The Annals of Probability
1980, Vol. 8, No. 6, 1189–1190

THERE ARE NO BOREL SPLIFs

By D. Blackwell

University of California, Berkeley

There is no Borel function f, defined for all infinite sequences of 0's and 1's, such that for every sequence X of 0-1 random variables that converges in probability to a constant c, we have $f(x) = c$ a.s.

1. Introduction and summary. Simons (1971), studying weak and strong consistency of a sequence of estimates, introduced the concept *probability limit identification function* (PLIF). A PLIF is a real-valued function f, defined for all infinite sequences of real numbers, such that for every sequence $X = (X_1, X_2, \cdots)$ of random variables that converges in probability, say to X^*, we have $f(X) = X^*$ with probability 1. Whether PLIFs exist was left open, but Simons showed that PLIFs exist iff special PLIFS (SPLIFs) do, where a SPLIF is a 0-1 valued function f, defined for all infinite sequences of 0's and 1's, such that for every sequence X of 0-1 random variables that converges in probability to a constant c (necessarily 0 or 1) we have $f(X) = c$ a.s. Štěpán (1973) showed that the continuum hypothesis implies that PLIFs exist.

Recently Simons suggested that SPLIFs may be an instance of a rule I mentioned: if, although functions of a certain class have been shown to exist, no one has ever produced one, then it is likely that (a) there are no Borel functions in the class and (b) a 0-1 law will be useful in proving (a). Simons' suggestion turns out to be correct; the main result of this note is

Theorem 1. *There are no Borel* SPLIFs.

Our proof uses Oxtoby's category 0-1 law and

Theorem 2. *Call a set S of finite sequences of* 0's *and* 1's *"dense" if every finite sequence of* 0's *and* 1's *is an initial segment of some element of S. For any "dense" S there is a sequence $X = (X_1, X_2, \cdots)$ of* 0-1 *variables such that X converges to* 0 *in probability and $P\{X$ has infinitely many initial segments in $S\} = 1$.*

2. Proofs. We first show how Theorem 1 follows from Theorem 2. If f is a SPLIF, so is g, defined by $g(x_1, x_2, \cdots) = \limsup_{n\to\infty} f(Z_1, \cdots, Z_n, x_{n+1}, \cdots)$, where $Z_1 = Z_2 = \cdots = Z_n = 0$. Moreover, g is a *tail function*: $g(x') = g(x)$ whenever x' and x agree in almost all coordinates. So it suffices to show that no Borel tail function h is a SPLIF. Oxtoby's (1971) category 0-1 law asserts that if E is a tail set with the Baire property, i.e., differs from some open set by a set of first category, then either E or its complement contains a dense G_δ. Assume, as we may without loss of generality, that $E = \{h = 1\}$ contains a dense G_δ set H. According to a result of Wolfe (1955, Section 2) every G_δ set H has the form $J(S)$ for some set S of finite sequences of 0's and 1's, where $J(S)$ denotes those infinite sequences with infinitely many initial segments in S. It is not hard to see that $J(S)$ is dense iff S is "dense" in the sense of Theorem 2. Theorem 2 then implies that h is not a SPLIF.

Theorem 2 follows from the

Lemma. *For any "dense" S and any sequence $\epsilon_1, \epsilon_2, \cdots$ of positive numbers, there is a sequence $0 = n_0 < n_1 < n_2 < \cdots$ of integers and a sequence $X = (X_1, X_2, \cdots)$ of* 0-1 *random variables such that, for all $k \geq 1$,* (a) *$P\{X_n = 1\} \leq \epsilon_k$ for $n_{k-1} < n \leq n_k$ and* (b) *$P\{(X_1, \cdots, X_n) \in S$ for some n with $n_{k-1} < n \leq n_k\} = 1$.*

Received May 31, 1979; revised August 1979.

AMS 1970 *subject classifications.* Primary 28A20; secondary 28A05.

Key words and phrases. Convergence in probability, Borel function.

The proof is by induction on k. Having defined $X_1, \cdots, X_n$ for $n \leq n_k$ satisfying (a) and (b) we define, for each sequence t of 0's and 1's of length n_k, an integer $N = N(t) > n_k$ and the conditional distribution of $(X_{n_k+1}, \cdots, X_N)$ given that $(X_1, \cdots, X_{n_k}) = t$, so that (a) and (b) will be satisfied, given that $(X_1, \cdots, X_{n_k}) = t$, with $n_{k+1} = N$. We then put $n_{k+1} = \max N(t)$ over all t of length n_k, and put $X_n = 0$ for $N(t) < n \leq n_{k+1}$. To define N and the conditional distribution of $X_{n_k+1}, \cdots, X_N$ given that $(X_1, \cdots, X_{n_k}) = t$, choose an integer $R > 1/\epsilon_{k+1}$ and R sequences $s_1, \cdots, s_R$ such that (1) for $2 \leq r \leq R$, s_r has at least $|s_{r-1}|$ initial 0's, where $|s|$ denotes the length of s and (2) $ts_r \in S$ for $1 \leq r \leq R$, where ts_r denotes the sequence consisting of t followed by s_r. Denote by t_r the sequence of length $|s_R|$ obtained from s_r by adding terminal 0's. Then putting $N = n_k + |s_R|$ and $(X_{n_k+1}, \cdots, X_N) = t_U$, where U is uniform on $1, \cdots, R$, gives a sequence with the required properties.

REFERENCES

[1] Oxtoby, John C. (1971). *Measure and Category.* Springer-Verlag.
[2] Simons, Gordon (1971). Identifying probability limits. *Ann. Math. Statist.* **42** 1429–1433.
[3] Štěpán, Josef (1973). The probability limit identification function exists under the continuum hypothesis. *Ann. Probability* **1** 712–715
[4] Wolfe, Philip (1955). The strict determinateness of certain infinite games. *Pacific J. Math.* **5** 841–847.

Department of Statistics
University of California
Berkeley, California 94720