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THE DEGENERATE NEUMANN PROBLEM AND DEGENERATE DIFFUSIONS WITH VENTTSEL'S BOUNDARY CONDITIONS

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A stochastic solution of the Neumann problem is obtained, when the second order elliptic operator L is degenerate at the boundary of the domain. Let D be a domain in R^n with the smooth boundary ∂D, and the second order elliptic operator L be defined in R^n. We construct a diffusion $X^r(t) = (\mathfrak{G}^r, D(\mathfrak{G}^r))$ in $\bar{D}$ such that (i) $D(\mathfrak{G}^r) \supset D(A^r) = \{f \in C^2(D); \partial f/\partial \nu = 0 \text{ for } x \in D\}$, (ii) $f \in D(A^r) \Rightarrow \mathfrak{G}^r f = Lf$. With that diffusion, the stochastic solution of our Neumann problem is defined, and the existence and the uniqueness conditions of that are obtained. The analytic meaning of our stochastic solution is explained. The diffusions in $\bar{D}$, satisfying the other Venttsel boundary conditions are also constructed, which are useful for the degenerate third boundary value problems.

This paper treats the following Neumann problem in the case where the elliptic second order differential operator L is possibly degenerate at $D \cup \partial D$:

$$\text{(N)} \qquad Lu = F \quad \text{in } D, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = G \quad \text{in} \quad \overline{\Sigma_2 \cup \Sigma_3} \subset \partial D,$$

where D is an open bounded domain in R^n with the smooth boundary ∂D, $L = ½ \sum_{i,j=1}^n a_{ij}(x)(\partial^2/\partial x_i \partial x_j) + \sum_{i=1}^n b_i(x)(\partial/\partial x_i)$, ν is the inward normal vector to ∂D, and a part of the boundary $\overline{\Sigma_2 \cup \Sigma_3}$ is defined later.

When L is nondegenerate in $D \cup \partial D$, Ikeda [6] gave the stochastic solution of (N), and Freidlin [3] extended Ikeda's result to the case where L is degenerate inside D. Thus, our interest comes to the remaining case where L may be degenerate at $D \cup \partial D$, especially at the boundary ∂D.

As far as the author knows, no study has been done with respect to such a Neumann problem. But, the particular Dirichlet problem

$$Lu = F \quad \text{in } D, \quad \text{and} \quad u = G \quad \text{in } \overline{\Sigma_2 \cup \Sigma_3} \subset \partial D$$

is called the "Fichera problem" when L is degenerate at ∂D. It has been studied by analytic methods [2, 9, 14, 15, etc.] and by probabilistic methods [4, 5, 19, etc.]. So we apply to our Neumann problem (N) the results which have been obtained in the Fichera problem.

In order to precisely formulate the Fichera problem, Fichera [2] and the others [9, 14, etc.] divided the boundary ∂D into four disjoint subsets:

$$\Sigma_3 = \{x \in \partial D; (\textstyle\sum_{i,j=1}^n a_{ij}\nu_i\nu_j)(x) > 0\}$$

$$\Sigma_2 = \{x \in \partial D - \Sigma_3; \left[\sum_{i,j=1}^n (b_i - \frac{1}{2}\frac{\partial}{\partial x_j} a_{ij})\nu_i\right](x) < 0\}$$

$$\Sigma_1 = \{x \in \partial D - \Sigma_3; [\textstyle\sum_{i,j=1}^n (b_i - \frac{1}{2}\frac{\partial}{\partial x_j} a_{ij})\nu_i](x) > 0\}$$

$$\Sigma_0 = \{x \in \partial D - \Sigma_3; [\textstyle\sum_{i,j=1}^n (b_i - \frac{1}{2}\frac{\partial}{\partial x_j} a_{ij})\nu_i](x) = 0\}.$$

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On the other hand, Pinsky [17] has made clear the probabilistic meaning of $\Sigma_0 - \Sigma_3$. Let $\tau \equiv \inf\{t \geqq 0; x(t) \notin D\}$, where $x(t)$ is the diffusion corresponding to L. He proved that $P_x\{x(\tau) \in (\Sigma_0 \cup \Sigma_1)^0, \tau < \infty\} = 0$ for any $x \in D$. Roughly speaking, diffusions behave near $\Sigma_0 - \Sigma_3$ like the natural boundary, the entrance, the exit, and the regular in the one dimensional case, respectively. Our approach is based on those properties of $\Sigma_0 - \Sigma_3$.

