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CHARACTERIZING THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM

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Asymptotic upper and lower bounds are obtained for the uniform measure of the rate of convergence in the central limit theorem using a variety of norming constants. For many distributions the upper and lower bounds are of the same order of magnitude. As easy corollaries we deduce extensive generalizations of the classical characterizations of the rate of convergence in terms of series and order of magnitude conditions.

1. Introduction. Let $X, X_1, X_2, \cdots$ be independent and identically distributed variables with distribution function F, and set $S_n = \sum_1^n X_i$. If $c > 0$ and d are constants, the uniform distance between the probability law of $(S_n - d)/c$ and the standard normal law is defined by

$$\Delta_n(c, d) = \sup_x |P(S_n \le cx + d) - \Phi(x)|,$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2}\, du$. When F is in the domain of attraction of the normal law (designated here by $F \in \text{DN}$) we have $\Delta_n(c_n, d_n) \to 0$ for suitable constants c_n and d_n, and when F is in the domain of partial attraction (designated by $F \in \text{DPN}$),

$$\liminf_{n\to\infty} \Delta_n(c_n, d_n) = 0.$$

In this paper we obtain explicit measures of the rate of convergence for a variety of norming constants c_n and d_n. Our results are in the spirit of Osipov (1968), who showed that when $E|X|^3 = \infty$, $E(X^2) = 1$ and $E(X) = 0$,

$$\Delta_n(n^{1/2}, 0) \asymp \delta_n$$
$$= E[X^2 I(|X| > n^{1/2})] + n^{-1}E[X^4 I(|X| \le n^{1/2})] + n^{-1/2}|E[X^3 I(|X| \le n^{1/2})]| \tag{1}$$

if X has an absolutely continuous distribution, and $\Delta_n(n^{1/2}, 0) \asymp \delta_n + n^{-1/2}$ *if* X is lattice. (The relation $\Delta_n \asymp \delta_n$ asserts that Δ_n/δ_n is bounded away from zero and infinity as $n \to \infty$.) Indeed, it follows easily from our Theorem 2 that (1) holds whenever $E(X^2) = 1$, $E(X) = 0$ and

$$\liminf_{x\to\infty} x^3 P(|X| > x) > 0,$$

and from Theorem 1 that

$$\inf_{c,d} \Delta_n(c, d) \asymp nP(|X| > n^{1/2}) + n^{-1}E[X^4 I(|X| \le n^{1/2})] + n^{-1/2}|E[X^3 I(|X| \le n^{1/2})]|$$

under the same conditions.

Our aim is to provide measures of the rate of convergence which are sufficiently general to imply the well known characterizations in terms of series and orders of magnitude; for these and related results see Friedman, Katz and Koopmans (1966), Ibragimov (1966, 1967), Heyde (1967, 1969, 1970, 1973, 1975), Heyde and Leslie (1972), Davis (1968), Galstyan (1971a, b), Egorov (1973), Lifshits (1976) and Maejima (1978). Therefore we state our results a little differently from Osipov. Section 2 considers the case of finite variance, and some results for variables without finite variance are presented in Section 3. Our result there implies a rate of

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convergence for distributions in DPN. The proofs of our main results are deferred until Section 4.

By way of notation we set $H(x) = P(|X| \le x)$ and let $I(E)$ denote the indicator function of the event E. The symbols C, C_1 and C_2 denote generic positive constants, not necessarily the same at each appearance.

2. The case of finite variance. Suppose $E(X^2) = 1$ and $E(X) = 0$, and set $\mu_n = E[XI(|X| \le n^{1/2})]$ and $\sigma_n^2 = E[X^2 I(|X| \le n^{1/2})]$. Our first result characterizes the rate of convergence using optimal norming constants.

THEOREM 1. *We have*

$$\limsup_{n\to\infty} \Delta_n(n^{1/2}\sigma_n, n\mu_n)/\{nP(|X| > n^{1/2}) + n^{-1}E[X^4 I(|X| \le n^{1/2})] + n^{-1/2}|E[X^3 I(|X| \le n^{1/2})]| + n^{-1/2}\} < \infty, \tag{2}$$

and for any $\lambda_1, \lambda_2, \lambda_3 > 0$,

$$\liminf_{n\to\infty} \{\inf_{c,d} \Delta_n(c, d) + n^{-1}\}/\{nP(|X| > \lambda_1 n^{1/2}) + n^{-1}E[X^4 I(|X| \le \lambda_2 n^{1/2})] + n^{-1/2}|E[X^3 I(|X| \le \lambda_3 n^{1/2})]|\} > 0. \tag{3}$$

The most commonly used norming constants are $c_n = n^{1/2}$ and $d_n = 0$, which generally provide a slightly inferior rate of convergence.

