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# SIZES OF ORDER STATISTICAL EVENTS OF STATIONARY PROCESSES[1]

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Given a process $\{X_i\}$, any permutation $\sigma:[1, n] \to [1, n]$ determines an order statistical event $A(\sigma) = \{X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)}\}$. How many events $A(\sigma)$ are needed to form a union whose probability exceeds $1 - \epsilon$? This question is answered in the case of stationary ergodic processes with finite entropy.

**1. Introduction.** One of the key properties of independent, identically distributed continuous random variables $Y_i$, $1 \le i \le n$, is that the order statistical events defined by

$$\{\omega : Y_{\sigma(1)} < Y_{\sigma(2)} < \cdots < Y_{\sigma(n)}\}$$

have the same probability $1/n!$ for any permutation $\sigma:[1, n] \to [1, n]$. The main objective of this paper is to determine the extent to which this property is retained asymptotically for processes which are only assumed to be stationary and ergodic.

To set the problem precisely, we suppose that $\{X_i\}_{i=1}^{\infty}$ is a strictly stationary, ergodic process defined on the probability space $(\Omega, \mathscr{F}, P)$. We will also use the representation of such a process by $X_i(\omega) = f(T^{-i+1}\omega)$, $1 \le i < \infty$, where $T:\Omega \to \Omega$ is an ergodic measure preserving transformation and $f:\Omega \to \mathbb{R}$ is a measurable map. To avoid inessential messiness, we also restrict attention to processes which satisfy the continuity property

$$P(X_i = X_j) = 0, \qquad i \ne j. \tag{1.1}$$

Our approach to the analysis of the order statistical events is motivated by the Shannon-McMillan Breiman theorem, and particularly the phrasing of that result in terms of the equipartition property ([1], page 135, [9], page 35 (6.3)). Loosely speaking, that phrasing tells one in terms of the entropy of $T$ just how many sets of a certain type are needed to cover most of $\Omega$.

To establish a comparable result for the order statistical events, we let $Q_n(F)$ be defined for any $F \in \mathscr{F}$ by

$$Q_n(F) = |\{\sigma : X_{\sigma(1)}(\omega) < X_{\sigma(2)}(\omega) < \cdots < X_{\sigma(n)}(\omega), \quad \text{for some } \omega \in F\}|.$$

Here $|S|$ denotes the cardinality of the set $S$, so $Q_n(F)$ is equal to the least number of order statistical events

$$A_\sigma = \{\omega : X_{\sigma(1)}(\omega) < X_{\sigma(2)}(\omega) < \cdots < X_{\sigma(n)}(\omega)\} \quad \text{which one needs to cover } F.$$

The quantity of main interest is now defined for $\epsilon > 0$ by

$$Q_n^*(\epsilon) = \min_{E \cdot P(E) \le \epsilon} Q_n(\Omega \backslash E), \tag{1.2}$$

so $Q_n^*(\epsilon)$ is the least number of $A_\sigma$ which will cover a set of probability $1 - \epsilon$.

To familiarize $Q_n^*(\epsilon)$, we note that if the $\{X_i\}_{i=1}^{\infty}$ are i.i.d. and satisfy (1.1), then $Q_n^*$ is equal to the least integer greater than $(1 - \epsilon)n!$.

In this particular example $\{X_i\}_{i=1}^{\infty}$ is a process with infinite entropy, and $Q_n^*(\epsilon)$ is near its

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a priori upper bound. One intuitively expects that $Q_n^*(\epsilon)$ should be of smaller order than $n!$ for processes with finite entropy. We show more precisely that in that case $Q_n^*(\epsilon)$ is, in fact, exponentially smaller.

THEOREM 1. *For any stationary ergodic process* $\{X_i\}_{i=1}^{\infty}$ *which has finite entropy and satisfies* $P(X_i = X_j) = 0$, $i \neq j$, *there is for any* $\epsilon > 0$ *a sequence of positive reals* $\rho_n$ *tending to zero for which*

$$Q_n^*(\epsilon) \leq (n!)\rho_n^n, \qquad \textit{for all} \quad n \geq 1. \tag{1.3}$$

Before giving the complementing result, two comments are in order. In the first place we note there are many processes satisfying the hypotheses, since if $f$ satisfies $P\{\omega : f^{-1}(\omega) = y\} = 0$ for all $y \in \mathbb{R}$, the condition $P(X_i = X_j) = 0$, $i \neq j$, trivially holds for any measure preserving $T$. Also, transformations of finite entropy not only abound but play key roles in such distinct subjects as statistical mechanics and the metrical theory of diophantine approximation ([1]).

