The Annals of Probability
1980, Vol. 8, No. 6, 1148–1156

SOME PATH PROPERTIES OF *p*TH ORDER AND SYMMETRIC STABLE PROCESSES

BY STAMATIS CAMBANIS AND GRADY MILLER

University of North Carolina, Chapel Hill and U.S. Army Materiel Systems Analysis Activity

Necessary and sufficient conditions are presented for the measurability of pth order or symmetric stable stochastic processes and for the integrability or absolute continuity of sample paths of symmetric stable processes. Also obtained are sufficient conditions for absolute continuity of pth order processes.

1. Introduction. This paper extends to pth order or symmetric α-stable (SαS) processes certain results known for second order or Gaussian processes by appropriate modification of the proofs of the latter. Specifically, we give necessary and sufficient conditions for the measurability of a pth order or a SαS process (Section 3), necessary and sufficient conditions for the integrability of almost all paths of a SαS process (Section 4), and sufficient and necessary and sufficient conditions for almost sure path absolute continuity for pth order and for SαS processes, respectively (Section 5). The zero-one laws for Gaussian processes in [7] are results which generalize to SαS processes immediately in view of Dudley and Kanter (1974) (see also Fernique (1973)).

2. Definitions and preliminary results. Most of the material in this section is from Section 2 of [7], where a more detailed treatment is presented. Let $\xi = \{\xi_t, t \in T\}$ be a stochastic process with underlying probability space $(\Omega, \mathscr{F}, P)$ such that $\xi_t \in L_p(\Omega)$ for all $t \in T$, where $1 < p < \infty$, and let $l(\xi)$ be the space of all finite linear combinations of elements of $\{\xi_t, t \in T\}$. Then we call ξ a pth *order process* and define a norm on $l(\xi)$ by

$$\|\zeta\| = (\mathscr{E}|\zeta|^p)^{1/p}, \qquad \zeta \in l(\xi).$$

The *linear space* $\mathscr{L}(\xi)$ of the process ξ is the completion of $l(\xi)$ with respect to this norm, i.e., in $L_p(\Omega)$. If $\mathscr{M}$ is a closed subspace of $L_p(\Omega)$, (such as $\mathscr{L}(\xi)$), then for fixed $\zeta \in \mathscr{M}$ the expression

$$C_\zeta(\eta) = \mathscr{E}[\eta(\zeta)^{p-1}], \qquad \eta \in \mathscr{M},$$

defines a continuous linear functional on $\mathscr{M}$, which by Hölder's inequality has norm $\|C_\zeta\|_{\mathscr{M}^*} = \|\zeta\|^{p-1}$. (Note: when raising a number u to a power q we use the convention $(u)^q = |u|^q \operatorname{sign}(u)$.)

An important subclass of pth order processes is the family of symmetric α-stable (SαS) stochastic processes with $1 < \alpha \le 2$. When $\alpha = 2$ these are the familiar zero mean Gaussian processes. For $1 < \alpha < 2$, the SαS processes are defined by consistent finite dimensional distributions with characteristic functions (ch.f.'s) of the form

$$\phi(y) = \exp\left\{-\int_S |\langle x, y\rangle|^\alpha \Gamma(dx)\right\}, \qquad y \in R^n,$$

where Γ is a uniquely determined (Kanter (1973), page 36) finite symmetric measure on the Borel subsets of the unit sphere $S = \{x \in R^n: \langle x, x\rangle = 1\}$ (Kuelbs (1973), page 264). Following

Received May 31, 1978; revised April 4, 1979.

[1] This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-75-2796.

AMS 1970 *subject classification.* Primary 60G17.

Key words and phrases. Symmetric stable processes, pth order processes, measurable processes, sub-Gaussian processes, absolute continuity of paths, integrability of paths.

Paulauskas (1976), page 357, Γ is called the *spectral measure* of the distribution. In particular, if ζ is a SαS random variable, then there exists some $b_\zeta \geq 0$ such that $\mathscr{E}(e^{ir\zeta}) = \exp\{-b_\zeta |r|^\alpha\}$ for all $r \in R$.

The map $\zeta \to b_\zeta^{1/\alpha}$ defines a norm on a linear space of SαS random variables (Schilder (1970), page 413); and if $1 < p < \alpha$, this norm is related to the usual $L_p(\Omega)$ norm by

$$(1) \qquad (\mathscr{E}|\zeta|^p)^{1/p} = cb_\zeta^{1/\alpha}$$

where c is a constant depending on p and α ([7]). The linear space $\mathscr{L}(\xi)$ of a SαS process ξ is therefore the completion of $l(\xi)$ with respect to either norm, and it can be seen from the form of the multivariate ch.f. that $\mathscr{L}(\xi)$ is a family of jointly SαS random variables.

