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ON THE EFFECT OF COLLISIONS ON THE MOTION OF AN ATOM IN R^1

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A joint generalization of Harris-Spitzer's and Szatzschneider's one-dimensional collision models is given and for the path of a given particle an approximation is obtained.

0. Introduction. One major aim of nonequilibrium statistical mechanics is the derivation of the main physical processes from microscopic assumptions. A very important example of such processes is the Brownian motion. For the one-dimensional case, Spitzer [7] proved in 1969 that if

(a) at time 0 a system of atoms, i.e., particles with equal masses, is given in R^1 according to a stationary Poisson process;

(b) and independently of the positions the atoms are given i.i.d. velocities;

(c) and the atoms collide or, in other words, they exchange velocities when they meet; then, under the same normalization as used in the Wiener process approximation of the random walk, the path of a single particle tends to the Wiener process in $C[0, 1]$. This model was introduced in 1965 by Harris [4], who also proved the existence of the motion and obtained a Gaussian approximation for the position of the particle after a long time.

In 1975 Szatzschneider [8] investigated this motion with the modification that he supposed that at time 0 the particles were situated in an almost deterministic way (almost) on the lattice of integers. He obtained the surprising result that in this case, in general, a non-Wiener Gaussian process approximates the motion of a single particle.

We give a joint generalization of the two results. Namely, we suppose that at time 0 the atoms are situated in points of a two-sided renewal process and we obtain that at a given initial velocity distribution the approximating Gaussian process depends on the first two moments of the renewal times (Theorem 1). The limit process can also be obtained for a wider class of initial positions. In Theorem 2 we only require that, roughly speaking, the process of initial positions, when suitably normalized, converges weakly to a $C(-\infty, +\infty)$ process.

In Section 1 we describe our model and state the results. They are proven in Section 2. Section 3 contains additional remarks and comments.

1. Results. Suppose that the family $x_i : i \in Z (Z = Z^1$ is the lattice of integers) of functions $x_i : [0, \infty) \to R^1$ satisfies the following conditions:

(i) x_i is continuous on $[0, \infty)$ $(i \in Z)$;

(ii) $x_i(0) \leq x_{i+1}(0)$;

(iii) for every $t \geq 0$

$$\lim_{i \to -\infty} \max_{\tau \in [0,t]} x_i(\tau) = -\infty;$$

$$\lim_{i \to +\infty} \min_{\tau \in [0,t]} x_i(\tau) = \infty.$$

Then, by a suitable modification of the existence theorem of Harris [4], there exists a unique family $y_i : i \in Z$ of functions $y_i : [0, \infty) \to R^1$ such that

(1) y_i is continuous on $[0, \infty)$ $(i \in Z)$;

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(2) $y_i(0) = x_i(0) \qquad (i \in Z)$;
(3) $y_i(t) \leq y_{i+1}(t)$;
(4) for every $t \geq 0$

$$\lim_{i\to-\infty}\max_{\tau\in[0,t]} y_i(\tau) = -\infty$$

$$\lim_{i\to\infty}\min_{\tau\in[0,t]} y_i(\tau) = \infty;$$

(5) $$\cup_{i=-\infty}^{+\infty} \mathrm{Gr}(y_i) = \cup_{i=-\infty}^{+\infty} \mathrm{Gr}(x_i),$$

where $\mathrm{Gr}(z)$ denotes the graph of the function z, i.e., $Gr(z) = \{(u, t): z(t) = u \text{ for some } t \geq 0\}$, and the equality of the unions is understood in the strong sense, that is we require for every (u, t) that

$$\mathrm{card}\{i: y_i(t) = u\} = \mathrm{card}\{i: x_i(t) = u\}.$$

The content of this theorem is the following: if the x_i's denote the motions of atoms in the absence of collision, then the y_i's describe the motions of the same particles under collision since under collision the order of particles remains invariant.

(A) Suppose that the $x_i(0)$'s form a renewal process, i.e., the nonnegative rv's $x_{i+1}(0) - x_i(0)$ are i.i.d. with some common distribution K, and $x_0(0) = 0$;

(B) $x_i(t) = x_i(0) + v_i t (i \in Z)$ where the v_i's are i.i.d. rv's with a common distribution F; they are also independent of the initial positions $X(0) = \{x_i(0): i \in Z\}$;

(*C*) $Ev_i = 0$.

It is easy to see that conditions (A), (B), (C) imply that (i), (ii) and (iii) hold with probability 1 and consequently we can give the motion $y_i(t)$, $t \geq 0$, $i \in Z$ of the system with collisions with probability 1. We will be interested in the path of the atom with label 0 so we denote $y(t) = y_0(t)$.

