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CORRECTION TO "CRITERIA FOR RECURRENCE AND EXISTENCE OF INVARIANT MEASURES FOR MULTIDIMENSIONAL DIFFUSIONS"[1]

BY R. N. BHATTACHARYA

The University of Arizona

The note contains a construction of (possibly explosive) diffusions whose coefficients satisfy Stroock-Varadhan-type regularity conditions, but are unbounded. The appropriate state space here is the one-point compactification of R^k, and the sample paths of the diffusions are shown to be a.s. continuous on this state space.

The construction of the probability measure $P_{t,x}$ corresponding to unbounded coefficients $a(t, x)$, $b(t, x)$ outlined in Section 2 of [1] is wrong. Here is a correct proof for the general nonhomogeneous case. Assume

(A1) $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ are Borel measurable and bounded on compacts,

(A2) for each $N > 0$ there exists a function $\delta_N(r) \downarrow 0$ as $r \downarrow 0$ such that $\| a(t, x) - a(t, y) \| \le \delta_N(|x - y|)$ whenever $0 \le t \le N$, $|x| \le N$, $|y| \le N$,

(A3) the smallest eigenvalue of $a(t, x)$ is bounded away from zero for each x on every finite interval $0 \le t \le T$.

Let $\Omega' = C([0, \infty): R^k)$, and define $\bar{\Omega}$ to be the set of all functions ω on $[0, \infty)$ into the one-point compactification $R^k \cup \{'\infty'\}$ of R^k, satisfying: there exists $\zeta(\omega)$, $0 \le \zeta(\omega) \le \infty$, such that

(i) the function $\omega(t)$ is continuous on $[0, \zeta(\omega))$ into R^k,

(ii) $\limsup |\omega(t)| = \infty$ as $t \uparrow \zeta(\omega)$, if $\zeta(\omega) < \infty$,

(iii) $\omega(t) = '\infty'$ for $\zeta(\omega) \le t < \infty$.

Let us use the same symbol $X(t, \cdot)$ to denote the map: $\omega \to \omega(t)$ on $\bar{\Omega}$ and its restriction to Ω'. On $\bar{\Omega}(\Omega')$ let $\bar{\mathscr{M}}_t^s(\mathscr{M}_t^s)$ and $\bar{\mathscr{M}}^s(\mathscr{M}^s)$ denote $\sigma\{X(t', \cdot): s \le t' \le t\}$, and $\sigma\{X(t', \cdot): t \ge s\}$, respectively. For each positive integer N define the stopping time $\theta_N(\omega) = \inf\{t \ge 0: |\omega(t)| \ge N\} \wedge N$. Fix $x_0 \in R^k$ and choose an integer $N_0 > |x_0|$. For integers $N \ge N_0$, let $P_{N,t,x}$ denote the probability measure on $(\Omega', \mathscr{M}^t)$ which solves the martingale problem of Stroock and Varadhan (1979) (Theorem 7.2.1) corresponding to coefficients $a_N(t, x) = a(t \wedge N, (1 \wedge N/|x|)x)$, $b_N(t, x) = b(t \wedge N, (1 \wedge N/|x|)x)$. Let π'_N denote the regular conditional probability under $P_{N,0,x_0}$ given $\mathscr{M}_{\theta_{N-1}} = \sigma\{X(t \wedge \theta_{N-1}, \cdot): t \ge 0\}$, as prescribed in Stroock and Varadhan (1979) (Theorem 7.2.1.). Define the transition function $\pi^N(\cdot, \cdot)$ by $\pi^N(\omega, B) = \pi'_N(\omega_0, B_\omega)$, where ω_0 is any element of Ω' satisfying $\omega_0(s) = \omega(s)$ for $0 \le s \le \theta_{N-1}(\omega)$, and $B_\omega = \{\omega' \in \Omega' \cap B: \omega'(s) = \omega(s) \text{ for } 0 \le s \le \theta_{N-1}(\omega)\}$, for all $\omega \in \bar{\Omega}$, $B \in \bar{\mathscr{M}}_{\theta_N}$. By Theorem 1.1.9 in Stroock and Varadhan (1979), there exists a unique probability measure P_{0,x_0} on $(\bar{\Omega}, \bar{\mathscr{M}}^0)$ such that π^N is the regular conditional probability distribution (on $\bar{\mathscr{M}}_{\theta_N}$) given $\bar{\mathscr{M}}_{\theta_{N-1}}$. In the same manner one may construct $P_{t,x}$ on $(\bar{\Omega}, \bar{\mathscr{M}}_t)$. To prove a.s. $(P_{0,x})$ continuity of the sample paths one needs to show that