In the sequel when ξ is a pth order process, $\mathscr{L}(\xi)$ will have the usual $L_p(\Omega)$ norm, and when ξ is a SαS process, $\mathscr{L}(\xi)$ will be normed either by the $L_p(\Omega)$ norm, $1 < p < \alpha$, or by $b_\zeta^{1/\alpha}$. As the latter is the more natural choice for a norm in the SαS case, it will be convenient to let $\|\zeta\|$ denote $b_\zeta^{1/\alpha}$ when ζ belongs to a SαS family, while retaining the notation $\|\zeta\| = (\mathscr{E}|\zeta|^p)^{1/p}$ when ζ is assumed only to have a finite pth moment.

For any two jointly SαS random variables η and ζ, we define

$$C_\zeta(\eta) = \int_S x_1(x_2)^{\alpha-1}\Gamma\,(dx)$$

where Γ is the spectral measure of the distribution of (η, ζ). If ξ is a SαS process, then for each fixed $\zeta \in \mathscr{L}(\xi)$ the function C_ζ: $\eta \to C_\zeta(\eta)$ defines a continuous linear functional on $\mathscr{L}(\xi)$, which by Hölder's inequality has norm $\|C_\zeta\|_{\mathscr{L}(\xi)^*} = \|\zeta\|^{\alpha-1}$.

The use of C_ζ to denote a continuous linear functional in both the pth order and the SαS cases will produce some economy of expression when the particular form of the functional is not important. As in [19], we shall refer to $C_\zeta(\eta)$ as the *covariation of* η *with* ζ.

When a particular pth order process $\xi = \{\xi_t, t \in T\}$ is being considered, the covariation of ξ_t with $\zeta \in \mathscr{L}(\xi)$ will be denoted by $C_\zeta(t)$ and the covariation of ξ_s with ξ_t by $C_{\xi\xi}(s, t)$, $s, t \in T$. If ζ is a SαS random variable and $\{\xi_t, t \in T\}$ a SαS process, then by definition of covariation

$$(2) \qquad \begin{aligned} C_\zeta(\zeta) &= \|\zeta\|^\alpha \\ C_{\xi\xi}(t, t) &= \|\xi_t\|^\alpha. \end{aligned}$$

The following property of the covariation functional is proved in [7].

PROPOSITION 2.1. *If* ξ *is a* pth *order or a* SαS *process and if* A *is a continuous linear functional on* $\mathscr{L}(\xi)$, *then there exists a unique* $\zeta \in \mathscr{L}(\xi)$ *such that* $A = C_\zeta$.

If $\eta = \{\eta_t, t \in T\}$ is a zero mean Gaussian process with covariance function R and ζ is an independent positive random variable with Laplace transform $W(\lambda) = \exp\{-\lambda^{\alpha/2}\}$, i.e., a positive stable random variable of index $\alpha/2$ (Feller (1966), page 427), then the finite dimensional distributions of the process $\xi = \zeta^{1/2}\eta = \{\zeta^{1/2}\eta_t, t \in T\}$ have ch.f.'s of the form

$$\mathscr{E}\exp\{i\textstyle\sum_{n=1}^N r_n\zeta^{1/2}\eta_{t_n}\} = W[2^{-1}\sum_{m,n=1}^N r_m r_n R(t_m, t_n)] = \exp\{-2^{-\alpha/2}[\sum_{m,n=1}^N r_m r_n R(t_m, t_n)]^{\alpha/2}\}.$$

These finite dimensional distributions are multivariate SαS (Paulauskas (1976), page 359) and specify an interesting family of SαS processes called sub-Gaussian processes (Bretagnolle, et al., (1966), page 251). While the theorems in this paper imply certain results for sub-Gaussian processes ξ, these results can be obtained immediately from the analogous theorems for Gaussian processes η in [5], [6], and [21], and the representation $\xi = \zeta^{1/2}\eta$.

3. Measurability of pth order processes. Let $\xi = \{\xi_t, t \in T\}$ and $\eta = [\eta_t, t \in T\}$ be stochastic processes on the probability space $(\Omega, \mathscr{F}, P)$, where T is a Borel subset of a complete separable metric space and $\mathscr{B}(T)$ denotes the Borel subsets of T. The process η is a *modification* of ξ if $P\{\xi_t = \eta_t\} = 1$ for all $t \in T$; η is called *measurable* if $(t, \omega) \to \eta_t(\omega)$ is a product measurable map from $T \times \Omega$ into R. The existence of a measurable modification is frequently of interest in the study of path properties.

The following condition for the existence of a measurable modification is contained in Cohn (1972). We shall use ξ to denote the map $t \to \xi_t$ and shall specify the range space when required for clarity. Let $\mathscr{M}$ be the space of all real-valued random variables on $(\Omega, \mathscr{F}, P)$, and define a metric ρ on $\mathscr{M}$ by

$$\rho(\zeta_1, \zeta_1) = \mathscr{E}\left(\frac{|\zeta_1 - \zeta_2|}{1 + |\zeta_1 - \zeta_2|}\right)$$

for all $\zeta_1, \zeta_2 \in \mathscr{M}$. Then ρ metrizes the topology of convergence in probability. *The process ξ has a measurable modification if and only if the map ξ from T to $\mathscr{M}$ is Borel measurable* (in which case it has a separable range).