THEOREM 1. *If* $E(x_{i+1}(0) - x_i(0)) = \mu$, $(0 < \mu < \infty)$. $D^2(x_{i+1} - x_i) = \sigma^2 < \infty$ *then the processes* $\psi_A(t) = A^{-1/2}y(At)$ *tend to a Gaussian process* $\gamma(t)$, *as* $A \to \infty$ *in the sense of weak convergence in* $C[0, \infty)$, *where* γ *is determined by*

$$E\gamma(s)\gamma(t) = \mu^{-1}E|v|\min(s, t) + \mu^{-3}(\sigma^2 - \mu^2)\, E\min(s|v|, t|v'|)\chi\{vv' > 0\},$$

$$E\gamma(s) = 0.$$

$(s, t \geq 0)$ *and* v *and* v' *are* i.i.d. rv's *with common distribution* F. ($\chi\{\ \}$ *denotes the indicator variable of the event in brackets.*)

Theorem 1 will be proven as a consequence of a more general theorem. Denote

$$\nu(x) = \mathrm{card}\{i: x_i(0) \in (0, x)\} \quad \text{if } x > 0$$
$$= -\mathrm{card}\{i: x_i(0) \in [x, 0]\} \quad \text{if } x \leq 0$$

and define the processes

$$S_A(t) = A^{-1/2}(\nu(At) - \mu^{-1}At)$$

for $A > 1$, $-\infty < t < \infty$. Instead of assumption (A), we need the following ones:

(A1) $\lim_{|n|\to\infty} n^{-1}x_n(0) = \mu$ with probability 1, where μ is a positive rv.

(A2) There exists a process $S(t)$, $-\infty < t < \infty$ with trajectories in $C(-\infty, +\infty)$ such that $S_A(t)$ converges to $S(t)$ in the sense of weak convergence in $D(-\infty, +\infty)$.

(A3) $\sup_t[(1 + |t|)^{-1}|S_A(t)|]$ is stochastically bounded in A.

As in the previous case, it is not difficult to see that (A1), (B) and (C) imply the existence of the collision model with probability 1 and we shall keep denoting $y(t) = y_0(t)$ and $\psi_A(t) = A^{-1/2}y(At)$. Also, conditions (A2) and (A3) involve that

(1.1) $$\sup_t[(1 + |t|)^{-1}|S(t)|] < \infty \text{ a.s.}$$

Hence, by (C), the integral $\int_{-\infty}^{\infty} S(t)F(dt)$ exists and is finite with probability 1.

In the proof of the tightness we shall also need an additional assumption.

(D) For every positive T, the processes $\xi_r^T(t) = S(t+r)$, $-T \le t \le T$ are tight in $C[-T, T]$ uniformly in r, $-\infty < r < +\infty$. This condition obviously holds in the most important particular case when S has stationary increments.

Denote by $\Phi_\Sigma(y)(y \in R^N)$ the N-dimensional normal distribution function with expectation vector 0 and covariance matrix Σ. Now we can formulate our

THEOREM 2. *If conditions* (A1)-(A3), (B) *and* (C) *are satisfied, then the finite dimensional distributions of the process* $\psi_A(t)$ *converge, as* $A \to \infty$, *to the finite dimensional distributions of a process* $\beta(t)$, *given as follows*

$$P(\beta(t_j) < w_j, j = 1, \cdots, N) = E\Phi_\Sigma(u_1, \cdots, u_N)$$

where the covariance matrix $\Sigma = (\sigma_{ij})_{i,j=1,\ldots,N}$ *is determined by*

$$\sigma_{ij} = \min(t_i, t_j)E|v| - E[\min(t_i|v|, t_j|v'|)\chi\{vv' > 0\}]$$

and

$$u_j = \int S(-vt_j)F(dv) + w_j.$$

The random variables v *and* v' *in this expression are* i.i.d. *with common distribution function* F.

If, moreover, (D) *holds, too, then the processes* $\Psi_A(t)$ *converge to* $\beta(t)$ *in* $C[0, \infty)$ *in the weak sense.*

The limit process β, in general, is not Gaussian. However, if S is a Gaussian process, then so is β. If $\mu =$ const. and $S(t) = \sigma W(t)$ with σ a positive constant, then the limit process reduces to that of Theorem 1.

An interesting question is when the limit process is Markovian. If β is Gaussian, then the results of Timoszyk [9] can be used to answer this question. For example, it can be shown similarly as it is done in [8] that the limit process γ of Theorem 1 is Markovian if and only if either $\mu = \sigma$ or for some $a > 0$ $F = \frac{1}{2}[\delta_{-a} + \delta_a]$ where δ_a denotes the degenerate distribution at point a.

2. Proofs. Let us recall that a sequence of probability measures given on $D(I)$ $(C(I))$, where I is a finite or infinite interval $(I \subset R^1)$ converges weakly if and only if, for some sequence of compact intervals I_n such that $I_n \subset I_{n+1}$, $\cup I_n = I$ the projections of the measures onto $D(I_n)(C(I_n))$ converge weakly for every n.

We shall need the following simple consequences of condition (A1):

$$\sup_y(1 + |y|)^{-1}|\nu(y)| < C_1\mu^{-1} \tag{2.1}$$

$$\lim_{A\to\infty} A^{-1}\nu(Ay_A) = y\mu^{-1} \tag{2.2}$$

where C_1 is a constant and $\lim_{A\to\infty} y_A = y \ne 0$. For $y > 0$, these relations follow from the inequality $X_{\nu(y)} < y \le X_{\nu(y)+1}$ and (A1).