In Section 1, we construct the diffusion X on the upper half space $\bar{G}$ of R^n with the infinitesimal operator $(\mathfrak{G}, D(\mathfrak{G}))$ such that

(i) $D(\mathfrak{G}) \supset D(A) \equiv \{f \in C_b^2(\bar{G}); Af = 0 \text{ at } \partial G\}$,
(ii) $f \in D(A) \Rightarrow f = Lf$ on $\bar{G}$,

where A is the Venttsel type boundary operator defined by (1.1). In the case $\partial G = \Sigma_3$, Watanabe [21] constructed such the diffusion stochastically. Thus our construction is done in the case $\partial G \neq \Sigma_3$. In Section 2, we discuss the usefulness of Venttsel's boundary condition. A Venttsel's boundary condition corresponds to a diffusion in the case of Σ_3, but it is not necessarily true in the case of $\Sigma_0 - \Sigma_2$.

In Section 3, we construct the diffusion in $\bar{D}$ associated with the boundary condition (3.2). Using that diffusion, we give the stochastic solution of (N) in Section 4. The meaning of the stochastic solution of (N) (Definition 4.3) is natural from the viewpoint of the probabilist, but its analytic meaning is not clear. In Section 5, we discuss that problem. In the Appendix, we replace the stochastic conditions of the main theorems (Theorems 4.1 and 4.2) by the conditions on the coefficients of L.

Preliminaries. Let D be a subset in R^n, and let its boundary be denoted by ∂D. The ith component of the vector x is denoted by x_i, and the transformed matrix of the matrix A is denoted by A^*. $D_i f$, $D_{ij} f$ $(i, j = 1, \cdots, n)$ denote $\partial f/\partial x_i$, $\partial^2 f/\partial x_i \partial x_j$, respectively. $C_b(\bar{D})$ is the set of bounded continuous functions on $\bar{D}$, $C_b^1(\bar{D})$ is the set of $C_b(\bar{D})$-functions with bounded continuous first derivatives on $\bar{D}$, and $C_b^2(\bar{D})$ is similarly the set of those with bounded second derivatives.

$(\Omega, F, P: F_t)$ is a complete probability space, where $\{F_t\}_{t \geqq 0}$ are increasing right continuous sub σ-fields of F and each F_t contains all P-null sets. $B(t) = (B_1(t), \cdots, B_n(t))$ is an F_t-adapted n-dimensional Brownian motion. Let τ be an F_t-adapted Markov time such that $P\{\tau < \infty\} > 0$; then $\hat{B}(t) \equiv B(t + \tau) - B(\tau)$ is an F_t-adapted Brownian motion on $(\hat{\Omega}, \hat{F}, \hat{P}: \hat{F}_t)$, where $\hat{\Omega} = \{\omega; \tau(\omega) < \infty\}$, $\hat{F} = F/\hat{\Omega}$, $\hat{F}_t = F_{t+\tau}/\hat{\Omega}$, and $\hat{P}(\cdot) = P(\cdot)/P(\hat{\Omega})$ (see [22]).

Let (X_t, P_x) be a diffusion on $\bar{D}$, with the infinitesimal operator $(\mathfrak{G}, D(\mathfrak{G}))$ of (X_t, P_x) defined as follows:

$$D(\mathfrak{G}) = \quad u \in C_b(\bar{D});\ v \in C_b(\bar{D}) \text{ such that } u(X_t) - u(X_0) - \int_0^t v(X_s)\,ds$$

$$\text{is a continuous } L^2\text{-martingale on } (\Omega, F, P_x, F_t) \text{ for any } x \in \bar{D};$$

$$\mathfrak{G}u = v.$$

In the definition $\tau \equiv \inf\{t \geqq 0; X_t \in B\}$, it is understood that $\tau \equiv \inf\{t \geqq 0; X_t \in B\}$ if $\{\ \} \neq \phi$, and $\tau \equiv \infty$ if $\{\ \} = \phi$.