For pth order process we now obtain further equivalent conditions for the existence of a measurable modification. The proof is omitted since a similar line of argument is used in [5].

THEOREM 3.1. *Let $\xi = \{\xi_t, t \in T\}$ be a* pth *order process with $p > 1$ or a* SαS *process with $1 < \alpha < 2$, and let $\mathscr{L}(\xi)$ be the linear space of the process. Then the following are equivalent*:

(i) *The process ξ has a measurable modification.*

(ii) *The map ξ: $T \to \mathscr{L}(\xi)$ has separable range and is such that, for every $t_0 \in T$, $\| \xi_t - \xi_{t_0} \|$ is $\mathscr{B}(T)$-measurable.*

(iii) *The map ξ: $T \to \mathscr{L}(\xi)$ is Borel measurable.*

(iv) *$\mathscr{L}(\xi)$ is separable and for every $\zeta \in \mathscr{L}(\xi)$ the function $C_\zeta(t)$ is Borel measurable.*

It should be noted that this result does not reflect the fact that the existence of a measurable modification is a property of the two-dimensional distributions of the process, as shown in Hoffmann-Jørgensen (1973).

COROLLARY 3.2 *A stochastic process ξ as in Theorem* 3.1 *has a measurable modification under each of the following three conditions*:

(i) *ξ is a weakly continuous process.*

(ii) *T is an arbitrary interval and the strong left* (*right*) *limit of ξ exists at all but countably many $t \in T$.*

(iii) *T is an arbitrary interval and ξ is a* SαS *process with independent increments.*

PROOF. (i) If ξ is a weakly continuous process, then $C_\zeta(t)$ is a continuous (hence measurable) function of t for every $\zeta \in \mathscr{L}(\xi)$. To see the separability of $\mathscr{L}(\xi)$, let T^* be a countable dense subset of T and let $\mathscr{N}$ be the space of all rational linear combinations of elements in $\{\xi_t, t \in T^*\}$. Then $\mathscr{N}$ is a countable dense subset of $\mathscr{L}(\xi)$ by Rudin (1973), Theorem 3.12, and the existence of a measurable modification follows from (iv) of Theorem 3.1.

(ii) Parts (i) and (ii) (a) of the proof in Bulatović and Ašić (1976) for second order processes hold with no alteration for the process ξ and show that the set T_1 of all points of discontinuity of ξ is countable. Let $T_2 \subset T - T_1$ be a countable dense subset of T. Then the space of all rational linear combinations of elements in $\{\xi_t, t \in T_1 \cup T_2\}$ is a countable dense subset of $\mathscr{L}(\xi)$. It is clear that $C_\zeta(t)$ is Borel measurable for every $\zeta \in \mathscr{L}(\xi)$ since it is a continuous function on $T - T_1$. Thus ξ has a measurable modification again by (iv) of Theorem 3.1.

(iii) If ξ is SαS with independent increments, then $F(t) = \| \xi_t \|^\alpha$ is an increasing function (Schilder (1970)) which therefore has at most countably many points of discontinuity. Let T_1 be the set of all points of discontinuity of F, and let $T_2 \subset T - T_1$ be a countable dense subset of T. From the relationship $\| \xi_s - \xi_t \|^\alpha = | F(s) - F(t) |$ for all $s, t \in T$ (Schilder (1970)), it is easy to see that the space of all rational linear combinations of elements in $\{\xi_t, t \in T_1 \cup T_2\}$ is a countable dense subset of $\mathscr{L}(\xi)$. For the measurability of $C_\zeta(t)$, $\zeta \in \mathscr{L}(\xi)$, we note that C_ζ has at most countably many discontinuities, since

$$| C_\zeta(s) - C_\zeta(t) | \leq \| \zeta \|^{\alpha - 1} | F(s) - F(t) |^{1/\alpha}. \qquad \square$$

For a general SαS process ξ it appears that product measurability of $C_{\xi\xi}(s, t)$ and separability of $\mathscr{L}(\xi)$ are not sufficient for the existence of a measurable modification, though we do not have a counterexample. Since $\| \xi_t - \xi_{t_0} \|$ cannot be expressed in terms of $C_{\xi\xi}(t, t_0)$ for $1 < \alpha < 2$, we cannot express the condition for a measurable modification in terms of $C_{\xi\xi}(t, t_0)$, except in the sub-Gaussian case (Gaussian when $\alpha = 2$) where

$$\| \xi_t - \xi_{t_0} \|^2 = C_{\xi\xi}(t, t)^{2/\alpha} - 2C_{\xi\xi}(t_0, t_0)^{(2-\alpha)/\alpha} C_{\xi\xi}(t, t_0) + C_{\xi\xi}(t_0, t_0)^{2/\alpha}.$$