Let us introduce the notations

$$B_A(t, w) = \operatorname{card}\{k, k \le 0, \quad x_k + tAv_k \ge A^{1/2}w\}$$

$$C_A(t, w) = \operatorname{card}\{k, k > 0, \quad x_k + tAv_k < A^{1/2}w\}$$

$$z_A(t, w) = A^{-1/2}[B_A(t, w) - C_A(t, w)]$$

where $x_k = x_k(0)$, $0 \le t \le 1$, $w \in R^1$. The following observation (cf. Harris [4]) is very important: the events $z_A(t, w) < 0$ and $\psi_A(t) < w$ agree. This means in particular that

$$P(\psi_A(t_j) < w_j, j = 1, 2, \cdots, N) = P(z_A(t_j, w_j) < 0, j = 1, 2, \cdots, N). \tag{2.3}$$

We shall investigate the right-hand side of the identity (2.3).

Denote by $\mathscr{X}$ the σ-algebra generated by the random variables x_n, $n \in Z$. Let us observe that, because of the independence of the random sequences $\{x_n, n \in Z\}$ and $\{v_n, n \in Z\}$, we can write

$$\begin{aligned} P(z_A(t_j, w_j) < \alpha_j, j = 1, \cdots, N \mid \mathscr{X}) \\ = P\Big(A^{-1/2}\Big[\textstyle\sum_{l=-\infty}^{0} \chi\Big\{v_l \geq \frac{A^{1/2}w_j - x_l}{At_j}\Big\} \\ - \textstyle\sum_{l=1}^{\infty} \chi\Big\{v_l < \frac{A^{1/2}w_j - x_l}{At_j}\Big\}\Big] < \alpha_j, j = 1, \cdots, N\Big) \end{aligned} \tag{2.4}$$

for arbitrary real numbers $\alpha_1, \cdots, \alpha_N$. We also have

$$E(z_A(t, w) \mid \mathscr{X}) = A^{-1/2}\Big\{\textstyle\sum_{l=-\infty}^{0}\Big[1 - F\Big(\frac{A^{1/2}w - x_l}{At}\Big)\Big] - \sum_{l=1}^{\infty} F\Big(\frac{A^{1/2}w - x_l}{At}\Big)\Big\}. \tag{2.5}$$

Here

$$\begin{aligned} \textstyle\sum_{l=1}^{\infty} F\Big(\frac{A^{1/2}w - x_l}{At}\Big) &= \int_0^{\infty}\int_{-\infty}^{\infty} \psi\{x_l + vAt < A^{1/2}w\}F(dv)\nu(dx) \\ &= \int_{-\infty}^{w/A^{1/2}t} \nu(A^{1/2}w - vAt)F(dv) \end{aligned}$$

and similarly

$$\textstyle\sum_{l=-\infty}^{0}\Big[1 - F\Big(\frac{A^{1/2}w - x_l}{At}\Big)\Big] = -\int_{w/A^{1/2}t}^{\infty} \nu(A^{1/2}w - vAt)F(dv).$$

Consequently, by (2.5)

$$\begin{aligned} E(z_A(t, w) \mid \mathscr{X}) &= -A^{-1/2}\int \nu(A^{1/2}w - vAt)F(dv) \\ &= -\int S_A(-vt + A^{-1/2}w)F(dv) - w, \end{aligned} \tag{2.6}$$

where we need that $Ev = 0$.

To prove the theorem we need several lemmas. We can and do assume that $\mu = 1$ a.s.

LEMMA 1. *Under the assumptions* (A1), (B) *and* (C), *the conditional distribution of the multidimensional random variable*

$$z_A(t_j, w_j) - E(z_A(t_j, w_j) \mid \mathscr{X}) \qquad j = 1, 2, \cdots, N \tag{2.7}$$

with respect to the σ-algebra $\mathscr{X}$ is asymptotically normal with the covariance matrix $\Sigma = (\sigma_{ij})$, $i, j = 1, \cdots, N$, where

$$\sigma_{ij} = E|v|\min(t_i, t_j) - E[\min(t_i|v|, t_j|v'|)\,\chi\{vv' > 0\}]. \tag{2.8}$$

Lemma 1 says, in particular, that the conditional distribution of the expression in (2.7) has a limit independent of the condition.

PROOF. Because of (2.4), (2.5) and the independence of the v_i's the multidimensional central limit theorem implies that the conditional distribution of the expression in (2.7) is asymptotically normal with expectation zero and covariance matrix$(\sigma_{i,j}^{A})$ $i, j = 1, 2, \cdots, N$, where

$$\sigma_{i,j}^A = A^{-1} \sum_{k=-\infty}^{\infty} \left[F\left(\min \left(\frac{A^{1/2}w_i - x_k}{At_i}, \frac{A^{1/2}w_j - x_k}{At_j}\right)\right) - F\left(\frac{A^{1/2}w_i - x_k}{At_i}\right) F\left(\frac{A^{1/2}w_j - x_k}{At_j}\right)\right]. \tag{2.9}$$

In order to prove Lemma 1, it remains to show that $\sigma_{i,j}^A \to \sigma_{i,j}$ given by (2.8). Consider, for example, the sum

$$\frac{1}{A} \sum_{k=1}^{\infty} F\left(\frac{A^{1/2}w_i - x_k}{At_i}\right) F\left(\frac{A^{1/2}w_j - x_k}{At_j}\right).$$