1. Degenerate diffusion with the Venttsel boundary condition. Define $G = \{x \in R^n; x_n > 0\}$ and $\partial G = \{x \in R^n; x_n = 0\}$. Let $X(t) = (\mathfrak{G}, D(\mathfrak{G}))$ be a conservative diffusion in $\bar{G}$ with continuous trajectories at the boundary. Venttsel [20] proved that if a function $f(x) \in C_b^2(\bar{G}) \cap D(\mathfrak{G})$, then

(1.1) $\quad Af(x) \equiv [\tfrac{1}{2} \sum_{i,j=1}^n \alpha_{ij} D_{ij} f + \sum_{i=1}^n \beta_i D_i f - \gamma f + \delta D_n f](x) = 0 \qquad \text{for} \quad x \in \partial G,$

where $(\alpha_{ij}(x))$ is a symmetric nonnegative definite matrix and $\gamma(x)$, $\delta(x)$ are nonnegative functions. Conversely, Watanabe [21] and others [6, etc.] constructed the diffusion $X(t)$

$= (\mathfrak{G}, D(\mathfrak{G}))$ on $\bar{G}$ such that

$$(1.2)\qquad \begin{aligned}&\text{(i) } D(\mathfrak{G}) \supset D(A) \equiv \{f \in C_b^2(\bar{G});\, Af(x) = 0 \quad \text{for} \quad x \in \partial G\},\\ &\text{(ii) } f \in D(A) \quad \Rightarrow \mathfrak{G}f = Lf \quad \text{for} \quad x \in \bar{G},\end{aligned}$$

where (α_{ij}), (β_i), γ, and δ are suitably given on ∂G. However, they assumed that $\partial G = \Sigma_3$. In this section, without assuming that $\partial G = \Sigma_3$, we construct the diffusion on $\bar{G}$ which satisfies (1.2).

Our assumption is:

(A.1). $(a_{ij}(x))$ is a symmetric nonnegative definite matrix. $a_{ij}(x) \in C_b^2(R^n)$ for each $i, j = 1, \cdots, n$, and $(b_i(x))$ is Lipschitz continuous.

REMARK 1.1. Phillips and Sarason [16] proved that if $(a_{ij}(x))$ satisfies (A.1), then there is a Lipschitz continuous matrix $(\sigma_{ij}(x))$ such that $a = \sigma \cdot \sigma$.

THEOREM 1.1. *Let* (A.1) *hold. Assume that*
(B.1) $\partial G = \Sigma_2$. Then, there is a diffusion $X^r(t) = (\mathfrak{G}^r, D(\mathfrak{G}^r))$ on $\bar{G}$, such that

$$(1.3)\qquad \begin{aligned}&\text{(i) } D(\mathfrak{G}^r) \supset D(A^r) \equiv \{f \in C_b^2(\bar{G});\, D_n f(x) = 0 \quad \textit{for} \quad x \in \partial G\};\\ &\text{(ii) } f \in D(A^r) \quad \Rightarrow \mathfrak{G}^r f = Lf \quad \textit{for} \quad x \in \bar{G}.\end{aligned}$$

PROOF. *Step* 1. By (B.1), we have

$$(1.4)\qquad \sigma_{ni}(x) = 0,\ b_n(x) < 0 \quad \text{for} \quad x \in \partial G \qquad (i = 1, \cdots, n).$$

The following stochastic differential equation has the unique solution $x(t{:}x) = (x_1(t{:}x), \cdots, x_n(t{:}x))$ in R^n:

$$(1.5)\qquad x_i(t) - x_i = \int_0^t \Sigma_j \sigma_{ij}(x(s))\, dB_j(s) + \int_0^t b_i(x(s))\, ds \qquad (i = 1, \cdots, n).$$