THEOREM 2. *We have*

$$\limsup_{n\to\infty} \Delta_n(n^{1/2}, 0)/\{E[X^2 I(|X| > n^{1/2})] + n^{-1}E[X^4 I(|X| \le n^{1/2})] + n^{-1/2}|E[X^3 I(|X| \le n^{1/2})]| + n^{-1/2}\} < \infty, \tag{4}$$

and for any $\lambda_1, \lambda_2, \lambda_3 > 0$,

$$\liminf_{n\to\infty} \{\Delta_n(n^{1/2}, 0) + n^{-1}\}/\{E[X^2 I(|X| > \lambda_1 n^{1/2})] + n^{-1}E[X^4 I(|X| \le \lambda_2 n^{1/2})] + n^{-1/2}|E[X^3 I(|X| \le \lambda_3 n^{1/2})]|\} > 0. \tag{5}$$

From Theorems 1 and 2 follow some important generalizations of the familiar characterizations of the rate of convergence in the central limit theorem, using either series or order of magnitude conditions. Let $\mathscr{F}$ denote the class of nonincreasing functions $f: [0, \infty) \to (0, \infty)$ such that $xf(x^2)$ is eventually nondecreasing, and let $\mathscr{G}$ denote the class of measurable functions $g: [0, \infty) \to (0, \infty)$ such that $x^{-1}g(x)$ is eventually nonincreasing and

$$\int_0^\infty x^{-2} g(x)\, dx < \infty. \tag{6}$$

With each $g \in \mathscr{G}$ we associate an absolutely continuous, nondecreasing function G defined by

$$G(x) = \int_1^x u^{-1} g(u)\, du, \qquad x \ge 1. \tag{7}$$

COROLLARY 1. *Suppose* $f \in F$. *The following three conditions are equivalent.*

$$\Delta_n(n^{1/2}\sigma_n, 0) = O(f(n)); \tag{8}$$

$$\inf_{c,d} \Delta_n(c, d) = O(f(n)); \tag{9}$$

(10)(i) $P(|X| > x) = O(x^{-2} f(x^2))$ *and*

(ii) $E[X^3 I(|X| \le x)] = O(xf(x^2))$ *as* $x \to \infty$.

The following two conditions are equivalent.

(11) $$\Delta_n(n^{1/2}, 0) = O(f(n));$$

(12)(i) $$E[X^2 I(|X| > x)] = O(f(x^2)) \quad \text{and}$$

(ii) $$E[X^3 I(|X \le x)] = O(xf(x^2)) \qquad \text{as } x \to \infty.$$

Note that (12)(i) implies (10)(i), and that if $x^{1-\delta}f(x^2)$ is eventually nondecreasing for some $\delta > 0$ then (10)(i) implies (10)(ii), since

$$E[|X|^3 I(|X \le x)] \le 3\int_0^x u^2[1 - H(u)]\,du \le C\int_0^x f(u^2)\,du$$
$$\le C_1 x^{1-\delta} f(x^2)\int_0^x u^{\delta-1}\,du = O(xf(x^2)).$$

See Ibragimov (1966, 1967) for results related to Corollary 1.

PROOF. Condition (8) obviously implies (9), and (10) follows from (9) via (3). Condition (10) implies that

(13) $$\Delta_n(n^{1/2}\sigma_n, n\mu_n) = O(f(n))$$

by (2), since $xf(x^2)$ is eventually nondecreasing and, integrating by parts,

$$n^{-1}E[X^4 I(|X| \le n^{1/2})] \le Cn^{-1}\int_0^{n^{1/2}} xf(x^2)\,dx \le C_1 f(n).$$

From (13) follows (8), since

$$|\Delta_n(n^{1/2}\sigma_n, n\mu_n) - \Delta_n(n^{1/2}\sigma_n, 0)| \le \sup_x |\Phi(x + n^{1/2}\mu_n/\sigma_n) - \Phi(x)|$$

$$\le Cn^{1/2}|\mu_n| \le Cn^{1/2}E[|X| I(|X| > n^{1/2})] = C[nP(|X| > n^{1/2}) + n^{1/2}\int_{n^{1/2}}^\infty [1 - H(u)]\,du]$$
$$= O(f(n))$$

under (10). The equivalence of (11) and (12) is proved similarly.

COROLLARY 2. *Suppose $g \in \mathscr{G}$ and define G as in (7). The following three conditions are equivalent.*

(14) $$\Sigma n^{-1}G(n^{1/2})\Delta_n(n^{1/2}\sigma_n, 0) < \infty;$$

(15) $$\Sigma n^{-1}G(n^{1/2})\{\inf_{c,d}\Delta_n(c, d)\} < \infty;$$

(16)(i) $$E[X^2 G(|X|)] < \infty \quad \text{and}$$

(ii) $$\int_1^\infty u^{-2}G(u)\left|\int_{\{|x|\le u\}} x^3\,dF(x)\right|du < \infty.$$

The following two conditions are equivalent.