If we set $\sigma(s, t) = \| \xi_s - \xi_t \|$, then of course product measurability of $\sigma(s, t)$ implies measurability of $\sigma(t, t_0)$ in t, for each fixed t_0. Conversely, if $\mathscr{L}(\xi)$ is separable and $\sigma(\cdot, t_0)$ is measurable for each $t_0 \in T$, then (ii) of Theorem 3.1 implies the existence of a measurable modification η and we can apply Fubini's theorem to show that $\sigma(s, t) = \| \eta_s - \eta_t \|$ is product measurable. Hence condition (ii) in Theorem 3.1 may be written in the more symmetric form:

(ii)′. *The map* $\xi: T \to \mathscr{L}(\xi)$ *has separable range and the function* $\sigma(s, t) = \| \xi_s - \xi_t \|$ *is* $\mathscr{B}(T) \times \mathscr{B}(T)$*-measurable.*

Finally, we note that if ξ is SαS and $\mathscr{L}(\xi)$ is separable, then ξ has an integral representation of the type

$$\xi_t = \int_{-1/2}^{1/2} f_t(u)\, d\zeta_u,$$

where $\{\zeta_u, -\tfrac{1}{2} \le u \le \tfrac{1}{2}\}$ is a SαS process with independent increments, $\| \zeta_u \|^\alpha = F(u)$, and $f_t(\cdot) \in L_\alpha(dF)$ for all t (Kuelbs (1973), Theorem 4.2). And conversely, if ξ has such a spectral representation, then $\mathscr{L}(\xi)$ is separable since $\mathscr{L}(\zeta)$ is separable (Corollary 3.2 (iii)). In particular, *every measurable* SαS *process has such a spectral representation.*

4. Integrability of sample paths of SαS processes. In this section we apply a result due to DeAcosta (1975) to obtain a necessary and sufficient condition for almost all sample paths of an SαS process to belong to $L_p(T, \mathscr{A}, \nu)$, $1 < p < \alpha$. The referee pointed out that the sample path integral in Lemma 4.2 and the probability measure μ in the proof of Theorem 4.3 could be proved SαS by a simple axiomatic approach without the use of Lemma 4.1. The approach presented here permits a slightly stronger statement of Lemma 4.2 and takes advantage of a shorter proof suggested by the referee for Lemma 4.1.

LEMMA 4.1. *Let* $(T, \mathscr{A}, \nu)$ *be a finite measure space, and let* $\xi = \{\xi_t, t \in T\}$ *be a measurable* pth *order process with* $1 < p < \infty$ *and with* $\| \xi_t \| \le M < \infty$ *for all* $t \in T$. *For any element* $f \in L_q(T, \mathscr{A}, \nu)$, *where* $(1/p) + (1/q) = 1$, *the sample path integral* $\int_T f(t)\, \xi_t(\omega) \nu(dt)$ *belongs to* $\mathscr{L}(\xi)$.

PROOF. From the measurability of ξ and Fubini's theorem we get that $\xi_t(\omega) \in L_p(T, \mathscr{A}, \nu)$ a.s., since

$$\mathscr{E} \int_T | \xi_t(\omega) |^p \nu(dt) = \int_T \| \xi_t \|^p \nu(dt) \le M^p \nu(T) < \infty.$$

Therefore $X = \int_T f(t)\, \xi_t(\omega)\, \nu(dt)$ belongs to $L_p(\Omega)$, since

$$\mathscr{E} | X |^p \le \left[\int_T |f(t)|^q\, \nu(dt) \right]^{p/q} \mathscr{E} \int | \xi_t(\omega) |^p\, \nu(dt).$$

Suppose now (by way of contradiction) that $X \notin \mathscr{L}(\xi)$. Then by Proposition 2.1 there exists $\zeta \notin \mathscr{L}(\xi)$ with $C_\zeta(\xi_t) = 0$ for all $t \in T$, but $C_\zeta(X) \ne 0$. However,

$$C_\zeta(X) = \int_T f(t)\, C_\zeta(\xi_t)\, \nu(dt) = 0. \qquad \square$$

We next state a stronger version of Lemma 4.1 for a SαS process. The proof follows from a truncation argument suggested by Shepp and used in Rajput (1972), Proposition 3.2, where a similar result is proved for Gaussian processes ($\alpha = 2$).