Let $\tau_1 \equiv \inf\{t > 0, x_n(t) = 0\}$; then $\tilde{B}(t) = (\tilde{B}_1(t), \cdots, \tilde{B}_{n-1}(t)) = (B_1(t + \tau_1) - B_1(\tau_1), \cdots, B_{n-1}(t + \tau_1) - B_{n-1}(\tau_1))$ is a $F_{t+\tau_1}$-adapted $(n-1)$-dimensional Brownian motion. Let $x(t{:}x) = (x_1(t{:}x), \cdots, x_{n-1}(t{:}x))$ be the unique solution of the stochastic differential equation

$$(1.6)\qquad \begin{aligned}\tilde{x}_i(t) - x_i = &\int_0^t \textstyle\sum_{j=1}^{n-1} \sigma_{ij}(\tilde{x}_1(s), \cdots, \tilde{x}_{n-1}(s), 0)\, d\tilde{B}_j(s)\\ &+ \int_0^t b_i(\tilde{x}_1(s), \cdots, \tilde{x}_{n-1}(s), 0)\, ds \qquad (i = 1, \cdots, n-1).\end{aligned}$$

Define

$$\begin{aligned}X_i^r(t{:}x) &= x_i(t{:}x) && \text{for} \quad t \leqq \tau_1\\ &= x_i(t - \tau_1; x(\tau_1{:}x)) && \text{for} \quad t > \tau_1\\ X_n^r(t{:}x) &= x_n(t{:}x) && \text{for} \quad t \leqq \tau_1\\ &= 0 && \text{for} \quad t > \tau_1.\end{aligned}$$

Step 2. It is easy to show that $X^r(t)$, defined above, is a diffusion on $\bar{G}$. Thus we prove that $X^r(t)$ satisfies (1.3). Let $\tau_1' = \tau_1 \wedge N$ with a constant N, and $M_i(t) = B_i(t \wedge \tau_1')$ $i = 1, \cdot\cdot, n$ and $N_i(t) = B_i(t \vee \tau_1') - B(\tau_1')$ $i = 1, \cdots, n-1$. $M_i(t)$ and $N_i(t)$ are continuous L^2-martingales for $t \in [0, N]$ such that $d\langle M_i\rangle(t) = d(t \wedge \tau_1')$ and $d\langle N_i\rangle(t) = d(t \vee \tau_1')$ (see

[10]). Our $X^r(t)$ satisfies the following stochastic differential equation for $t \in [0, N]$:

$$dX_i^r(t) = \sum_{j=1}^n \sigma_{ij}(X^r(t))(dM_j(t) + dN_j(t)) + b_i(X^r(t))(d(t \wedge \tau_1') + d(t \vee \tau_1')), \qquad i = 1, \cdots, n-1$$

$$dX_n^r(t) = \sum_{j=1}^n \sigma_{nj}(X^r(t))\, dM_j(t) + b_n(X^r(t))\, d(t \wedge \tau_1').$$

By the generalized Itô formula, we have for any $f \in C_b^2(\bar{G})$

$$f(X^r(t)) - f(X^r(0)) = \int_0^t (Lf)(X^r(s))\, d(s \wedge \tau_1') + \int_0^t [\tfrac{1}{2} \textstyle\sum_{i,j,k=1}^{n-1} \sigma_{ik}\sigma_{jk} D_{ij} f + \sum_{i=1}^{n-1} b_i D_i f](X^r(s))\, d(s \vee \tau_1')$$

$$+ \text{martingales} = \int_0^t (Lf)(X^r(s))\, ds - \int_0^t [b_n D_n f](X^r(s))\, d(s \vee \tau_1') + \text{martingales}, \quad \text{for} \quad t \in [0, N].$$

For the last equality, we use (1.4). Noting (1.4), we have

$$f \in D(\mathfrak{G}^r) \Leftrightarrow D_n f(x_1, \cdots, x_{n-1}, 0) = 0.$$

It is also clear that (ii) holds.