$$P_{0,x}(\{\omega \in \bar{\Omega}: \zeta(\omega) < \infty, \liminf_{s \uparrow \zeta(\omega)} |\omega(s)| < \infty\}) = 0. \tag{1}$$

Choose positive numbers N_1, N_2 with $|x| < N_1 < N_2$. Consider the balls $B_1 = B(0:N_1)$, $B_2 = B(0:N_2)$. Define $\eta_1 = \inf\{t \ge 0: X(t, \cdot) \in R^k \backslash B(0:N_2)\}$, $\eta_{2i} = \inf\{t \ge \eta_{2i-1}: X(t, \cdot) \in \bar{B}_1\}$, $\eta_{2i+1} = \inf\{t \ge \eta_{2i}: X(t, \cdot) \in R^k \backslash B(0:N_2)\}$ $(i = 1, 2, \cdots)$. By inequality (2.1) on page 87 in Stroock and Varadhan (1979), given any $T > 0$ there exists $h > 0$ such that

$$\delta \doteq \sup_{y \in \bar{B}_1, 0 \le s \le T} P_{s,y}(\sup_{s \le t \le s+h} |X(t, \cdot) - X(s, \cdot)| \ge N_2 - N_1) < 1. \tag{2}$$

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By (2) and the strong Markov property,

$$\begin{aligned}
&P_{0,x}(\eta_{2i+1} \le T \quad \text{for all } i) \\
&= P_{0,x}(\eta_{2i+1} - \eta_{2i} \le h \quad \text{for all sufficiently large } i,\ \eta_{2i+1} \le T \quad \text{for all } i) \\
(3)\qquad &\le \textstyle\sum_{j=0}^{\infty} P_{0,x}(\eta_{2i+1} - \eta_{2i} \le h \quad \text{for all } i \ge j,\ \eta_{2i+1} \le T \quad \text{for all } i) \\
&= \textstyle\sum_{j=0}^{\infty} [\lim_{n\to\infty} P_{0,x}(\eta_{2i+1} - \eta_{2i} \le h \quad \text{for } j \le i \le n + j,\ \eta_{2i+1} \le T \quad \text{for all } i)] \\
&\le \textstyle\sum_{j=0}^{\infty} [\lim_{n\to\infty} \delta^n] = 0.
\end{aligned}$$

Since this is true for all T, $P_{0,x}(\lim_{i\to\infty}\eta_{2i+1} = \infty) = 1$. Now $\{\zeta < \infty\} \cap \{\lim_{i\to\infty} \eta_{2i+1} = \infty\} \subset \{\zeta < \infty, \eta_{2i-1} < \infty \text{ and } \eta_{2i} = \infty \text{ for some } i\} \subset \{\zeta < \infty, \liminf_{s\uparrow\zeta} | X(s, \cdot) | > N_1\}$. This implies

$$P_{0,x}(\zeta < \infty, \liminf_{s\uparrow\zeta} | X(s, \cdot) | \le N_1) = 0. \tag{4}$$

Since N_1 may be chosen arbitrarily large, (1) is proved.

REFERENCES

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[2] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes.* Springer, New York.