(17) $$\Sigma n^{-1}g(n^{1/2})\Delta_n(n^{1/2}, 0) < \infty;$$

(18)(i) $$E[X^2 G(|X|)] < \infty \quad \text{and}$$

(ii) $$\int_1^\infty u^{-2}g(u)\left|\int_{\{|x|\le u\}} x^3\,dF(x)\right|du < \infty.$$

If $x^{\delta-1}g(x)$ is eventually nonincreasing for some $\delta > 0$ then (16)(i) implies (16)(ii) and (18)(ii). Note that $2G$ eventually dominates g, since if $x^{-1}g(x)$ is nonincreasing for $x > x_0$ then

$$G(x) \geq \int_{x_0}^{x} u^{-1}g(u)\,du \geq x^{-1}g(x)(x - x_0) \sim g(x). \tag{19}$$

Therefore (14) implies that

$$\Sigma n^{-1}g(n^{1/2})\Delta_n(n^{1/2}\sigma_n, 0) < \infty,$$

and (16)(ii) implies (18)(ii). See Lifshits (1976) and the references therein for results related to Corollary 2.

PROOF. If (15) holds it follows from (3) that

$$\infty > \textstyle\sum_1^\infty G(n^{1/2})P(|X| > n^{1/2}) \geq C \displaystyle\int_1^\infty G(u^{1/2})\,du \int_{u^{1/2}}^\infty dH(x)$$

$$= 2C \int_1^\infty dH(x) \int_1^x uG(u)\,du.$$

Now, (19) implies that with $a(x) = x^{-3/2}G(x)$ we have

$$a'(x) = x^{-5/2}(g(x) - 3G(x)/2) \leq 0$$

for large x, so that $x^{-3/2}G(x)$ is eventually nonincreasing. Therefore

$$\infty > \int_1^\infty dH(x) \int_1^x u^{5/2}[u^{-3/2}G(u)]\,du \geq C \int_1^\infty x^2 G(x)\,dH(x),$$

proving (16)(i). Condition (16)(ii) is proved similarly. Conversely from (16) we deduce that

$$\Sigma G(n^{1/2})P(|X| > n^{1/2}) \leq C \int_1^\infty dH(x) \int_1^x uG(u)\,du \leq C \int_1^\infty x^2 G(x)\,dH(x) < \infty,$$

$$\Sigma n^{-2}G(n^{1/2})E[X^4 I(|X| \leq n^{1/2})] \leq C \int_1^\infty x^4\,dH(x) \int_x^\infty u^{-3}G(u)\,du$$

$$\leq C_1 \int_1^\infty x^{5/2}G(x)\,dH(x) \int_x^\infty u^{-3/2}\,du$$

$$\leq C_2 \int_1^\infty x^2 G(x)\,dH(x) < \infty, \qquad \text{and}$$

$$\Sigma n^{-3/2}G(n^{1/2}) \leq C \int_1^\infty u^{-2}G(u)\,du \leq C \int_1^\infty u^{-2}g(u)\,du < \infty,$$

on integrating by parts and using (6). The result (2) now implies that

$$\Sigma n^{-1}G(n^{1/2})\Delta_n(n^{1/2}\sigma_n, n\mu_n) < \infty,$$

and as before this implies (14). The equivalence of (17) and (18) is proved similarly.

Egorov (1973) and Heyde (1973) characterized the condition of finite variance by showing that if F is any distribution in DN, then $E(X^2) < \infty$ if and only if

$$\Sigma n^{-1}\{\inf_{c,d} \Delta_n(c, d)\} < \infty.$$

(A new proof of this result is presented in Section 3. We should point out that the result proved by Heyde is actually rather stronger, in that he did not assume that $F \in$ DN.) This leads us to ask whether there exists a sequence $\epsilon_n \to 0$ such that whenever $E(X) = 0$ and $E(X^2) = 1$, $\Sigma n^{-1}\epsilon_n \Delta_n(n^{1/2}, 0) < \infty$ although $\Sigma n^{-1}\epsilon_n = \infty$. Cohn (1974) has obtained such a rate of convergence for the law of the iterated logarithm. We show next that no such rate can exist for the central limit theorem. This negative answer spotlights the faster rate of convergence obtained using the norming constants $n^{1/2}\sigma_n$ instead of $n^{1/2}$.

COROLLARY 3. *Suppose $g \in G$ and that for all absolutely continuous, symmetric X with $E(X^2) = 1$, $\Sigma n^{-1} g(n^{1/2})\Delta_n(n^{1/2}, 0) < \infty$. Then $\Sigma n^{-1} g(n^{1/2}) < \infty$.*

PROOF. If false then

$$\infty = \sum_{n=2}^{\infty} \int_{n-1}^{n} n^{-1} g(n^{1/2})\, du \le C \sum_{n=2}^{\infty} \int_{n-1}^{n} u^{-1} g(u^{1/2})\, du = 2C \int_{1}^{\infty} u^{-1} g(u)\, du.$$

Therefore we may choose an absolutely continuous, symmetric X with $E(X^2) = 1$ but $E[X^2 G(|X|)] = \infty$. For such a distribution we have by Corollary 2 that $\Sigma n^{-1} g(n^{1/2})\Delta_n(n^{1/2}, 0) = \infty$.

3. Characterizations without the assumption of finite variance. In this section we remove the restriction $E(X^2) < \infty$ and obtain characterizations of the rate of convergence of the form

$$\Delta_n(c_n, d_n) \asymp nP(|X| > c_n).$$

Thus the rate is expressed as an explicit function of tail behaviour. In the case $E(X^2) < \infty$, all the results here may be obtained from Section 2.