It is equal to

$$\frac{1}{A} \int_0^{\infty} F\left(\frac{A^{1/2}w_i - x}{At_i}\right) F\left(\frac{A^{1/2}w_j - x}{At_j}\right) \nu(dx)$$

$$= \frac{1}{A} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi\{x + vAt_i < A^{1/2}w_i\}\chi\{x + v'At_j < A^{1/2}w_j\} F(dv) F(dv')\nu(dx)$$

$$= \int_{-\infty}^{W_i/A^{1/2}t_i} \int_{-\infty}^{W_j/A^{1/2}t_j} \frac{\nu(A \min\{A^{-1/2}w_i - t_i v, A^{-1/2}w_j - t_j v'\})}{A} F(dv) F(dv')$$

where we changed the order of integration. By (2.2) and the Lebesgue convergence theorem, the last integral tends to

$$\int_{-\infty}^{0} \int_{-\infty}^{0} \min\{t_i |v|, t_j |v'|\} F(dv) F(dv')$$

$$= E[\min(t_i |v|, t_j |v'|)\chi\{v, v' < 0\}].$$

Taking the limit is legitimate since, by (2.1), the integrand is less than $A^{-1} |\nu(A^{1/2}w_i - At_i v)| \leq C_1(t_i |v| + 1)$. Similarly, the same sum for $k \leq 0$ tends to

$$E[\min(t_i |v|, t_j |v'|)\chi\{v, v' > 0\}].$$

Analogously,

$$\frac{1}{A} \sum_{k=1}^{\infty} F\left(\min\left\{\frac{A^{1/2}w_i - x_k}{At_i}, \frac{A^{1/2}w_j - x_k}{At_j}\right\}\right)$$

$$= \int_{-\infty}^{A^{-1/2}\min\{w_i/t_i, w_j/t_j\}} \frac{1}{A} \nu(A \min\{A^{-1/2}w_i - t_i v, A^{-1/2}w_j - t_j v\}) F(dv)$$

$$\to \min(t_i, t_j) Ev^-$$

and, again, the same sum for $k \leq 0$ tends to $\min(t_i, t_j)Ev^+$.

LEMMA 2. *Under the assumptions of Theorem 2,*

(a) *The realizations of the process* $\Theta(t) = \int S(-vt)F(dv)$, $-\infty < t < \infty$ *are continuous* a.s.

(b) *For every* w, *the process* $\Theta_A(t) = \int S_A(-vt + A^{-1/2}w)F(dv)$, $-\infty < t < \infty$ *converges to* $\Theta(t)$ *in the sense of the weak convergence in* $D(-\infty, \infty)$, *as* $A \to \infty$.

(c) *The conditional expectation vector* $E(z_A(t_j, w_j) | \mathscr{X})$, $j = 1, \cdots, N$ *of the random variables* $z_A(t_j, w_j)$, $j = 1, \cdots, N$ *tends in distribution to the vector* $-u = -(u_1, \cdots, u_N)$

where

$$u_j = \int S(-vt_j)F(dv) + w_j.$$

REMARK. Similar statements as (a) and (b) hold for the processes $\Theta^+(t) = \int_0^\infty S(-vt)F(dv)$ and $\Theta_A^+(t) = \int_0^\infty S_A(-vt)F(dv)$ (and for the processes $\Theta^-(t) = \int_{-\infty}^0 S(-vt)F(dv)$ and $\Theta_A^-(t) = \int_{-\infty}^0 S_A(-vt)F(dv)$ respectively).

PROOF.

$$\Theta(t) = \int \left[\frac{S(-vt)}{1+|v|t}\,\frac{1+|v|t}{1+|v|}\right](1+|v|)F(dv).$$

The expression in square brackets is bounded and continuous in t a.s. and, since $E|v| < \infty$, the assertion follows from the Lebesgue theorem.

(b) We prove the weak convergence in $[-T, T]$. Let $\epsilon, \eta > 0$. By (A2), for every positive ϵ a number K_ϵ can be chosen such that on a set H_ϵ of measure exceeding $1-\epsilon$ we have

$$\frac{|S_A(t)|}{1+|t|}, \frac{|S(t)|}{1+|t|} < K_\epsilon$$

for any t and A. Thus, if v_0 is large enough, then, by $E|v| < \infty$, everywhere on H_ϵ

$$\int_{|v|>v_0} S_A\left(-vt + \frac{w}{A^{1/2}}\right)F(dv) \le \eta$$

$$\int_{|v|>v_0} S(-vt)F(dv) < \eta$$

for every $|t| \le T$. Finally, by (A2), $S_A(t + w/A^{1/2})$ tends weakly to $S(t)$ in $D[-v_0T-1, v_0T+1]$ implying that $\int_{|v|\le v_0} S_A(-vt + w/A^{1/2})F(dv)$ converges weakly to $\int_{|v|\le v_0} S(-vt)F(dv)$. In fact, the continuous mapping theorem (cf. [1], Theorem 5.1) applies, since the functional ψ defined on $D[-v_0T-1, v_0T+1]$ by $\psi(x) = \int_{|v|\le v_0} x(-vt)F(dv)$ is continuous on $C[-v_0T-1, v_0T+1]$. Hence the statement.