Second, we note (1.3) is equivalent to saying $Q_n^*(\epsilon) = o(\rho^n n!)$ for each $\rho > 0$. The phrasing of Theorem 1 was chosen in view of the next result which makes precise the sense in which Theorem 1 is best possible.

THEOREM 2. *For any positive* $\rho_n$ *which tend to zero there is a stationary, ergodic process* $\{X_i\}_{i=1}^{\infty}$ *with* $P(X_i = X_j) = 0$, $i \neq j$, *which has* <u>*zero*</u> *entropy, and which satisfies*

$$Q_n^*(\epsilon) \geq (n!)\rho_n^n$$

*for infinitely many* $n$ *and any* $\epsilon < 1$.

The preceding theorem is easily seen to be a consequence of the next result which shows that the underlying $T$ plays a surprisingly small role in determining $Q_n^*(\epsilon)$.

THEOREM 3. *Given any ergodic measure preserving transformation* $T$ *on a nonatomic probability space* $(\Omega, \mathscr{F}, P)$, *and given any* $\rho_n > 0$ *tending to zero, there is an* $f : \Omega \to [0, 1]$ *which satisfies*

$$P(\{\omega : f^{-1}(\omega) = x\}) = 0, \qquad \forall\, x \in [0, 1], \tag{1.4}$$

*and*

$$Q_n^*(\epsilon) \geq (n!)\rho_n^n \tag{1.5}$$

*for infinitely many* $n$ *and any* $\epsilon < 1$.

In the next section we give the proof of Theorem 1 as a consequence of a counting argument and the application of the Shannon-McMillan-Breiman theorem to an appropriately chosen partition.

The proof of Theorem 3 is more subtle and makes use of a generalization of a combinatorial structure known as de Bruijn sequences.

Since the construction provides a technique for building copies of a finite sequence of independent random variables inside a general stationary process, the construction should be useful in a variety of problems.

Finally, in the fourth section a brief speculation on the theory of order statistical events is ventured.

**2. The upper bound method.** For any measure preserving transformation $S : \Omega \to \Omega$ and any partition $\mathscr{P} = \{P_i\}_{i=1}^{s}$ of $\Omega$, the sets given by

$$\cap_{j=0}^{n-1} \{\omega : S^{-j}(\omega) \in P_{i_j}\} \tag{2.1}$$

for some $1 \leq i_j \leq s$ will be called the <u>$n$-$p$-$S$ name</u> associated with the <u>$n$-$p$-$S$ alias</u> $(i_1, i_2,$

$\cdots, i_j\}$. If $S$ is ergodic and has entropy $H(S) < \alpha$, the Shannon-McMillan-Breiman theorem says there is an $n_0 = n_0(\epsilon, \alpha, S, p)$ such that for $n \geq n_0$ there is a collection of $2^{\alpha n}$ of the $n$-$p$-$S$ names whose union has probability at least $1 - \epsilon$.

Since $P(X_i = X_j) = 0$, $i \neq j$, the disjoint sets $P_\sigma$ given by the permutations of $\{1, 2, \cdots, k\}$ by

$$P_\sigma = \{\omega : X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(k)}\}$$

have union $\Omega$ (except for a set of measure zero). The partition $p = \{P_\sigma\}$ can be related usefully to the possible orderings of $\{X_i\}_{i=1}^{kn}$.