LEMMA 4.2. *Let $(T, \mathscr{A}, \nu)$ be a σ-finite measure space, and let $\xi = \{\xi_t, t \in T\}$ be a measurable* SαS *process with $\xi(\cdot, \omega) \in L_p(T, \mathscr{A}, \nu)$* a.s., *where $1 < p < \alpha$. Then for every element $f \in L_q(T, \mathscr{A}, \nu)$, where $(1/p) + (1/q) = 1$, the sample path integral $\int_T f(t)\xi_t(\omega)\nu(dt)$ belongs to $\mathscr{L}(\xi)$ (and is thus a* SαS *random variable).*

THEOREM 4.3. *Let $(T, \mathscr{A}, \nu)$ be a σ-finite measure space, let $\{\xi_t, t \in T\}$ be a measurable* SαS *process, and suppose that $L_p(T, \mathscr{A}, \nu)$ is separable, where $1 < p < \alpha$. Then $\int_T |\xi_t|^p \nu(dt) < \infty$* a.s. *if and only if $\int_T \mathscr{E}(|\xi_t|^p)\nu(dt) < \infty$.*

PROOF. Sufficiency is clear. For the necessity, define the map $\Phi: \Omega \to L_p(T, \mathscr{A}, \nu)$ by

$$\begin{aligned} \Phi(\omega) &= \xi(\cdot, \omega) \quad \text{if} \quad \xi(\cdot, \omega) \in L_p(T, \mathscr{A}, \nu), \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Then Φ is a measurable map from $(\Omega, \mathscr{F})$ to $(L_p, \mathscr{B}(L_p))$, since the process ξ is measurable and $L_p(T, \mathscr{A}, \nu)$ is separable. Thus ξ induces a measure μ on $L_p(T, \mathscr{A}, \nu)$ by $\mu(B) = P\Phi^{-1}(B)$ for all $B \in \mathscr{B}(L_p)$. If $f \in L_q(T, \mathscr{A}, \nu)$, $(1/p) + (1/q) = 1$, and if f^* denotes the element of $L_p^*(T, \mathscr{A}, \nu)$ corresponding to $f \in L_q(T, \mathscr{A}, \nu)$, then we have that $f^*(\xi(\cdot, \omega)) = \int_T f(t)\xi_t(\omega)\nu(dt)$ defines an element of $\mathscr{L}(\xi)$ by Lemma 4.2. Therefore $\mu(f^*)^{-1}$ is an SαS distribution on R, since for every Borel subset B of R,

$$\begin{aligned} \mu(f^*)^{-1}(B) &= P\,\Phi^{-1}\{x \in L_p(T, \mathscr{A}, \nu): f^*(x) \in B\} \\ &= P\{\omega \in \Omega: f^*(\xi(\cdot, \omega)) \in B\}. \end{aligned}$$

Thus μ is an SαS measure on $L_p(T, \mathscr{A}, \nu)$, so that by DeAcosta (1975), Theorem 3.2,

$$\int_T \mathscr{E}(|\xi_t|^p)\nu(dt) = \int_\Omega \|\xi(\cdot, \omega)\|_{L_p(T)}^p\, p(d\omega) = \int_{L_p(T)} \|x\|_{L_p(T)}^p\, \mu(dx) < \infty. \qquad \square$$

If $(T, \mathscr{A}, \nu) = ([a, b], \mathscr{B}[a, b], \text{Leb})$, then Theorem 4.3 holds for $p = 1$. The only alteration required to the proof is to take $q = \infty$ and use a result in Doob (1953), page 64, instead of Lemma 4.2.

By equations (1) and (2) it follows that

$$\mathscr{E}(|\xi_t|^p) = c^p\, \|\xi_t\|^p = c^p[C_{\xi\xi}(t, t)]^{p/\alpha};$$

so the necessary and sufficient condition of Theorem 4.3 can be written $\int_T [C_{\xi\xi}(t, t)]^{p/\alpha}\, \nu(dt) < \infty$.

5. **Absolute continuity of sample paths.** In this section we obtain sufficient conditions for the sample paths of a pth order process to be absolutely continuous and show these conditions to be also necessary when the process is SαS. The argument used is from [6] where second order processes are considered.

If $\mathbb{B}$ is a Banach space with norm $\|\cdot\|$ and $T = [a, b]$ is an interval of R, then we write $L_1[T, \mathbb{B}]$ for the space of Borel measurable functions $f: T \to \mathbb{B}$ such that $\|f(t)\| \in L_1(T, \text{Leb})$. We call $f: T \to \mathbb{B}$ absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every disjoint family $\{(s_k, t_k)\}_{k=1}^n$ of subintervals of T, $\sum_{k=1}^n (t_k - s_k) \le \delta$ implies that $\sum_{k=1}^n \|f(t_k) - f(s_k)\| \le \epsilon$. Then f is absolutely continuous if and only if it can be expressed in terms of a Bochner integral

$$f(t) = f(a) + \int_a^t \hat{f}(s)\, ds, \qquad t \in T,$$

where $\hat{f} \in L_1[T, \mathbb{B}]$ (Brézis (1973), Appendice).