(c) By using the representation (2.6), a proof similar to the previous one gives this statement, too.

PROOF OF THEOREM 2. The convergence of the finite dimensional distributions easily follows from Lemmas 1 and 2. It is sufficient to observe that if $\epsilon > 0$, then for sufficiently large A, we have

$$|P(z_A(t_j, w_j) < 0, j = 1, \cdots, N) - E\Phi_\Sigma(-Ez_A(t_j, w_j), \quad j = 1, \cdots, N\,|\,\mathscr{X})| < \epsilon$$

by Lemma 1 and the continuity of the normal distribution, and also

$$|E(\Phi_\Sigma(-Ez_A(t_j, w_j), j = 1, \cdots, N)|\,\mathscr{X}) - E\Phi_\Sigma(u_1, \cdots, u_N)| < \epsilon$$

by Lemma 2. Now (2.3) yields the desired statement.

Now we turn to the proof of the tightness of the ψ_A's. For simplicity, it will be proven in $C[0, 1]$. For $f \in D[0, \infty)$ and $0 < \delta < 1$ denote $\omega(f, \delta) = \sup_{0\le t\le s\le t+\delta\le 1}|f(s) - f(t)|$, which is the well-known modulus of continuity of the function f, taken on the interval $[0, 1]$. The tightness will be verified if we show that, for every positive ϵ and η, there exists a positive δ and a number A_0 such that

$$P(\omega(\psi_A, \delta) > \epsilon) < \eta, A \ge A_0.$$

Introduce the processes

$$Q_A^r(t) = A^{-1/2}[\textstyle\sum_{x_i<r} \chi\{x_i + v_iAt > r\} - \sum_{x_i>r} \chi\{x_i + v_i + At < r\}],$$

which is the normed difference of the number of left crossings and the number of right crossings of the level r. First we reduce the proof of the tightness of ψ_A to the proof of the tightness of Q_A^0. Since this reduction uses elements of a similar reduction given by Szatzschneider, our proof here will be concise, and the reader is referred to [8] for more details. Let $0 < \delta < 1$ and denote by M the maximal integer less than δ^{-1}. By denoting $t_k = k\delta$ if $0 \le k \le M$ and $t_{M+1} = 1$ we can write

$$P(\omega(\psi_A, \delta) > \epsilon) \le P(\cup_{k=0}^M \{\sup_{s\in[t_k,t_{k+1}]} |y(As) - y(At_k)| > \frac{\epsilon}{3} A^{1/2}\}).$$

Let $G = \{|y(At_k)| < f(\delta)A^{1/2}, k = 0, 1, \cdots, M+1\}$, where $f(\delta)$ is a function satisfying $\lim_{\delta\to 0} f(\delta) = \infty$ and it will be specified later. Then

$$P(\omega(\psi_A, \delta) > \epsilon) \le P(\cup_{k=0}^M \{\sup_{s\in[t_k,t_{k+1}]}(y(As) - y(At_k)) > \frac{\epsilon}{3} A^{1/2}\}, G)$$
$$+ P\left(\cup_{k=0}^M \left\{\inf_{s\in[t_k,t_{k+1}]}(y(As) - y(At_k)) < -\frac{\epsilon}{3} A^{1/2}\right\}, G\right) + P(G^C).$$

Now

$$\left\{\sup_{s\in[t_k,t_{k+1}]}(y(As) - y(At_k)) > \frac{\epsilon}{3} A^{1/2}\right\}$$
$$\subset \{\sup_{s\in[t_k,t_{k+1}]}(Q_A^{y(At_k)+(\epsilon/3)A^{1/2}}(s) - Q_A^{y(At_k)+(\epsilon/3)A^{1/2}}(t_k))$$
$$> A^{-1/2} N_{y(At_k),\, y(At_k)+(\epsilon/3)A^{1/2}}\}$$

where $N_{y(At_k),y(At_k)+(\epsilon/3)A^{1/2}}$ denotes the number of particles being in the interval $[y(At_k), y(At_k) + (\epsilon/3)A^{1/2}]$ at the time At_k.

LEMMA 3. *Let $t \in [0, 1]$ and $I = [a, b)$ be given and fixed and denote $\varphi_A = \varphi_A(t, I) = \Sigma_i \chi\{x_i(At)A^{-1/2} \in I\}$. Then*

$$\lim_{A\to\infty} A^{-1/2}\varphi_A = |I|$$

in probability.

PROOF. The statement is a straightforward consequence of Lemma 2, since $A^{-1/2}\varphi_A = z_A(t, a) - z_A(t, b)$.