THEOREM 1.2. *The diffusion $X^r(t)$, obtained in Theorem 1.1, is unique in the following sense. Let $Y(t) = (\mathfrak{G}', D(\mathfrak{G}'))$ be a diffusion on $\bar{G}$ such that*

$$\text{(1.7)} \qquad \text{(i) } D(\mathfrak{G}') \supset D(A^r) \qquad \text{(ii) } f \in D(A^r) \Rightarrow \mathfrak{G}' f = Lf.$$

Then the processes $X^r(t)$ and $Y(t)$ are stochastically equivalent.

PROOF. Let $\phi(x)$ be a C_b^2-function such that $\phi(x) = (x_n)^2$ for $|x| \leqq 1$, and $\phi(x) \in D(A^r)$. Let $Y(t)$ be a diffusion associated with (1.7). By (A.1) and (B.1), there are positive constants ϵ and c such that $b_n(x) \leqq -c < 0$ and $0 \leqq a_{nn}(x) \leqq c(x_n)^2$ for $|x| \leqq \epsilon$. Set $\tau^\epsilon \equiv \inf\{t \geqq 0; |Y(t)| = \epsilon\}$. From the definition of the infinitesimal operator, we have

$$\phi(Y(t \wedge \tau^\epsilon)) - \phi(Y(0)) = \int_0^{t \wedge \tau^\epsilon} (L\phi)(Y(s))\, ds + \text{martingales}.$$

Take expectations of both sides:

$$E_0[Y_n(t \wedge \tau^\epsilon)]^2 = E_0\left[\int_0^{t \wedge \tau^\epsilon} (a_{nn}(Y(s)) + 2Y_n(s) b_n(Y(s)))\, ds\right] \leqq -cE_0\left[\int_0^{t \wedge \tau^\epsilon} (Y_n(s))^2\, ds\right].$$

Thus, we obtain

$$\text{(1.8)} \qquad P_{(x_1, \cdots, x_{n-1}, 0)}[Y_n(t) = 0 \text{ for any } t \geqq 0] = 1.$$

Let $\phi_i(x)$ and $\phi_{ij}(x)$ $(i, j = 1, \cdots, n-1)$ be C_b^2-functions such that they belong to $D(A^r)$ and $\phi_i(x) = x_i$, $\phi_{ij}(x) = x_i x_j$ for $|x| \leqq 1$. Applying the Itô formula to $\phi_i(x)$ and $\phi_{ij}(x)$, we obtain the martingale $M(t)$ on ∂G such that

$$M_i(t) = Y_i(t) - Y_i(0) - \int_0^t b_i(Y_1(s), \cdots, Y_{n-1}(s), 0)\, ds$$

$$\langle M_i, M_j \rangle (t) = \frac{1}{2} \int_0^t a_{ij}(Y_1(s), \cdots, Y_{n-1}(s), 0)\, ds.$$

According to [22], if we have the suitable Brownian motion $\bar{B}(t)$, then it follows that

$$\begin{aligned} dY_i(t) &= \textstyle\sum_{j=1}^{n-1} \sigma_{ij}(Y_1(t), \cdots, Y_{n-1}(t), 0)\, d\bar{B}_j(t) \\ &\quad + b_i(Y_1(t), \cdots, Y_{n-1}(t), 0)\, dt, \qquad i = 1, \cdots, n-1 \\ dY_n(t) &= 0. \end{aligned} \tag{1.9}$$

Since the solution of (1.9) is unique, we have $Y(t) = X^r(t)$ in ∂G in the sense of stochastic equivalence. By a similar method in G, the uniqueness is proved.

Define the differential operator A on ∂G by (1.1). Let $(\alpha_{ij}(x))_{i,j=1}^n$, $(\beta_i(x))_{i=1}^n$, $\gamma(x)$, and $\delta(x)$ be given in ∂G as follows:

(A.2) (α_{ij}) is a nonnegative definite symmetric matrix. $\gamma(x)$ and $\delta(x)$ are nonnegative functions (see [20]).