LEMMA 2.1. *For any $n$-$p$-$T^k$ name $A$ we have*

$$Q_{nk}(A) \leq (nk)!/(k!)^n.$$

PROOF. First consider $n = 2$. The 2-$p$-$T^k$ name $A$ has an associated sequence $(i_1, i_2)$; and $i_1$ determines the ordering of $R_1 = \{X_1(\omega), X_2(\omega), \cdots, X_k(\omega)\}$, while $i_2$ determines the order of $R_2 = \{X_{k+1}(\omega), X_{k+2}(\omega), \cdots, X_{2k}(\omega)\}$. To count the possible orderings of $R_1 \cup R_2$, we note the set $R_1$ determines $k + 1$ intervals $(-\infty, X_{(1)})$, $(X_{(1)}, X_{(2)})$, $\cdots$, $(X_{(n)}, \infty)$ where $\{X_{(i)}\}_{i=1}^k$ are the order statistics of $\{X_i\}_{i=1}^k$. Since there are $\binom{2k}{k}$ ways of putting the order statistics of $\{X_i\}_{i=k+1}^{2k}$ into the $k + 1$ intervals, we have

$$Q_{2k}(A) \leq \binom{2k}{k} = (2k)!/(k!)^2.$$

In general, we see for $R_j = \{X_{jk+1}, X_{jk+2}, \cdots, X_{(j+1)k}\}$ and $0 \leq j < n$ that $\bigcup_{j=0}^t R_j$ determines $(t + 1)k + 1$ intervals into which the order statistics of $R_{t+1}$ can be placed in $\binom{(t+1)k}{k}$ ways. Making the sequential choices we have

$$Q_{nk}(A) \leq \binom{2k}{k}\binom{3k}{k} \cdots \binom{nk}{k} = (nk)!(k!)^{-n},$$

which completes the lemma. □

To prove Theorem 1 we need to bound the number of $n$-$p$-$T^k$ names which are needed to cover a set of probability $1 - \epsilon$. We first note that any $nk$-$T$-$p$ name is contained in some $n$-$T^k$-$p$ name because for any alias $\{i_j\}_{j=0}^{nk-1}$ one has

$$\bigcap_{j=0}^{n-1} \{\omega : T^{-jk}\omega \in P_{i_{jk}}\} \supset \bigcap_{j=0}^{nk-1} \{\omega : T^{-j}\omega \in P_{i_j}\}.$$

The Shannon-McMillan-Breiman theorem applied to the ergodic transformation $T$ with entropy $H(T) < \alpha < \infty$ says there is a collection $\mathscr{C}$ of $2^{\alpha nk}$ of the $nk$-$p$-$T$ names whose union contains $\Omega \backslash E$ with $P(E) < \epsilon$ for all $n \geq n_0 = n_0(\epsilon, p)$. By the preceding remark this also implies there is a collection $\mathscr{C}'$ of $2^{\alpha nk}$ of the $n$-$p$-$T^k$ names with the same property.

We now see

$$Q_{nk}(\Omega \backslash E) \leq \textstyle\sum_{A \in \mathscr{C}'} Q_{nk}(A) \leq 2^{\alpha nk}(nk)!(k!)^{-n}$$

where the last inequality follows from Lemma 2.1. For any $k$ we can write $m = nk + r$ with $1 \leq r \leq k$ one has

$$Q_m^*(\epsilon) \leq Q_{(n+1)k}^*(\epsilon) \leq 2^{\alpha(n+1)k}((n + 1)k)!(k!)^{-n-1}$$

provided $n \geq n_0$.

Since $(n + 1)k!/m! \leq ((n + 1)k)^k$, and $k! \geq k^k e^{-k}$, we have

$$Q_m^*(\epsilon) \leq m![2^{(\alpha+1)m}((n + 1)k)^k(k/e)^{-k(n+1)}\}$$

and the fixed integer $k$ was arbitrary, so

$$Q_m^*(\epsilon) = O(\rho^m m!)$$

for all $\tau > 0$. The implied constant depends not only on $H(T)$ but on $T$ through the $n_0$ given by the SMB theorem. As noted earlier, this last relation is sufficient to imply Theorem 1.

**3. The lower bound method.** The first lemma required for Theorem 3 is the so-called strong form of Rohlin's lemma ([8], page 22) which provides a systematic method for applying combinatorial constructions to stationary processes.

