfrak{D}(\alpha)$ for some $\alpha \in (0,2)$, then there exist a sequence of $k \times k$ matrices $\{A_n\}$ and a sequence $\{\mathbf{b}_n\}$ in $\mathbb{R}^k$ such that the joint limit distribution of $A_n\Sigma_{j=1}^n \mathbf{Z}_j + \mathbf{b}_n$ exists and is that of $\mathbf{U}, \mathbf{V}$, where $\mathbf{U}$ and $\mathbf{V}$ are independent m- and $(k-m)$-dimensional random vectors respectively, $\mathbf{U}$ having a joint normal distribution, and $\mathbf{V}$ being operator-stable with a Lévy spectral measure determined by two orbits.*

PROOF. The hypothesis implies that there exists a slowly varying function $T(x)$ such that $P[|X_1^k| \geq x] = x^{-\alpha}T(x)$. If $k\alpha/j > 2$, one easily verifies that $E(|X_1^j|^2) < \infty$, i.e., $X_1^j \in \mathfrak{D}_\eta(2)$. If $k\alpha/m = 2$, it follows that $P[|X_1^m| \geq x] = x^{-2}Q(x)$, where $Q(x)$ is a slowly varying function. By Theorem 4 in [11], $X_1^m \in \mathfrak{D}(2)$. In this case, if $X_1^m \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$, there exists a slowly varying function $L(x)$ such that $L(x)$ is nondecreasing, $L(x) \to \infty$ as $x \to \infty$ and such that the sequence $\{n^{\frac{1}{2}}L(n)\}$ are normalizing coefficients for $\{X_n^m, n = 1, 2, \cdots\}$. We denote $L_m(n) = 1$ if $X_1^m \in \mathfrak{D}_\eta(2)$ and $= L(n)$ if $X_1^m \in \mathfrak{D}(2) \setminus \mathfrak{D}_\eta(2)$. Also by hypothesis there exists a slowly varying function $S(x)$ such that $\{n^{1/\alpha}S(n)\}$ serve as normalizing coefficients for X_1^k and such that $P[|X_1|^k \geq n^{1/\alpha}S(n)] \sim 1/n$. For $m < j \leq k$, $X_1^j \in \mathfrak{D}(k\alpha/j)$ with corresponding normalizing coefficients $\{n^{j/k\alpha}S^{j/k}(n)\}$. Let $A_n = (a_{ij}^{(n)})$ be a $k \times k$ diagonal matrix where

$$\begin{aligned} a_{ii}^{(n)} &= 1/n^{\frac{1}{2}} && \text{if } 1 \leq i \leq m-1 \\ &= 1/n^{\frac{1}{2}}L_m(n) && \text{if } i = m \\ &= n^{-i/k\alpha}S^{-i/k}(n) && \text{if } m+1 \leq i \leq k. \end{aligned}$$

We now prove that every subsequence of $S = A_n\Sigma_{j=1}^n \mathbf{Z}_j +$ (some) $\mathbf{d}_n$ converges in law to $\binom{\mathbf{U}}{\mathbf{V}}$. Let $\mathbf{U}_n$ denote the first m coordinates of $\mathbf{S}_n$, and let $\mathbf{V}_n$ denote the last $k - m$ coordinates of $\mathbf{S}_n$. By Theorem 1, $\mathbf{U}_n$ converges in law to $\mathbf{U}$, and, by Theorem 2, $\mathbf{V}_n$ converges in law to $\mathbf{V}$. This implies that the sequence of distributions $\{F_n\}$ of $\{S_n\}$ is tight, and hence every convergent in the wide sense subsequence $\{F_{t_n}\}$ of $\{F_n\}$ converges completely to a distribution function. Take an arbitrary convergent subsequence $F_{t_n} \to_c F$. Since $\{\{A_{t_n}\mathbf{Z}_1, \cdots, A_{t_n}Z_{t_n}\}\}$ is an infinitesimal system, it follows that F is infinitely divisible (see, e.g., [10]). Since $\mathbf{U}_{t_n}$ converges in law to $\mathbf{U}$ and $\mathbf{V}_{t_n}$ converges in law to $\mathbf{V}$, it follows by Lemma 2 that F is the joint distribution of independent random vectors $\mathbf{U}, \mathbf{V}$ where $\mathbf{U}$ and $\mathbf{V}$ are as is claimed in the statement of Theorem 3. Since F does not depend on the particular subsequence, it follows by Theorem 2.3 of Billingsley [1] that $F_n \to_c F$. □

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] DARLING, D. A. (1975). Note on a limit theorem. *Ann. Probability* **3** 876–878. (Correction to Note on a limit theorem. *Ann. Probability* **4** 697.)

[3] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications, Vol. II*. Wiley, New York.

[4] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Theorems for Sums of Independent Random Variables*. (Translated by K. L. Chung). Addison-Wesley, Cambridge.

[5] GREENWOOD, P. and RESNICK, S. (1978). A bivariate stable characterization and domains of attraction. Unpublished manuscript.

[6] HAHN, M. and KLASS, M. (1978). Matrix normalization of sums of i.i.d. random vectors in the domain of attraction of a multivariate normal. *Ann. Probability* **8** 262–280.

[7] HUDSON, W. N. and TUCKER, H. G. (1974). Limit theorems for variational sums. *Trans. Amer. Math. Soc.* **191** 404–426.

[8] HUDSON, W. N. and TUCKER, H. G. Asymptotic independence in the multivariate central limit theorem. *Ann. Probability* **7** 662–671.

[9] PRÉKOPA, A. and RENYI, A. (1956). On the independence in the limit of sums depending on the same sequence of independent random variables. *Acta Math. Acad. Sci. Hungar.* **7** 319–326.

[10] RVAČEVA, E. L. (1954). On domains of attraction of multi-dimensional distributions. *L'vov. Gos. Univ. Učen. Zap. Ser. Meh.-Mat.* **6**. (English translation in *Selected Trans. Math. Statist. Probability* **2** (1962) 183–204.)

[11] SHARPE, M. (1969). Operator-stable probability distributions on vector groups. *Trans. Amer. Math. Soc.* **136** 51–65.

[12] TUCKER, H. G. (1968). Convolutions of distributions attracted to stable laws. *Ann. Math. Statist.* **39** 1381–1390.

[13] TUCKER, H. G. (1971). On asymptotic independence of the partial sums of the positive parts and negative parts of independent random variables. *Advances Appl. Probability* **3** 404–425.

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