(A.3) $[-b_n(x)\alpha_{ij}(x) + \delta(x)a_{ij}(x)]/[-\gamma(x)b_n(x) + \delta(x)]$, $i, j = 1, \cdots, n-1$, are $C_b^2(\partial G)$-functions.

(A.4) $[-b_n(x)\beta_i(x)]/[-\gamma(x)b_n(x) + \delta(x)]$, $i = 1, \cdots, n-1$, are Lipschitz continuous in ∂G.

THEOREM 1.3. *Assume that* (A.1)–(A.4) *and* (B.1) *hold. Then there is the diffusion* $X(t) = (\mathfrak{G}, D(\mathfrak{G}))$ *on* $\bar{G}$ *such that* (1.2) *holds.*

PROOF. *Step* 1. Since $a_{ni} = a_{in} = 0$ in ∂G $(i = 1, \cdots, n)$, heuristic considerations indicate that in ∂G,

$$\frac{\partial}{\partial x_n} = \frac{1}{b_n(x)} [L - \tfrac{1}{2} \textstyle\sum_{i,j=1}^{n-1} a_{ij}(x) D_{ij} - \sum_{i=1}^{n-1} b_i(x) D_i]. \tag{1.10}$$

Substituting (1.10) into "$Af = 0$ for $x \in \partial G$", we have

$$Lf(x) = \tfrac{1}{2} \textstyle\sum_{i,j=1}^{n-1} \hat{\alpha}_{ij}(x) D_{ij} f(x) + \sum_{i=1}^{n-1} \hat{\beta}_i(x) D_i f(x) \qquad \text{for} \quad x \in \partial G$$

where we set for $x \in \partial G$

$$\hat{\alpha}_{ij}(x) = [-b_n(x)\alpha_{ij}(x) + \delta(x)a_{ij}(x)]/[-\gamma(x)b_n(x) + \delta(x)]$$

$$\hat{\beta}_i(x) = [-b_n(x)\beta_i(x) + \delta(x)b_i(x)]/[-\gamma(x)b_n(x) + \delta(x)].$$

By (A.1) – (A.4), (B.1), and (1.4), it follows that $(\hat{\alpha}_{ij}(x))$ is a nonnegative definite symmetric matrix and belongs to $C_b^2(\partial G)$, and $(\hat{\beta}_i(x))$ is Lipschitz continuous. By Remark 1.1, there is a Lipschitz continuous $(\hat{\sigma}_{ij}(x))$ on ∂G such that $\hat{\alpha} = \hat{\sigma}\,\hat{\sigma}$. We consider the following stochastic differential equation on ∂G:

$$\hat{x}_i(t) - \hat{x}_i(0) = \int_0^t \textstyle\sum_{i,j=1}^{n-1} \hat{\sigma}_{ij}(\hat{x}(s))\, d\tilde{B}_j(s) + \displaystyle\int_0^t \hat{\beta}_i(x(s))\, ds \qquad (i = 1, \ldots, n-1) \tag{1.11}$$

where $\tilde{B}(t) = (\tilde{B}_1(t), \ldots, \tilde{B}_{n-1}(t)) \equiv (B_1(t + \tau_1) - B_1(\tau_1), \ldots, B_{n-1}(t + \tau_1) - B_{n-1}(\tau_1))$ is the $F_{t+\tau_1}$-adapted Brownian motion. Let $\hat{x}(t:\hat{x}(0))$ be the solution of (1.11). We define

$$\begin{aligned} X_i(t:x) &= x_i(t:x) && \text{for} \quad t \leqq \tau_1 \qquad (i = 1, \ldots, n-1) \\ &= \hat{x}_i(t - \tau_1 : x(\tau_1 : x)) && \text{for} \quad t \geqq \tau_1 \\ X_n(t:x) &= x_n(t:x) && \text{for} \quad t \leqq \tau_1 \\ &= 0 && \text{for} \quad t \geqq \tau_1. \end{aligned}$$