Let $V(x) = \int_{\{|u| \le x\}} u^2\, dF(u)$ and $a(x) = \sup\{a \mid a^{-2} V(a) \ge x^{-1}\}$. For all distributions, $x^{-2} V(x) \to 0$ as $x \to \infty$, and so $a(x)$ is well defined for large x. For such values of x

$$xa(x)^{-2} V(a(x)) = 1, \tag{20}$$

even for continuous distributions. Let

$$a_n = a(n),\ v_n = E[XI(|X| \le a_n)] \quad \text{and} \quad b_n^2 = n(V(a_n) - v_n^2).$$

Our first result provides an upper bound on the rate of convergence. To obtain (22) it is not necessary to assume that $F \in$ DPN.

THEOREM 3. *If*

$$\lambda^3 \liminf_{x\to\infty} P(|X| > \lambda x)/P(|X| > x) \to \infty \tag{21}$$

as $\lambda \to \infty$ then

$$\limsup_{n\to\infty} \Delta_n(b_n, nv_n)/nP(|X| > a_n) < \infty. \tag{22}$$

If $F \in$ DN then $nP(|X| > a_n) \to 0$, and if $F \in$ DPN then

$$\liminf_{n\to\infty} nP(|X| > a_n) = 0. \tag{23}$$

Condition (21) implies that $E|X|^3 = \infty$. To see this, observe that we may choose $\lambda > 1$ and x_λ so large that for all $x \ge x_\lambda$,

$$P(|X| > \lambda x) > \lambda^{-3} P(|X| > x).$$

Then for $x \geq x_\lambda$,

$$\int_{\lambda x}^{\infty} u^2 P(|X| > u)\, du = \lambda^3 \int_x^{\infty} u^2 P(|X| > \lambda u)\, du > \int_x^{\infty} u^2 P(|X| > u)\, du,$$

which is impossible unless $E|X|^3 = \infty$.

Condition (21) holds for many common distributions with $E|X|^3 = \infty$. For example, if $P(|X| > x) = x^{-\alpha}L(x)$ where L is slowly varying at infinity then (21) holds if and only if $\alpha < 3$.

If $E(X^2) = \infty$ (or if $E|X| < \infty$ and $E(X) = 0$) then $\nu_n^2/V(a_n) \to 0$ (see page 80 of Ibragimov and Linnik (1971)) and so $b_n^2 \sim nV(a_n) = a_n^2$. Condition (21) now implies that $P(|X| > a_n)/P(|X| > b_n)$ is bounded as $n \to \infty$, and it follows from (22) that

$$\limsup_{n\to\infty} \Delta_n(b_n, n\nu_n)/nP(|X| > b_n) < \infty.$$

Next we obtain a lower bound of the same order of magnitude as the upper bound in Theorem 3. In this case it is necessary to restrict our attention to $F \in \mathrm{DN}$.

THEOREM 4. *If $F \in \mathrm{DN}$ then for any constants c_n such that $\Delta_n(c_n, d_n) \to 0$ for suitable d_n,*

$$\liminf_{n\to\infty} \{\inf_{c,d} \Delta_n(c, d) + n^{-1}\}/nP(|X| > \lambda c_n) > 0 \tag{24}$$

for all $\lambda > 0$.

As a corollary we easily deduce the characterization of $E(X^2) < \infty$ first obtained by Egorov (1973) and Heyde (1973). (Heyde's result is actually rather stronger than that proved here.)

COROLLARY 4. *Suppose $F \in \mathrm{DN}$. Then $\sum_1^\infty n^{-1} \{\inf_{c,d} \Delta_n(c, d)\} < \infty$ if and only if $E(X^2) < \infty$.*

PROOF. Sufficiency follows from Corollary 2. To prove necessity, we may assume that (24) holds for an increasing sequence $\{c_n\}$. Then

$$\begin{aligned} \infty > \textstyle\sum_1^\infty P(|X| > c_n) &= \textstyle\sum_{n=1}^\infty \sum_{j=n}^\infty P(c_j < |X| \leq c_{j+1}) \\ &= \textstyle\sum_{j=1}^\infty jP(c_j < |X| \leq c_{j+1}) \geq \sum_{j=1}^\infty jP(c_{j-1} < |X| \leq c_j) - 2, \end{aligned} \tag{25}$$

where $c_0 = 0$. *Now,*

$$V(c_n) = \textstyle\sum_{j=1}^n \displaystyle\int_{\{c_{j-1} < |x| \leq c_j\}} x^2\, dF(x) \leq \textstyle\sum_{j=1}^n c_j^2 P(c_{j-1} < |X| \leq c_j).$$

But $c_n^{-2}V(c_n) \sim n^{-1}$ and so

$$V(c_n) \leq C \textstyle\sum_{j=1}^n jV(c_j)P(c_{j-1} < |X| \leq c_j). \tag{26}$$

If $E(X^2) = \infty$ then $V(x) \to \infty$, and the inequalities (25) and Kronecker's lemma imply that

$$V(c_n)^{-1} \textstyle\sum_{j=1}^n jV(c_j)P(c_{j-1} < |X| \leq c_j) \to 0,$$

contradicting (26). Therefore $E(X^2) < \infty$.

