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B-SPACES ARE STANDARD BOREL

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Separated B-spaces introduced by E. B. Dynkin are just standard Borel and thus here we have an intrinsic definition of standard Borel spaces.

We consider a measurable space $(\Omega, \mathbf{F})$ where $\mathbf{F}$ is countably generated and contains singletons. Following Dynkin [1978], we say that a countable family W of bounded functions—always real valued—on $(\Omega, \mathbf{F})$ is a support family if the following two conditions hold:

(A) If (μ_n) is a sequence of probability measures on $\mathbf{F}$ and if for each $f \in W$, $\lim_n \int f\, d\mu_n = l(f)$ exists then there is a probability μ on $\mathbf{F}$ such that for each $f \in W$, $l(f) = \int f\, d\mu$.

(B) If a class H of functions on Ω contains W and is closed under addition, multiplication by constants, and bounded convergence then H contains all bounded measurable functions (f_n converges boundedly to f means f_n converges to f pointwise and the functions f_n are uniformly bounded).

$(\Omega, \mathbf{F})$ is said to be a B-space if it has a support family. From Condition (B) it follows that if $(\Omega, \mathbf{F})$ is a B-space and W is a support family then W generates $\mathbf{F}$. As noted by Dynkin every standard Borel space is a B-space. We now show that the converse is also true, that is, every B-space is a standard Borel space. We find this observation helpful in understanding the content of Dynkin [1978].

So, let $(\Omega, \mathbf{F})$ have a support family $W = \{f_1, f_2, \cdots\}$. Let M be the space of all probabilities on $(\Omega, \mathbf{F})$. The σ-field $\mathbf{M}$ on M is the usual one generated by all functions of the form $p \mapsto p(A)$ for $A \in \mathbf{F}$. By (B), $\mathbf{M}$ is also the σ-field generated by the functions $p \mapsto \int f_i\, dp$ $(i \geq 1)$. Since $\mathbf{F}$ is countably generated it is easy to see that the collection D of point masses belongs to $\mathbf{M}$ and further $(D, D \cap \mathbf{M})$ is Borel isomorphic to $(\Omega, \mathbf{F})$. We shall now topologize M as follows: consider the map $e: p \mapsto (\int f_1\, dp, \int f_2\, dp, \cdots)$ from M to $R \times R \times \cdots = R^\infty$. By (B) e is a 1-1 map and by (A) the range of e, denoted by $e(M)$, is closed in R^∞. Thus identifying M with $e(M)$ we make M into a complete separable metric space. Let $\bar{\mathbf{M}}$ be the Borel σ-field of M. Since the Borel σ-field on R^∞ is generated by the coordinate maps, $\bar{\mathbf{M}}$ is generated by the maps $p \mapsto \int f_i\, dp$ $(i \geq 1)$. This implies that $\bar{\mathbf{M}} = \mathbf{M}$ and so $(M, \mathbf{M})$ is a standard Borel space. As remarked earlier $(\Omega, \mathbf{F})$ is isomorphic to a Borel subspace of $(M, \mathbf{M})$ so that $(\Omega, \mathbf{F})$ is also a standard Borel space. This completes the proof of the observation.

We assumed $\mathbf{F}$ to be countably generated. If $(\Omega, \mathbf{F})$ has a countable support family then Condition (B) implies that $\mathbf{F}$ is countably generated. We assumed $\mathbf{F}$ to contain singletons. This is no restriction because otherwise we can look at the space of atoms.

The referee has remarked that the concept of B-space was introduced in Dynkin [1971] and was used extensively in Dynkin and Yushkevich (1979).

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