Step 2. It is easy to see that $X(t)$ is the diffusion on $\bar{G}$. Let $M_i(t)$, $N_i(t)$ be the same martingales as in Step 2 of the proof of Theorem 1.1; then $X(t)$ satisfies, for $t \in [0, N]$,

$$dX_i(t) = \sum_{j=1}^{n-1} \sigma_{ij}(X(t))\, dM_j(t) + \sum_{j=1}^{n-1} \hat{\sigma}_{ij}(X(t))\, dN_j(t)$$
$$+ b_i(X(t))\, d(t \wedge \tau_1') + \hat{\beta}_i(X(t))\, d(t \vee \tau_1') \qquad (i = 1, \ldots, n-1)$$
$$dX_n(t) = \sum_{j=1}^{n-1} \sigma_{nj}(X(t))\, dM_j(t) + b_n(X(t)) d(t \wedge \tau_1').$$

By using the generalized Itô formula, we conclude that $X(t)$ satisfies (1.2).

THEOREM 1.4. *Let* (A.1) *hold. Assume that*
(B.2) $\partial G = \Sigma_1$, *or*
(B.3) $\partial G = \Sigma_0$.
Then there is a diffusion $X(t) = (\mathfrak{G}_j, \mathrm{D}(\mathfrak{G}_j))$ on $\bar{G}$, *associated with* (1.2).

PROOF. Let $X(t)$ be the solution of (1.5). Since (B.2) or (B.3) holds,

(1.12) $$P_x[X(t) \notin \partial G \quad \text{for} \quad t \geqq 0] = 1 \quad \text{for any} \quad x \in G \text{ (see [15])}.$$

By Tanaka's method (see, for example, [5]), we have

$$P_x[X_n(t) \geqq 0 \quad \text{for} \quad t \geqq 0] = 1 \quad \text{for any} \quad x \in \partial G.$$

Thus $X(t)$ is the diffusion on $\bar{G}$, and it is clear that $X(t)$ satisfies (1.2).

2. Supplement to Venttsel's boundary condition. If $\partial G = \Sigma_3$, then the unique diffusion in $\bar{G}$ satisfies (1.3) and it satisfies (1.3) only. But this is not necessarily true if $\partial G \neq \Sigma_3$. If $\partial G = \Sigma_2$, then the diffusion $X^r(t)$ constructed in Theorem 1.1 satisfies (1.3). But $X^r(t)$ satisfies the other representation of Venttsel's boundary condition. In fact, let $\alpha(x)$ and $\beta(x)$ be $C_b(\partial G)$-functions such that

(2.1) $$\alpha(x), \beta(x) \geqq 0, \quad \text{and} \quad \alpha(x) + \beta(x) > 0 \quad \text{for any} \quad x \in \partial G.$$

Set for $f \in C_b^2(\partial G)$ and $x \in \partial G$

$$A_{\alpha,\beta} f(x) = \alpha(x)\, [\tfrac{1}{2} \textstyle\sum_{i,j=1}^{n-1} a_{ij} D_{ij} f + \sum_{i=1}^{n-1} b_i D_i f - Lf](x)$$
$$+ \beta(x) D_n f(x) = [\tfrac{1}{2} \textstyle\sum_{i,j} \alpha\, a_{ij} D_{ij} f + \sum_i \alpha\, b_i\, D_i f - \alpha L f + \beta D_n f](x).$$

By (2.1), the coefficients of $A_{\alpha,\beta}$ satisfy (A.2), and the boundary condition

(2.2) $$A_{\alpha,\beta} f = 0 \quad \text{for} \quad x \in \partial G$$

is Venttsel's type. Set $D(A_{\alpha,\beta}) \equiv \{f \in C_b^2(\bar{G})\text{: (2.2) holds}\}$; then $X^r(t)$ satisfies

(2.3)
(i) $D(\mathfrak{G}_j^r) \supset D(A_{\alpha,\beta})$;
(ii) $f \in D(A_{\alpha,\beta}) \Rightarrow \mathfrak{G}_j^r f = Lf \quad \text{for} \quad x \in \bar{G}$.