Set $p = [f(\delta)]^{-1}A^{1/2}$, $h_l = lp$ and $g_k = \max\{h_l : h_l \le y(At_k)\}$. Further we restrict our attention to trajectories in $G' = G \cap \{$for every $k \le M$ at time At_k $\min_{|l|\le f(\delta)^2} N_{[h\in, h_{l+1}]} > [f(\delta)]^{-1/2}A^{1/2}\}$. Then, by Lemma 3 (cf. [8] page 164)

$$\left\{\sup_{s\in[t_k,t_{k+1}]}(y(As) - y(At_k)) > \frac{\epsilon}{3} A^{1/2}\right\} \subset \{\sup_{s\in[t_k,t_{k+1}]}(Q_A^{g_k}(s) - Q_A^{g_k}(t_k)) > (\epsilon/12)\}$$

and $|k| \le [f(\delta)]^2$. Consequently

$$P\left(\cup_{k=0}^M \left\{\sup_{s\in[t_k,t_{k+1}]}(y(As) - y(At_k)) > \frac{\epsilon}{3} A^{1/2}\right\}, G\right)$$
$$\le P\left(\cup_{|l|\le[f(\delta)]^2} \cup_{k=0}^M \sup_{s\in[t_k,t_{k+1}]}\left\{(Q_A^{lp}(s) - Q_A^{lp}(t_k)) > \frac{\epsilon}{12}\right\}\right)$$
$$\le P\left(\cup_{|l|\le[f(\delta)]^2}\left\{\omega(Q_A^{lp}, \delta) > \frac{\epsilon}{12}\right\}\right)$$
$$\le [f(\delta)]^2 \mathit{sup}_{|l|\le[f(\delta)]^2} P\left(\omega(Q_A^{lp}, \delta) > \frac{\epsilon}{12}\right).$$

Next we show that for any positive ϵ and η there exist a positive δ and an A_0 such that

$$P(\omega(Q_A^0, \delta) > \epsilon) < \eta \qquad A \ge A_0. \tag{2.10}$$

By assumption (D), our proof will work for any Q_A^r uniformly in r. Since (2.10) implies the existence of a function $f(\delta)$, defined for $\delta \in (0, 1)$ and satisfying $\lim_{\delta\to 0} f(\delta) = \infty$ such that for every positive ϵ and η there exist a positive δ and an A_0 for which

$$[f(\delta)]^2 P(\omega(Q_A^0, \delta) > \epsilon) < \eta \qquad A \geq A_0.$$

(2.10) will also give the desired tightness of the ψ_A's.

Put $L(t) = \Sigma_{i<0}\chi_i\{x_i + v_i t > 0\}$ and $R(t) = \Sigma_{i>0}\chi_i\{x_i + v_i t < 0\}$ and also $Q_A(t) = Q_A^0(t)$. Clearly

$$P(\omega(Q_A, \delta) > \epsilon) = EP(\omega(Q_A, \delta) > \epsilon \mid \mathscr{X}).$$

We also have the decomposition

$$Q_A(t) = A^{-1/2}[L(At) - E(L(At) \mid \mathscr{X})] - A^{-1/2}[R(At) - E(R(At) \mid \mathscr{X})]$$
$$+ A^{-1/2}[E(L(At) \mid \mathscr{X}) - AtEv^+] - A^{-1/2}[E(R(At) \mid \mathscr{X}) - AtEv^-]$$

where $v^+ = \max\{0, v\}$ and $v^- = (-v)^+$ and we used that, by (C), $Ev^+ = Ev^-$. Now it is easy to see that

$$A^{-1/2}[E(L(At) \mid \mathscr{X}) - AtEv^+] = \Theta_A^+(t)$$

and

$$A^{-1/2}[E(L(At) \mid \mathscr{X}) - AtEv^-] = \Theta_A^-(t).$$

Denote the first (and the second) term in the decomposition by $L_A(t)$ (and $R_A(t)$ respectively).

Then

$$P(\omega(Q_A, \delta) > \epsilon) \leq EP\left(\omega(L_A, \delta) > \frac{\epsilon}{4} \mid \mathscr{X}\right)$$
$$+ EP\left(\omega(R_A, \delta) > \frac{\epsilon}{4} \mid \mathscr{X}\right) + P\left(\omega(\Theta_A^+, \delta) > \frac{\epsilon}{4}\right) + P\left(\omega(\Theta_A^-, \delta) > \frac{\epsilon}{4}\right).$$

The third and fourth summands on the right-hand side are small only if δ is small and A is large. This follows from our remark after Lemma 2. Thus it is sufficient to show that, for every positive ϵ and η there exist a positive δ and an A_0 such that

$$EP(\omega(L_A, \delta) > \epsilon \mid \mathscr{X}) \leq \eta, A \geq A_o.$$

For simplicity suppose $Ev^+ = 1$.

An easy calculation yields the inequality

$$E((L_A(t) - L_A(s))^4 \mid \mathscr{X}) \leq K_1[A^{-2}\Sigma_{i<0}p_i + A^{-2}(\Sigma_{i<0}p_i)^2] \tag{2.11}$$

where $s \leq t$ and $p_i = p_i(s, t) = P(-(At)^{-1}x_i < v < (-As)^{-1}x_i)$, $i < 0$. Also we have

$$A^{-1}\Sigma_{i<0}p_i = A^{-1/2}(\Theta_A^+(t) - \Theta_A^+(s)) + (t - s). \tag{2.12}$$