THEOREM 5.1. *Let $\xi = \{\xi_t, t \in T = [a, b]\}$ be a separable process on $(\Omega, \mathscr{F}, P)$. If ξ is* pth *order with $p > 1$, then each of the following two equivalent conditions is sufficient for the sample paths of ξ to be absolutely continuous with probability one (and have a measurable* pth *order process as derivative):*

(i) *The map $T \to \mathscr{L}(\xi)$ defined by $t \to \xi_t$ is absolutely continuous.*

(ii) *The function $C_\zeta(t)$ is absolutely continuous for all $\zeta \in \mathscr{L}(\xi)$, for all $t \in T - T_0$ with* $\mathrm{Leb}(T_0) = 0$ *the derivative $C'_\zeta(t)$ exists for all $\zeta \in \mathscr{L}(\xi)$, and*

$$\int_T \| \dot{\xi}_t \| \, dt < \infty,$$

where for each $t \in T - T_0$, $\dot{\xi}_t$ is the unique element of $\mathscr{L}(\xi)$ such that the covariation of $\dot{\xi}_t$ with ζ equals $C'_\zeta(t)$ for all $\zeta \in \mathscr{L}(\xi)$.

If ξ is SαS *with $1 < \alpha \le 2$, then each of* (i) *and* (ii) *is necessary and sufficient for the sample paths of ξ to be absolutely continuous with probability one.*

PROOF. The equivalence of (i) and (ii) is contained in Brézis (1973), page 145. If (i) holds, then we have

$$\xi_t = \xi_a + \int_a^t \dot{\xi}_s \, ds$$

for all $t \in T$ where $\dot{\xi} \in L_1[T, \mathscr{L}(\xi)]$. By Theorem 3.1, $\dot{\xi}$ has a measurable modification, say η. Observe that

$$\mathscr{E} \int_a^b | \eta(t, \omega) | \, dt \le \int_a^b \| \eta(t, \omega) \|_{L_p(\Omega)} dt = \int_a^b \| \dot{\xi}_t \|_{L_p(\Omega)} dt < \infty;$$

so $\eta(\cdot, \omega) \in L_1(T, \mathrm{Leb})$ a.s., i.e., for every $\omega \in \Omega - \Omega_0$ with $P(\Omega_0) = 0$. Define X by

$$\begin{aligned} X(t, \omega) &= \xi(a, \omega) + \int_a^t \eta(s, \omega) \, ds, && t \in T, \omega \in \Omega - \Omega_0, \\ &= 0, && t \in T, \omega \in \Omega_0, \end{aligned}$$

and note that the sample paths of X are absolutely continuous.

Let $\dot{\xi}_n$ be a sequence of simple functions $T \to \mathscr{L}(\xi)$ such that

$$\int_a^t \| \dot{\xi}_n(s) - \dot{\xi}(s) \|_{L_p(\Omega)} ds \to 0$$

for all $t \in T$. Then

$$\begin{aligned} \mathscr{E} | \xi_t - X_t | &= \mathscr{E} \left| \int_a^t \dot{\xi}_s \, ds - \int_a^t \eta_s \, ds \right| = \lim_{n\to\infty} \mathscr{E} \left| \int_a^t \dot{\xi}_n(s) \, ds - \int_a^t \eta(s) \, ds \right| \\ &\le \lim_{n\to\infty} \int_a^t \mathscr{E} | \dot{\xi}_n(s) - \eta(s) | \, ds \le \lim_{n\to\infty} \int_a^t \| \dot{\xi}_n(s) - \eta(s) \|_{L_p(\Omega)} \, ds \to 0. \end{aligned}$$

Thus $P\{\xi_t = X_t\} = 1$ for all $t \in T$, and the result follows from the separability of ξ.

In [4] it is shown that for a separable Gaussian process ξ, at every fixed $t \in T$ the paths of ξ are continuous, or differentiable, with probability zero or one. Also, if ξ is measurable, then with probability one its paths have essentially the same points of differentiability and continuity. These same results follow for SαS processes (with no change in argument) by applying a zero-one law for stable measures from Dudley and Kanter (1974).

Let now ξ be SαS, $1 < \alpha \le 2$, and assume that with probability one its sample paths are absolutely continuous. We will show that (i) holds, and thus the proof of the theorem will be complete. By an argument similar to the final paragraph of [6], we get that for all $t \in T$

$$\xi(t, \omega) = \xi(a, \omega) + \int_a^t \eta(s, \omega)\, ds$$

where η is such that $\eta(\cdot, \omega) = \xi'(\cdot, \omega)$ a.e. [Leb] and

$$\int_T \| \eta_s \|_{L_1(\Omega)}\, ds < \infty.$$

Let $\{(s_k, t_k)\}_{k=1}^n$ be a family of disjoint subintervals of T. Then

$$\sum_{k=1}^n \| \xi_{t_k} - \xi_{s_k} \|_{L_1(\Omega)} = \sum_{k=1}^n \mathscr{E} \left| \int_{s_k}^{t_k} \eta(s, \omega)\, ds \right|$$
$$\le \sum_{k=1}^n \int_{s_k}^{t_k} \| \eta_s \|_{L_1(\Omega)}\, ds = \int_{\cup_{k=1}^n (s_k, t_k)} \| \eta_s \|_{L_1(\Omega)}\, ds.$$