4. The proofs.

PROOF OF THEOREM 1. First we demonstrate (2). Let $X_n' = XI(|X| \leq n^{1/2})$, $X_{ni}' = X_i I(|X_i| \leq n^{1/2})$ and $S_n' = \sum_{i=1}^n X_{ni}'$. Then

$$\begin{aligned} \Delta_n(n^{1/2}\sigma_n, n\mu_n) &\leq \sup_x |P(S_n' \leq n^{1/2}\sigma_n x + n\mu_n) - \Phi(x)| + nP(|X| > n^{1/2}) \\ &= \Delta_n' + nP(|X| > n^{1/2}), \end{aligned}$$

say. Therefore it suffices to prove (2) with Δ_n' replacing $\Delta_n(n^{1/2}\sigma_n, n\mu_n)$. Let

$$\alpha_n(t) = \exp[-\tfrac{1}{2}t^2\sigma_n^2(1 + \gamma_n(t)) + it\mu_n]$$

denote the characteristic function of X_n'. Writing $\log \alpha_n(t) = \alpha_n(t) - 1 + r_n(t)$ where $|r_n(t)| \le |\alpha_n(t) - 1|^2/[1 - |\alpha_n(t) - 1|]$ for $|\alpha_n(t) - 1| < 1$, we see that

$$\begin{aligned} \tfrac{1}{2}t^2\sigma_n^2|\gamma_n(t)| &= |\log \alpha_n(t) + \tfrac{1}{2}t^2\sigma_n^2 - it\mu_n| \\ &\le |E[\cos(tX_n') - 1 + \tfrac{1}{2}(tX_n')^2]| + |E[\sin(tX_n') - tX_n']| + |r_n(t)|. \end{aligned} \tag{27}$$

For $t > 0$ we have the estimates

$$E|\cos(tX_n') - 1 + \tfrac{1}{2}(tX_n')^2| \le E[|tX_n'|^4 I(|X_n'| \le t^{-1})] + E[|tX_n'|^2 I(|X_n'| > t^{-1})];$$

$$E|\sin(tX_n') - tX_n'| \le E[|tX_n'|^3 I(|X_n'| \le t^{-1})] + 2E[|tX_n'| I|X_n'| > t^{-1})]$$

and

$$\begin{aligned} |\alpha_n(t) - 1| &\le |E[\cos(tX_n') - 1]| + |E[\sin(tX_n') - tX_n']| + t|\mu_n| \\ &\le \tfrac{1}{2}t^2\sigma_n^2 + t^3E[|X|^3 I(|X| < t^{-1})] + 2tE[|X| I(|X| > t^{-1})] + t|\mu_n|. \end{aligned}$$

Therefore $|\alpha_n(t) - 1|$ may be made small uniformly in n by choosing t sufficiently small, and by (27) we have for all n and all small t,

$$\begin{aligned} |\gamma_n(t)| \le C\big(&t^2E[X^4 I(|X| \le t^{-1})] + E[X^2 I(|X| > t^{-1})] + tE[|X|^3 I(|X| \le t^{-1})] \\ &+ t^{-1}E[|X|\, I(|X| > t^{-1})] + t^2 + \{t^2E[|X|^3 I(|X| < t^{-1})]\}^2 + \{E[|X|\, I(|X| > t^{-1})]\}^2 + \mu\big). \end{aligned}$$

Hence we may suppose that there is an $\epsilon > 0$ such that for all $|t| \le \epsilon n^{1/2}$ and all sufficiently large n,

$$|\gamma_n(t/n^{1/2}\sigma_n)| \le \tfrac{1}{2}.$$

The variable $(S_n' - n\mu_n)/n^{1/2}\sigma_n$ has characteristic function

$$\beta_n(t) = \alpha_n(t/n^{1/2}\sigma_n)^n \exp(-itn^{1/2}\mu_n/\sigma_n),$$

and

$$\begin{aligned} |\beta_n(t) - e^{-1/2t^2}| &= |\exp[-\tfrac{1}{2}t^2\gamma_n(t/n^{1/2}\sigma_n)] - 1|\, e^{-1/2t^2} \\ &\le \tfrac{1}{2}t^2\, |\gamma_n(t/n^{1/2}\sigma_n)| \exp[-\tfrac{1}{2}t^2 + \tfrac{1}{2}t^2\, |\gamma_n(t/n^{1/2}\sigma_n)|] \le \tfrac{1}{2}t^2\, |\gamma_n(t/n^{1/2}\sigma_n)|\, e^{-t^2/4} \end{aligned} \tag{28}$$

if $|t| \le \epsilon n^{1/2}$. We also have the estimates

$$|\alpha_n(t) - 1| \le |E[1 - \cos tX_n']| + |E[\sin tX_n']| \le \tfrac{1}{2}t^2\sigma_n^2 + \sigma_n|t|;$$