Put $\rho = m^{-1}\delta$ where δ is a positive number and m is a natural number to be chosen later. If

$$\epsilon \leq \Sigma_{i<0}p_i(s, s + \rho) \tag{2.13}$$

for any $s \in [0, 1 - \rho]$, then (2.11) gives

$$E((L_A(t) - L_A(s))^4 \mid \mathscr{X}) \leq \frac{2K_1}{\epsilon}(A^{-1}\Sigma p_i)^2$$

whenever $0 \leq s < s + \rho \leq t \leq 1$. Consequently

$$P(\max_{i\leq m} |L_A(s + i\rho) - L_A(s)| > \lambda \mid \mathscr{X}) \geq \frac{K_2}{\lambda^4\epsilon}(A^{-1}\,\Sigma p_i(s, s + \delta))^2$$

(cf. Theorem 12.2 in [1]). On the other hand, if $0 \le s \le t \le s + \rho \le 1$, then we have

$$|L_A(t) - L_A(s)| \le |L_A(s+\rho) - L_A(s)| + A^{1/2}\rho + \omega(\Theta_A^+, \rho). \tag{2.14}$$

By our remark after Lemma 2, for every positive ϵ and η there exist a δ_0 and A_0 such that

$$P(\omega(\Theta_A^+, \delta_0) < \epsilon/4) > 1 - \eta.$$

Denote $\mathscr{J} = \{\omega(\Theta_A^+, \delta_0) < \epsilon/4\}$. Restrict our attention to the set $\mathscr{J}$. If $\delta < \delta_0$ is fixed then, by (2.12),

$$\frac{\Sigma p_i(s, s+\delta)}{A} \le \delta + A^{-1/2}\frac{\epsilon}{4},$$

which is less than 2δ for A large. thus, on $\mathscr{J}$

$$P(\max_{i \le m} |L_A(s + i\rho) - L_A(s)| > \epsilon \mid \mathscr{X}) \le \frac{K_2}{\epsilon^5} 4\delta^2.$$

If $(K_2/\epsilon^5)\, 4\delta^2 < \eta\delta$ and $A^{1/2}\rho < \epsilon/2$, then (2.14) and (2.15) imply that

$$P(\sup_{t \in [s, s+\delta]} |L_A(t) - L_A(s)| > 4\epsilon \mid \mathscr{X}) < \eta\delta$$

on $\mathscr{J}$. Of course, we should not forget condition (2.13). By (2.12) it will be satisfied if $\epsilon \le A\rho - A^{1/2}\omega(\Theta_A^+, \delta)$ or if $\epsilon \le A\rho - A^{1/2}(\epsilon/4)$. But it is an easy matter to check that, if δ is fixed, then, to any A large, there exists a natural number m such that

$$A^{1/2}\frac{\delta}{m} < \frac{\epsilon}{2} \quad \text{and} \quad \epsilon \le A\frac{\delta}{m} - A^{1/2}\frac{\epsilon}{4}.$$

This completes the proof of (2.16) involving that on $\mathscr{J}$

$$P(\omega(L_A, \delta) > 4\epsilon \mid \mathscr{X}) < \eta,$$

and, also, that

$$P(\omega(L_A, \delta) > 4\epsilon) = EP(\omega(L_A, \delta) > 4\epsilon) < 2\eta.$$

Hence the theorem.

PROOF OF THEOREM 1. First we show that the conditions of Theorem 1 imply (A1), (A2), (A3).

Relations (A1) and (A2) obviously hold with $S(t) = \sigma W(t)$, where $W(t)$ is a standard Wiener process on $-\infty < t < \infty$. In order to prove (A3) we need the following result of Heyde (see [5]).

THEOREM A. *Consider a sequence* $X_1, X_2, \cdots,$ *of* i.i.d. rv's *and its partial sums* $S_n = \sum_{j=1}^n X_j$. *Let* $EX_1 = 0$, $EX_1^2 = 1$, $G(x) = P(X_1 < x)$ *and* $G_n(x) = P(S_n < x\sigma_n n^{1/2})$, *where*

$$\sigma_n^2 = \int_{|x|<n^{1/2}} x^2 dG(x) - \left[\int_{|x|<n^{1/2}} x dG(x)\right]^2.$$

Then we have

$$\sum_{n=1}^{\infty} \sup_x |G_{2^n}(x) - \Phi(x)| < \infty.$$

We will apply Theorem A to the sequence $x_n(0) - n\mu = \sum_{k=1}^n (x_k(0) - x_{k-1}(0) - \mu)$. The sequence $x_k(0) - x_{k-1}(0) - \mu$, $k = 1, 2, \cdots$ is a sequence of i.i.d. random variables with expectation 0 and finite variance σ. We may assume that $\sigma > 0$ since the case $\sigma = 0$ is trivial. In the following estimations we shall apply Theorem A and a well-known estimate on the maximum of partial sums of independent random variables (see, e.g., [6] page 248.) Given any $\epsilon > 0$, the constants $c > 0$ and $A > 0$ can be chosen in such a way that the following relations hold

$$P\left(\sup_{u>1}\left|\frac{S_A(u)}{u}\right|>c\right)=P\left(|x_k(0)-k\mu|>\frac{ck}{A^{1/2}}\quad\text{for some}\quad k\geq A\right)$$

$$\leq 2\sum_{l=[\log A]}^{\infty}P\left(\sup_{2^l<k<2^{l+1}}\left|\sum_{j=2^l+1}^{2^{l+1}}(x_j(0)-x_{j-1}(0)-\mu)\right|>\frac{c}{2}\frac{2^l}{A^{1/2}}\right)$$

$$\leq 2\sum_{l=[\log A]}^{\infty}P\left(\left|\sum_{j=1}^{2^l}(x_j(0)-x_{j-1}(0)-\mu)\right|>\frac{c}{4}\frac{2^l}{A^{1/2}}\right)$$

$$\leq \sum_{l=[\log A]}^{\infty}\left[1-\Phi\left(\frac{c}{5}\frac{2^l}{\sigma A^{1/2}}\right)\right]+4\sum_{l=[\log A]}^{\infty}\sup|G_{2^l}(x)-\Phi(x)|<\epsilon$$

where $G(x)$ is the distribution of $(1/\sigma)[x_1(0)-\mu]$.