Therefore the map $t \to \xi_t$ is absolutely continuous by the absolute continuity of the indefinite integral since $\int_T \| \eta_s \|_{L_1(\Omega)}\, ds < \infty$. □

Theorem 5.1 with appropriate modifications gives conditions for paths to be absolutely continuous with derivatives in $L_p(T, \text{Leb})$. It can also be extended to give conditions for paths to have $(n-1)$ continuous derivatives with the $(n-1)$th derivative absolutely continuous with derivative in $L_p(T, \text{Leb})$.

The following corollary utilizes a stochastic integral introduced by Schilder (1970) and generalizes a well known result for stationary Gaussian processes to the (nonstationary) SαS case.

COROLLARY 5.2. *Let* $\{\zeta_\lambda, -\infty < \lambda < \infty\}$ *be a* SαS *process with independent increments and* $F(\lambda) = \| \zeta_\lambda \|^\alpha$ *a bounded function. Then a separable stochastic process* $\xi = \{\xi_t, a \le t \le b\}$ *defined by*

$$\xi_t = \int_{-\infty}^{\infty} e^{it\lambda}\, d\zeta_\lambda$$

has absolutely continuous sample paths with probability one if and only if

$$\int_{-\infty}^{\infty} |\lambda|^\alpha\, dF(\lambda) < \infty.$$

PROOF. Even though complex-valued random variables appear formally in this result, the proof can be carried through with methods previously developed in the real-valued case. If ξ has absolutely continuous sample paths with probability one, then Theorem 5.1 shows that ξ_t is weakly differentiable at every $t \in T - T_0$ with $\text{Leb}(T_0) = 0$. Thus $f(s, t) \equiv (s-t)^{-1}(e^{is\lambda} - e^{it\lambda})$ converges weakly in $L_\alpha(dF)$ as $s \to t$, so that the set $\{\|f(s, t)\|_{L_\alpha(dF)}\}_{s\to t}$ is bounded. Since $f(s, t) \to_{s\to t} i\lambda e^{it\lambda}$ for all λ, it follows that $f(s, t)$ converges weakly as $s \to t$ to $i\lambda e^{it\lambda}$ in $L_\alpha(dF)$ (Hewitt and Stromberg (1965), page 207). In particular,

$$\int_{-\infty}^{\infty} |\lambda|^\alpha\, dF(\lambda) = \| i\lambda e^{it\lambda} \|_{L_\alpha(dF)}^\alpha < \infty.$$

For the converse, observe that the space $\mathscr{M} = \{\zeta\colon \zeta = \int_{-\infty}^{\infty} g(\lambda)\, d\zeta_\lambda,\ g \in L_\alpha(dF)\}$ contains $\mathscr{L}(\xi)$, and therefore by Proposition 2.1, $\{C_\zeta\colon \zeta \in \mathscr{M}\}$ respresents the dual of $\mathscr{L}(\xi)$. Proposition

3.3 of [7] shows that for every $g \in L_\alpha(dF)$ and $\zeta = \int_{-\infty}^{\infty} g(\lambda)\, d\xi_\lambda$,

$$C_\zeta(t) = \int_{-\infty}^{\infty} e^{it\lambda}(g(\lambda))^{\alpha-1}\, dF(\lambda),$$

which is clearly absolutely continuous since $\int_{-\infty}^{\infty} |\lambda|^\alpha\, dF(\lambda) < \infty$. Moreover, $C'_\zeta(t)$ exists at every $t \in (a, b)$ and $\dot{\xi}_t \equiv \int_{-\infty}^{\infty} i\lambda e^{it\lambda}\, d\xi_\lambda$ is such that the covariation of $\dot{\xi}_t$ with ζ equals $C'_\zeta(t)$ for all $\zeta \in \mathscr{M}$. Since

$$\int_a^b \|\dot{\xi}_t\|\, dt = (b-a)\left(\int_{-\infty}^{\infty} |\lambda|^\alpha\, dF(\lambda)\right)^{1/\alpha} < \infty,$$

the paths of ξ are absolutely continuous with probability one by Theorem 5.1. □

It should be remarked that a SαS process $\xi = \{\xi_t, a \le t \le b\}$ with independent increments cannot have absolutely continuous sample paths (except in the trivial case where $F(t) = \|\xi_t\|^\alpha$ is a constant function). For, the claim is obvious if F is not absolutely continuous. In the case of absolutely continuous F, given any $\epsilon > 0$ and $\delta > 0$ it is possible to choose a finite family of disjoint subintervals $\{(s_k, t_k)\}_{k=1}^n$ such that $\sum_{k=1}^n (t_k - s_k) \le \delta$, but

$$\textstyle\sum_{k=1}^n \|\xi_{t_k} - \xi_{s_k}\| = \sum_{k=1}^n |F(t_k) - F(s_k)|^{1/\alpha} > \epsilon.$$