LEMMA 3.1. *Suppose $T$ is an ergodic, measure preserving transformation on a nonatomic probability space $(\Omega, \mathscr{F}, P)$. For any finite partition $\mathscr{H} = \{H_1, H_2, \cdots, H_s\}$ of $\Omega$ and for any real $\epsilon > 0$ and integer $m$ there is an $E \in \mathscr{F}$ with the following properties*:

$$E, T^{-1}E, \cdots, T^{-m+1}E \text{ are disjoint}; \tag{3.1}$$

$$P(\cup_{i=0}^{m-1} T^{-1}E) \geq 1 - \epsilon; \tag{3.2}$$

$$P((T^{-i}E) \cap H_j) = P(E)P(H_j), \qquad 0 \leq i < m, \quad 1 \leq j \leq s. \tag{3.3}$$

The second lemma we need is a graph theoretic result due to I. J. Good ([5], [6], page 95) which sharpens a well-known result of Euler.

LEMMA 3.2. *If $G$ is a connected directed graph, and if at each point of $G$ there are the same number of arcs going out as coming in, then there is a directed cycle in $G$ that goes through every arc of $G$ in its given direction, and uses no arc twice.*

As an application of Lemma 3.2, we will obtain the existence of what can be called $s$-ary de Bruijn sequences. To introduce these sequences, we recall the classic result of de Bruijn which says the following: given a positive integer $n$, there is a sequence of 0's and 1's of length $N = 2^n$, say $a_1 a_2 a_3 \cdots a_N$, such that the $n$-tuples $a_i a_{i+1} \cdots a_{i+n-1}$ are a complete list of all $2^n$ of the $n$-tuples with alphabet $\{0, 1\}$.

Here $a_1$ is understood to follow $a_N$, etc. in the cycle. For example, when $n = 3$, the cycle (00010111) contains every 3-tuple of 0's and 1's exactly once. We will use the following generalization where $\mathscr{A}$ is an alphabet of $s$ letters and $\mathscr{B}$ is the set of all ordered $k$-tuples of $\mathscr{A}$.

LEMMA 3.3. *There is a sequence $a_1 a_2 \cdots a_N$ with $N = s^k$ of the elements of $\mathscr{A}$ such that each element of $\mathscr{B}$ occurs exactly once in the set of $k$-tuples $(a_{r+1}, a_{r+2}, \cdots, a_{r+k})$, where $0 \leq r < s^k$ and $a_t = a_u$ if $t > s^k$ and $t - s^k = u$.*

PROOF. We define a directed graph $G$ whose edges are the ordered $(k-1)$-tuples formed by elements of $\mathscr{A}$. We have an edge from $(b_1 b_2 \cdots b_{k-1})$ to $(b_1', \cdots, b_{k-1}')$ provided $b_2 = b_1'$, $b_3 = b_{k-2}' = b_{k-1}$ and $b_{k-1}'$ is arbitrary. Every vertex has in-degree and out-degree equal to $s$, so Lemma 3.2 implies there is a cycle which traverses the edges of $G$ and uses each exactly once. From such a cycle the sequence of $a_i \in \mathscr{A}$ given by the successive $b_{k-1}$ is easily checked to satisfy the claim of the lemma.

The proof given of Lemma 3.3 is only a mild modification of the application given by Good [5] and which Bondy and Murty [2] page 181, relate to the design of an efficient computer drum. A completely different algorithmic proof of Lemma 3.3 was developed independently in recent work of Fredricksen and Mariorana [4].

We now proceed to prove Theorem 3 by applying the preceding lemmas infinitely many times.

We suppose now that $\{p_i\}_{i=1}^{\infty}$, $\{t_i\}_{i=1}^{\infty}$, $\{h_i\}_{i=1}^{\infty}$ are increasing sequences of positive integers and $\{\epsilon_i\}_{i=1}$ is a sequence of positive reals decreasing to zero. We will define a sequence of functions $g_0(\omega) \equiv 0$, $g_1(\omega)$, $g_2(\omega)$, $\cdots$, $g_k(\omega)$, $\cdots$ where each $g_k(\omega)$ will be defined via $p_k$, $t_k$,

$h_k$, $\epsilon_k$, and the preceding $g_j(\omega)$, $0 \le j < k$. Also, we should remark that each of the $g_i(\omega)$ will assume only finitely many values.