$$|\alpha_n(t) - 1| \le \tfrac{1}{2}t^2\sigma_n^2 + |t|^5E[|X|^5 I(|X| \le n^{1/2})] + |t|^3|E[X^3 I(|X| \le n^{1/2})]| + |t\mu_n|;$$

and

$$\mu_n^2 \le P(|X| > n^{1/12}),$$

the last by Hölder's inequality. Combining these we deduce that for large n, small ϵ (independent of n) and $|t| \le \epsilon n^{1/2}$,

$$\begin{aligned} n\,|r_n(t/n^{1/2}\sigma_n)| &\le nC\big(n^{-2}t^4 + n^{-5}\{E[|X|^5 I(|X| \le n^{1/2})]\}^2 t^{10} \\ &\quad + n^{-3}|E[X^3 I(|X| \le n^{1/2})]|^2 t^6 + P(|X| > n^{1/2})t^2\big) \\ &\le C\big(n^{-1}t^4 + n^{-1}E[X^4 I(|X| \le n^{1/2})]t^{10} + n^{-1/2}\,|E[X^3 I(|X| \le n^{1/2})]|\, t^6 \\ &\quad + nP(|X| > n^{1/2})t^2\big). \end{aligned}$$

Next we estimate

$$
\begin{aligned}
\tfrac{1}{2}t^2 \mid \gamma_n(t/n^{1/2}\sigma_n)| &= \mid n \log \alpha_n(t/n^{1/2}\sigma_n) + \tfrac{1}{2}t^2 - itn^{1/2}\mu_n/\sigma_n \mid \\
&\le \mid n\{\alpha_n(t/n^{1/2}\sigma_n) - 1\} + \tfrac{1}{2}t^2 - itn^{1/2}\mu_n/\sigma_n \mid + n \mid r_n(t/n^{1/2}\sigma_n) \mid \\
&= \mid nE[\cos(tX'_n/n^{1/2}\sigma_n) - 1 + \tfrac{1}{2}(tX'_n/n^{1/2}\sigma_n)^2] + inE[\sin(tX'_n/n^{1/2}\sigma_n) \\
&\qquad -(tX'_n n^{1/2}\sigma_n) + \tfrac{1}{6}(tX'_n/n^{1/2}\sigma_n)^3] \\
&\qquad - \tfrac{1}{6} in^{-1/2}\sigma_n^{-3}t^3 E[X^3 I(\mid X \mid \le n^{1/2})] \mid + n \mid r_n(t/n^{1/2}\sigma_n) \mid \\
&\le C(n^{-1}E[X^4 I(\mid X \mid \le n^{1/2})](t^4 + t^{10}) \\
&\qquad + n^{-1/2} \mid E[X^3 I(\mid X \mid \le n^{1/2})] \mid (\mid t \mid^3 + t^6) \\
&\qquad + nP(\mid X \mid > n^{1/2})t^2 + n^{-1}t^4).
\end{aligned}
\tag{29}
$$

Finally we note that the smoothing inequality,

$$
\Delta'_n \le C\left(\int_{-\epsilon n^{1/2}}^{\epsilon n^{1/2}} \mid t^{-1}[\beta_n(t) - e^{-t^2/2}] \mid dt + n^{-1/2}\right)
$$

(see e.g., Petrov (1975), Theorem 2, page 109), and the estimates (28) and (29) combine to prove that (2) holds with Δ'_n replacing $\Delta_n(n^{1/2}\sigma_n, n\mu_n)$.

In proving (3) we choose c_n and d_n so that $\inf_{c,d}\Delta_n(c, d) \ge \tfrac{1}{2}\Delta_n(c_n, d_n)$ and $c_n > 0$. The techniques used to prove the inequality (3.20) of Ibragimov (1966) imply that for fixed $z > 0$,

$$
\left| \int_0^1 (1 - t)(\psi_n(tz)e^{(tz)^2/2} - 1)\, dt \right| \le C\Delta_n(c_n, d_n),
$$

ψ_n being the characteristic function of $(S_n - d_n)/c_n$. (The result holds for all $z > 0$; see Heyde (1973).) Let $\phi(t) = \exp(-\tfrac{1}{2}t^2(1 + \gamma(t))]$ denote the characteristic function of X. Then

$$
\begin{aligned}
\psi_n(tz)e^{(tz)^2/2} - 1 &= \exp\{\tfrac{1}{2}(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - itzd_n/c_n\} - 1 \\
&= \tfrac{1}{2}(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - itzd_n/c_n \\
&\qquad + k_n(t, z) \mid \tfrac{1}{2}(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - itzd_n/c_n \mid^2,
\end{aligned}
$$

where

$$
\mid k_n(t, z) \mid \le \exp\{\mid \tfrac{1}{2}(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - itzd_n/c_n \mid\}
$$

and so is bounded uniformly in $t \in (0, 1]$ as $n \to \infty$. Therefore

$$
\begin{aligned}
&\left| \int_0^1 (1 - t)\{(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - 2itzd_n/c_n\}\, dt \right. \\
&\qquad \left. + l_n(z) \int_0^1 \left| \frac{1}{2}(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - itzd_n/c_n \right|^2 dt \right| \le C\Delta_n(c_n, d_n),
\end{aligned}
\tag{30}
$$