To see this case, let (a_1, b_1) be a subinterval of $[a, b]$ such that $b_1 - a_1 < \delta$, $F(b_1) - F(a_1) > 0$, and define

$$h = \left(\frac{F(b_1) - F(a_1)}{\alpha\,\epsilon}\right)^{\alpha/(\alpha-1)}$$

By the uniform continuity of F we can choose n so large that $|t - s| < \delta/n$ implies $|F(t) - F(s)| < h$, for all $s, t \in [a, b]$. Let $\{(s_k, t_k)\}_{k=1}^n$ be a partition of (a_1, b_1) into n subintervals of equal length. Then $\sum_{k=1}^n (t_k - s_k) = b_1 - a_1 < \delta$, but

$$\textstyle\sum_{k=1}^n |F(t_k) - F(s_k)|^{1/\alpha} > \sum_{k=1}^n \dfrac{F(t_k) - F(s_k)}{\alpha h^{(\alpha-1)/\alpha}} = \epsilon.$$

REFERENCES

[1] Bretagnolle, J., Dacunha-Castelle, D. and Krivine, J. L. (1966). Lois stables et espaces L^p. *Ann. Inst. Henri Poincaré, Ser. B* **2** 231–259.

[2] Brézis, H. (1973). *Opérateurs Maximaux Monotones*. North-Holland, Amsterdam.

[3] Bulatović, J. and Ašić, M. (1976). The separability of the Hilbert space generated by a stochastic process. *J. Multivariate Anal.* **7** 215–219.

[4] Cambanis, S. (1973). On some continuity and differentiability properties of paths of Gaussian processes. *J. Multivariate Anal.* **3** 420–434.

[5] Cambanis, S. (1975a). The measurability of a stochastic process of second order and its linear space. *Proc. Amer. Math. Soc.* **47** 467–475.

[6] Cambanis, S. (1975b). On the path absolute continuity of second order processes. *Ann. Probability* **3** 1050–1054.

[7] Cambanis, S. and Miller, G. (1980). Linear problems in pth order and stable processes. *SIAM J. Appl. Math.* To appear.

[8] Cambanis, S. and Rajput, B. S. (1973). Some zero-one laws for Gaussian processes. *Ann. Probability* **1** 304–312.

[9] Cohn, D. L. (1972). Measurable choice of limit points and the existence of separable and measurable processes. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **22** 161–165.

[10] DeAcosta, A. (1975). Stable measures and seminorms. *Ann. Probability* **3** 865–875.

[11] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[12] Dudley, R. M. and Kanter, M. (1974). Zero-one laws for stable measures. *Proc. Amer. Math. Soc.* **45** 245–252.

[13] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications, Vol. II*. Wiley, New York.

[14] Fernique, X. (1973). Une démonstration simple du théorème de R. M. Dudley et Marek Kanter sur

les lois zéro-un pour les measures stables. *Lecture Notes in Math.* **381** 78–79. Springer, Berlin.

[15] HEWITT, E. and STROMBERG, K. (1965). *Real and Abstract Analysis.* Springer, New York.

[16] HOFFMANN-JØRGENSEN, J. (1973). Existence of measurable modifications of stochastic processes. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **25** 205–207.

[17] KANTER, M. (1973). The L^p norm of sums of translates of a function. *Trans. Amer. Math. Soc.* **179** 35–47.

[18] KUELBS, J. (1973). A representation theorem for symmetric stable processes and stable measures on H. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **26** 259–271.

[19] MILLER, G. (1978). Properties of certain symmetric stable distributions. *J. Multivariate Anal.* **8** 346–360.

[20] PAULAUSKAS, V. J. (1976). Some remarks on multivariate stable distributions. *J. Multivariate Anal.* **6** 356–368.

[21] RAJPUT, B. S. (1972). Gaussian measures on L_p spaces, $1 \leq p < \infty$. *J. Multivariate Anal.* **2** 382–403.

[22] RUDIN, W. (1973). *Functional Analysis.* McGraw-Hill, New York.

[23] SCHILDER, M. (1970). Some structure theorems for the symmetric stable laws. *Ann. Math. Statist.* **41** 412–421.

DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
CHAPEL HILL, N.C. 27514

U.S. ARMY MATERIEL SYSTEMS ANALYSIS ACTIVITY
ABERDEEN PROVING GROUND
MARYLAND 21005