For notational convenience we temporarily write $p = p_k$, $t = t_k$, $h = h_k$, and $\epsilon = \epsilon_k$. We define $\mathscr{A}$ by $\{a : a = \sum_{r=1}^{p} \epsilon_r 2^{-r},\ \epsilon_r = 0 \text{ or } 1,\ 1 \le r \le p\}$, and note by Lemma 3.3 there is a sequence of $|\mathscr{A}|^t = 2^{tp}$ elements of $\mathscr{A}$ which form a cycle in which each ordered $t$-tuples with letters from $\mathscr{A}$ appears precisely once among the $(a_{i+1}, a_{i+2}, \cdots, a_{i+t})$ for $0 \le i < 2^{tp}$.

We now apply Lemma 3.1 to obtain an $E$ such that for $0 \le k < 2^{h2^{tp}}$ the sets $T^{-k}E$ are disjoint and their union has probability at least $1 - \epsilon$. For the finite partion $\mathscr{H}$ we take the partition given by the distinct values of the sum $\sum_{i=1}^{k-1} g_i(\omega)2^{-q_i}$ where $q_j = p_1 + p_2 + \cdots + p_j$, $1 \le j < k$.

We now define $g_k(\omega) = a_{i'}$ if $\omega \in T^{-i+1}$ for $1 \le i \le h2^{tp}$ and $i \equiv i' \bmod 2^{tp}$. Finally, we take $g_k(\omega) = 0$ if $\omega$ is in none of the $T^{-i+1}E$, $1 \le k \le h2^{tp}$.

The whole point of this construction is that now by setting $\Omega_k$ equal to the union of $T^{-i}E$ with $0 \le i < (h-1)2^{tp}$ and letting $P_k(A) = P(A \cap \Omega_k)/P(\Omega_k)$, we see that the random variables $g_k(\omega), g_k(T^{-1}\omega), \cdots, g_k(T^{-t+1}\omega)$ are independent in the probability space $(\Omega_k, P_k, \mathscr{F}_k)$ where $\mathscr{F}_k = \{A \cap \Omega_k : A \in \mathscr{F}\}$. Moreover, these random variables are also independent when conditioned on the $\sigma$-field given by the partition $\mathscr{H}$. To prove these assertions one only has to note that for any $H \in \mathscr{H}$ we have by (3.1) and (3.3)

$$\begin{aligned} P(\{ g_k(\omega) = s_0 2^{-p_k}, g_k(T^{-1}\omega) = s_{-1}2^{-p_k}, \cdots, g_k(T^{-t+1}\omega) = s_{-t+1}2^{-p_k}\} \cap H) \\ = (h-1)P(E)P(H)/(h-1)2^{tp}P(E) = 2^{-tp}P(H). \end{aligned}$$

Finally, we are able to define $f(\omega)$ by letting $q_k = \sum_{j=1}^{k} p_j$ and setting

$$f(\omega) = \textstyle\sum_{k=1}^{\infty} g_k(\omega)2^{-q_k}.$$

The sum representing $f(\omega)$ converges for all $\omega$, and one finds no difficulty in checking that for $X_i(\omega) = f(T^{-i+1}\omega)$ we have $P(X_i = X_j) = 0$ for all $i \ne j$.

To prove Theorem 3 we suppose that the sequence $\rho_n \downarrow 0$ and $\epsilon \in (0, 1)$ are given, and we will proceed to show that $\{p_i\}_{i=1}^{\infty}$, $\{t_i\}_{i=1}^{\infty}$, $\{h_i\}_{i=1}^{\infty}$, and $\{\epsilon_i\}_{i=1}^{\infty}$ can be chosen so that $Q_n^*(\epsilon) \ge n!\rho_n^n$ for infinitely many $n$.