The quantity $\sup|S_A(t)/t|$ can similarly be estimated. Thus (A3) is also satisfied. Now we can apply Theorem 2. It remains to prove that the process $\gamma(t)$ has the form given in Theorem 1.

Since $S(t)=\sigma W(t)$, the random variable $(u_1,\cdots,u_N)$ is Gaussian with expectation $(w_1,\cdots,w_N)$ and covariance

$$\operatorname{Cov}(u_i,u_j)=\sigma^2E\int_{-\infty}^{\infty}W(-t_iu)\,dF(u)\int_{-\infty}^{\infty}W(-t_ju)\,dF(u)$$

$$=\sigma^2\int_0^{\infty}\int_0^{\infty}\min(t_iu,t_jv)\,dF(u)\,dF(v)$$

$$+\int_{-\infty}^{0}\int_{-\infty}^{0}\min(t_i|u|,t_j|v|)\,dF(u)\,dF(v)$$

$$=\sigma^2E\min(s_i|v|,s_j|v'|)\chi\{vv'>0\}.$$

Let $\Sigma_1=(\operatorname{Cov}(u_i,u_j))$ $i,j=1,2,\cdots,N$.

Next we show that

$$E\Phi_\Sigma(u_1,\cdots,u_N)=\Phi_{\Sigma+\Sigma_1}(w_1,\cdots,w_N)$$

which completes the proof of Theorem 1 via Theorem 2.

Let $V=(v_1,\cdots,v_n)$ be a normal random variable with expectation 0 variance Σ and independent of $u=(u_1,\cdots,u_N)$. Introducing the notation $w=(w_1,\cdots,w_N)$ we obtain that

$$\Phi_{\Sigma+\Sigma_1^{(w)}}=P(V-(u-W)<w)=P(V<u)$$

$$=EP(V<u\mid u)=E\Phi_\Sigma(u)$$

as we claimed.

3. Remarks.

(a) Our methods seem to be applicable in a variety of one-dimensional generalizations, namely

1. If we allow interdependence among the velocities but their dependence is weak, then, by applying multidimensional central limit theorems and weak-convergence results for weakly dependent sequences, we can get easily the finite dimensional convergence of ψ_A while the proof of the tightness will be more elaborate. If there exists interdependence between the initial positions and velocities, the proof of a similar result seems to be more difficult.

2. As Harris' approach was not restricted to uniform motion neither is ours. For example, if $\mu=1$ and $S(t)=\sigma W(t)$, and the motion of each particle (in case of no collision) is not uniform but rather is $\beta W(t)$, $t>0$ with β a positive parameter and $W(t)$ the standard Wiener-process, then by denoting the path of the 0 labelled atom by $y(t)$ we

obtain that

$$A^{-1/4}y(At) \to \delta(t) \quad \text{when} \quad A \to \infty$$

where $\delta(t)$ is a Gaussian process with $E\delta(t) = 0$ and

$$E\delta(s)\delta(t) = 2\int_0^\infty P(W(s) > \beta^{-1}x)\,dx + 2(\sigma^2 - 1)\int_0^\infty P(W(s) > \beta^{-1}x)P(W(t) > \beta^{-1}x)\,dx.$$

Here the convergence is that of the finite dimensional distributions only. In the problem of tightness we encounter principal difficulties (cf. [3]).

(b) If instead of pointwise particles, we have hard rods, i.e., the particles are intervals of equal length (and of equal masses), the distances between the intervals are the same as the distances in the previous point particle model and the velocities also agree in the two models, then, except for a shift, the path of any fixed rod particle in the collision model will be the same as the path of the corresponding particle in the point particle model. This means that our theorems apply to this case as well.

(c) The fact that in a natural collision model we obtain a non-Wiener limit process for the path of a particle is unexpected. However, it may be the consequence of the invariance of the order of points in a one-dimensional collision model. Dobrushin has proposed to investigate a model in which the particles only collide with some positive probability $\pi(<1)$ if they meet, and with probability $1 - \pi$ they continue their motion without collision. The one-dimensional character of this unexpected phenomenon is supported by a result for two dimensions, where the atoms start from a lattice configuration, as in Szatzschneider's case, and nevertheless the path of a given atom tends to the Wiener process [2].

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Added in proof. Szatzschneider could drop a condition on the initial velocities, which was used by him in [8], in his 1978 paper "A version of the Harris-Spitzer random constant velocity model for infinite systems of particles," *Studia Math.* **63** 171–187.

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