where $\mid l_n(z) \mid$ is bounded as $n \to \infty$. But

$$
\begin{aligned}
&\int_0^1 \left| \frac{1}{2}(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - itzd_n/c_n \right|^2 dt \\
&\qquad = (1 + o(1)) \int_0^1 \mid \psi_n(tz)e^{(tz)^2/2} - 1 \mid^2 dt \le C \int_{-\infty}^{\infty} t^{-2} \mid \psi_n(t) - e^{-t^2/2} \mid^2 dt
\end{aligned}
$$

$$= C_1 \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2 \, dx \le C_1 \Delta_n(c_n, d_n) \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) | \, dx$$

$$\le C_2 \Delta_n(c_n, d_n),$$

using Parseval's equality, where $F_n(x) = P(S_n \le c_n x + d_n)$. It now follows from (30) that

$$\left| \int_0^1 (1 - t)\{(tz)^2[1 - nc_n^{-2}(1 + \gamma(tz/c_n))] - 2itzd_n/c_n\} \, dt \right| \le C\Delta_n(c_n, d_n), \tag{31}$$

and so

$$\left| \int_0^1 (1 - t)\{(tz)^2 nc_n^{-2}[\gamma(2tz/c_n) - \gamma(tz/c_n)] - itzd_n/c_n\} \, dt \right| \le C\Delta_n(c_n, d_n).$$

Now,

$$-\tfrac{1}{2}(tz/c_n)^2(1 + \gamma(tz/c_n)) = \log \phi(tz/c_n)$$
$$= \phi(tz/c_n) - 1 + m_n(t, z) \, | \phi(tz/c_n) - 1 |^2$$

where $| m_n(t, z) |$ is bounded uniformly in $t \in (0, 1]$ as $n \to \infty$, and also

$$\int_0^1 | \phi(tz/c_n) - 1 |^2 \, dt = O(n^{-2})$$

since $c_n \sim n^{1/2}$. Consequently

$$\left| \int_0^1 (1 - t)\{n[\phi(2tz/c_n) - 1 - 4(\phi(tz/c_n) - 1)] + 2itzd_n/c_n\} \, dt \right| \le C[\Delta_n(c_n, d_n) + n^{-1}]. \tag{32}$$

Taking real parts we see that

$$C[\Delta_n(c_n, d_n) + n^{-1}] \ge nE\left[\int_0^1 (1 - t)(3 - 4\cos(tzX/c_n) + \cos(2tzX/c_n)) \, dt \right]$$
$$= 2nE\left[\int_0^1 (1 - t)(1 - \cos(tzX/c_n))^2 \, dt \right].$$

The function $a(\theta) = \int_0^1 (1 - t)(1 - \cos \theta t)^2 \, dt$ is positive for $\theta > 0$, and $a(\theta) \to 3/4$ as $\theta \to \infty$. Also $a(\theta) \ge C\theta^4$ for $| \theta | \le 1$. From these observations we deduce that for any $\lambda > 0$,

$$C[\Delta_n(c_n \, d_n) + n^{-1}] \ge nE[a(X/c_n)I(| X | > \tfrac{1}{2} \lambda c_n)] \ge C_1 nP(| X | > \lambda n^{1/2}), \tag{33}$$

and

$$C[\Delta_n(c_n, d_n) + n^{-1}] \ge n^{-1}E[X^4 I(| X | \le \lambda n^{1/2})]. \tag{34}$$

(Let $z = 1/2\lambda$ for the last result.)

Returning to (31) and using an argument exactly analogous to that used to obtain (32), we see that

$$\left| \int_0^1 (1 - t)\left\{ \frac{1}{2}(tz)^2 + n[\phi(tz/c_n) - 1] - itzd_n/c_n \right\} dt \right| \le C[\Delta_n(c_n, d_n) + n^{-1}].$$

Taking imaginary parts we obtain

$$\left| nE\left[\int_0^1 (1-t)\sin(tzX/c_n)\,dt\right] - zd_n/6c_n \right| \le C[\Delta_n(c_n, d_n) + n^{-1}].$$

Now,

$$E\left[\int_0^1 (1-t)\sin(tzX/c_n)\,dt\right] = E\left[\left\{\int_0^1 (1-t)\sin(tzX/c_n)\,dt\right\} I(|X| > \lambda n^{1/2})\right]$$

$$+ E\left[\left\{\int_0^1 (1-t)(\sin(tzX/c_n) - (tzX/c_n) + \frac{1}{6}(tzX/c_n)^3\,dt\right\} I(|X| \le \lambda n^{1/2})\right]$$

$$-\frac{1}{6}(z/c_n)^3 E[X^3 I(|X| \le \lambda n^{1/2})] \int_0^1 t^3(1-t)\,dt + z\mu_n'/6c_n = A_n + B_n + C_n + z\mu_n'/6c_n,$$