For the intervals $I_s = [s2^{-q_{k-1}}, (s+1)2^{-q_{k-1}})$ with $0 \le s < 2^{q_{k-1}}$ we define random variables $\nu_s$ as the number of elements of $\mathscr{I}_s = \{i : X_i \in I_s,\ 1 \le i \le n\}$. The ordering of $\{X_i\}_{i \in \mathscr{I}_s}$ is completely determined by the ordering of $\{g_k(T^{-i+1}\omega)\}_{i \in \mathscr{I}_s}$ except for at most those $\omega \in \mathscr{T} = \{\omega : g_k(T^{-i+1}\omega) = g_k(T^{-j+1}\omega)$, for some $0 \le i < j \le n\}$. Using the $P_k$-independence of the $\{g_k(T^{-i+1}\omega)\}$ $1 \le i < n$ for $n \le t$ and the conditional independence given $\mathscr{H}$ we have

$$\begin{aligned} P(\{X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)}\} \cap H) \\ &\le \epsilon_k + h_k^{-1} + P(\{X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)}) \cap H \cap \Omega_k) \\ &\le \epsilon_k + h_k^{-1} + P_k(\{X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)}\} \cap H) \\ &\le \epsilon_k + h_k^{-1} + P_k(\mathscr{T}) + P_k(H)(\textstyle\prod_s \nu_s!)^{-1}. \end{aligned}$$

Since there are only $\binom{n}{2}$ places a tie can take place and the $P_k$-probability of any such tie is $2^{-q_k}$, we have

$$P(\mathscr{T}) \le (1 - \epsilon_k - h_k^{-1})^{-1}P_k(\mathscr{T}) \le 2\binom{n}{2}2^{-q_k} < n^2 2^{-1_k}.$$

Also, $P_k(H) = P(H)$ for all $H \in \mathscr{H}$ by (3.3); so summing over $\mathscr{H}$ we have

$$P(X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)}) \le 2^{q_{k-1}}(\epsilon_k + h_k^{-1}) + n^2 2^{-p_k} + \textstyle\prod_s(\nu_s!)^{-1}.$$

Setting $r = 2^{q_{k-1}}$ we have $\sum_{s=0}^{r-1} \nu_s = n$. Hence, $\prod_s(\nu_s!) = \prod_s \Gamma(\nu_s + 1) \ge \Gamma((n+r)/r)^r$ by the convexity of $\log \Gamma(x)$, [7], page 285. We begin with the bounds

$$(3.4)\quad Q_n^*(\epsilon) \geq (1-\epsilon)\{\max_\sigma P(X_{\sigma(1)} < X_{\sigma(2)} < \cdots < X_{\sigma(n)})\}^{-1}$$
$$\geq (1-\epsilon)n!\left\{\Gamma(n+1)\Gamma\left(\frac{n+r}{r}\right)^r + \Gamma(n+1)(2^{-p_k}n^2 + r\epsilon_k + rh_k^{-1})\right\}^{-1}.$$

Since $r = 2^{q_{k-1}}$ is fixed (as we choose $\epsilon_k, p_k, t_k$, and $h_k$), we invoke Stirling's formula to obtain $\Gamma(n+1)\Gamma((n+r)/r)^{-r} \leq (r+1)^n$ for $n \geq n_0(r)$. Since $\rho_n \downarrow 0$, we can now choose $t = t_k \geq n_0$ so that

$$(3.5)\quad (r+1)^{-t} \geq 2\rho_t^t(1-\epsilon)^{-1}.$$

Finally, we choose $\epsilon_k$, $h_k$, and $p_k$ so that

$$(3.6)\quad \{\Gamma(t+1)(2^{-p_k}t^2 + r\epsilon_k + rh_k^{-1})\}^{-1} \geq 2\rho_t^t(1-\epsilon)^{-1}.$$

By the elementary inequality $1/(a+b) \geq (½)\min\{1/a, 1/b\}$ applied to (3.4), (3.5), and (3.6) for each $t = t_k$, $k = 1, 2, \cdots$ we have

$$Q_n^*(\epsilon) \geq \rho_n^n(n!) \quad \text{for} \quad n = t_1, t_2, \cdots.$$

The proof of Theorem 3 is thus complete. □

**4. A brief speculation.** The only two cases where one has a reasonably complete understanding of $Q_n^*(\epsilon)$ are the case of continuous i.i.d. random variables and now in the case of continuous stationary ergodic processes with finite entropy. These two cases are in a way polar opposites of generality, and many important classes of processes lie in between.

Since many basic probabilistic events are simply unions of the order statistical events, it would seem to be of considerable interest to discover those cases in which a precise understanding of $Q_n^*(\epsilon)$ can be obtained.

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