say, where $\mu_n' = E[XI(|X| \le \lambda n^{1/2})]$. Since $|A_n| \le P(|X| > \lambda n^{1/2})$ and $|B_n| \le Cn^{-2}E[X^4 I(|X| \le \lambda n^{1/2})]$ then in view of (33) and (34) we have

$$C[\Delta_n(c_n, d_n) + n^{-1}] \ge |n(z/c_n)^3 E[X^3 I(|X| \le \lambda n^{1/2})] + 20z(d_n - n\mu_n')/c_n| = b_n(z),$$

say. By considering $b_n(2) - 2b_n(1)$ we deduce that

$$C[\Delta_n(c_n, d_n) + n^{-1}] \ge n^{-1/2} |E[X^3 I(|X| \le \lambda n^{1/2})]|,$$

and (3) follows from this, (33) and (34).

PROOF OF THEOREM 2. The result (4) may be proved directly from (2), and so we concentrate on (5). In view of (3) it suffices to prove that

$$E[X^2 I(|X| > \lambda n^{1/2})] \le C[\Delta_n(n^{1/2}, 0) + n^{-1}] = C[\Delta_n^* + n^{-1}],$$

say. Following the argument above we obtain successively

$$\left|\int_0^1 (1-t)(\phi(t/n^{1/2})^n e^{t^2/2} - 1)\,dt\right| \le C\Delta_n^*;$$

$$\left|\int_0^1 (1-t)\left[\frac{1}{2}t^2 + n(\phi(t/n^{1/2}) - 1)\right] dt\right| \le C[\Delta_n^* + n^{-1}];$$

and

$$C[\Delta_n^* + n^{-1}] \ge \left|\int_0^1 (1-t)\left\{\frac{1}{2}t^2 + nE[\cos(tX/n^{1/2}) - 1]\right\} dt\right| = nE[b(X/n^{1/2})],$$

where the function $b(\theta) = \int_0^1 (1-t)(\cos\theta t - 1 + \frac{1}{2}(\theta t)^2\,dt$ is positive for $\theta > 0$, and $b(\theta)/\theta^2 \to \frac{1}{24}$ as $\theta \to \infty$. Therefore for any $\lambda > 0$,

$$C[\Delta_n^* + n^{-1}] \ge nE[b(X/n^{1/2})I(|X| > \lambda n^{1/2})] \ge C_1 E[X^2 I(|X| > \lambda n^{1/2})].$$

PROOF OF THEOREM 3. Let $X_n'' = XI(|X| \le a_n)$, $X_{ni}'' = X_i I(|X_i| \le a_n)$, $S_n'' = \sum_1^n X_{ni}''$ and

$$\Delta_n'' = \sup_x |P(S_n'' - n\nu_n \le b_n x) - \Phi(x)|.$$

Then

$$\Delta_n(b_n, n\nu_n) \le \Delta_n'' + nP(|X| > a_n).$$

The Berry-Esseén theorem implies that

$$\begin{aligned}\Delta_n'' &\le 3nE|X_n'' - \nu_n|^3/[n(V(a_n) - \nu_n^2)]^{3/2} \\ &\le CnE|X_n''|^3/[nV(a_n)]^{3/2} = CnE|X_n''|^3/a_n^3,\end{aligned}$$

using (20). Theorem 1 of Maller (1977) asserts that if (21) holds (and $E|X|^3 = \infty$) then

$$\limsup_{x\to\infty} E[|X|^3 I(|X| \le x)]/x^3 P(|X| > x) < \infty.$$

Condition (22) follows from this and the estimates above.

Lévy's characterizations of DN and DPN are that $F \in$ DN if and only if

$$\rho(x) = x^2 P(|X| > x)/V(x) \to 0$$

as $x \to \infty$, and $F \in$ DPN if and only if $\liminf_{x\to\infty} \rho(x) = 0$. In the first case $xP(|X| > a(x)) = \rho(a(x)) \to 0$. In the second we may choose $y_k \to \infty$ such that $\rho(y_k) \to 0$. Let $x_k = y_k^2/V(y_k)$. Then $x_k \to \infty$, $y_k \le a(x_k)$ and

$$\rho(a(x_k)) = x_k P(|X| > a(x_k)) \le x_k P(|X| > y_k) = \rho(y_k) \to 0.$$

If n_k is the integer part of $x_k + 1$ then

$$\rho(a(x_k)) = x_k P(|X| > a(x_k)) \ge (1 - n_k^{-1})\rho(a_{n_k}).$$

This proves that $\rho(a_{n_k}) \to 0$, and implies (23).

Theorem 4 is established using the techniques leading to (33) in the proof of Theorem 1. By employing the usual symmetrization procedures it is only necessary to consider a symmetric distribution. Note that a symmetric X has characteristic function

$$\phi(t) = \exp[-\tfrac{1}{2}t^2 V(t^{-1})(1 + o(1))]$$

as $t \downarrow 0$, and that V is slowly varying at infinity (see pages 83, 91 of Ibragimov and Linnik (1971)). Bearing in mind the properties of slowly varying functions (see Seneta (1976)) only minor modifications to the proof of Theorem 1 are necessary.

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