Conversely, using the same method as in Theorem 1.2, we can prove that the diffusion associated with (2.3) is unique, i.e., $X^r(t)$.

On the other hand, if $\partial G = \Sigma_1$, then the diffusion on $\bar{G}$ associated with (1.3) is not necessarily unique. In fact, let $R^n = R^2$ and $Lf = \tfrac{1}{2}[D_{11} f + x_2^2 D_{22} f] + D_2 f$. By Theorem 1.4, the solution $X(t{:}x) = (X_1(t{:}x), X_2(t{:}x))$ of the stochastic differential equation

$$X_1(t{:}x) = x_1 + B_1(t)$$
$$X_2(t{:}x) = x_2 + \int_0^t X_2(s{:}x)\, dB_2(s) + t$$

is associated with (1.3). Now we define the diffusion $\hat{X}(t:x) = (\hat{X}_1(t:x), \hat{X}_2(t:x))$ as follows:

$$\begin{aligned} \hat{X}_1(t:x) &= x_1 + B_1(t) \\ \hat{X}_2(t:x) &= X_2(t:x) \qquad \text{for} \quad x_2 > 0 \\ &= 0 \qquad \text{for} \quad x_2 = 0. \end{aligned}$$

$\hat{X}(t:x)$ also satisfies (1.3), and $\hat{X}(t:x) \neq X(t:x)$ *for* $x \in \partial G$.

3. The construction of the diffusion $X^r(t)$ on $\bar{D}$. Let D be a bounded open domain in R^n with the smooth boundary ∂D. When L is nondegenerate in $\bar{D}$, Ikeda [6] gave the stochastic solution u of (N). Let $X^r(t)$ be the diffusion on $\bar{D}$, associated with (3.2), and $\theta^r(t)$ be the local time of $X^r(t)$ at ∂D. He proved that

$$u(x) = -E_x \int_0^\infty F(X^r(t))\, dt - E_x \int_0^\infty G(X^r(t))\, d\theta^r(t),$$

if the right-hand side above exists. In this section, we construct the diffusion $X^r(t)$ on $\bar{D}$ associated with (3.2) in the case that $\partial D \neq \Sigma_3$. We assume the following:

(C.1). There is a smooth function $\rho(x)$ such that

(i) $D = \{x \in R^n;\ \rho(x) > 0\}$
(ii) $\partial D = \{x \in R^n;\ \rho(x) = 0\}$
(iii) $|\nabla\rho(x)| = 1$ for $x \in \partial D$.

Because of (C.1), we have local charts (ψ, U) such that U is an open set and ψ is a smooth mapping of $x \in U \cap \partial D$ to $\psi(x) = (x_1', \ldots, x_n') \in R^n$, where

$$\begin{aligned} x_n' > 0 &\Leftrightarrow x \in U \cap D \\ x_n' = 0 &\Leftrightarrow x \in U \cap \partial D. \end{aligned} \tag{3.1}$$

(C.2). There are mutually disjoint, connected, and closed sets $\partial D_k (k = 0, \ldots, 3)$ such that $\partial D = U_k \partial D_k$ and $\partial D_k = \Sigma_k$.

THEOREM 3.1. *Let* (A.1), (C.1), *and* (C.2) *hold. Then there is the diffusion* $X^r(t) = (\mathfrak{G}_j^r, D(\mathfrak{G}_j^r))$ on $\bar{D}$ *such that*

$$\begin{aligned} &\text{(i) } D(\mathfrak{G}_j^r) \supset D(A^r) \equiv \left\{ f \in C_b^2(\bar{D});\ \left(\nabla\rho, \frac{\partial f}{\partial x}\right) = 0 \quad \textit{for} \quad x \in \partial D \right\} \\ &\text{(ii) } f \in D(A^r) \Rightarrow \mathfrak{G}_j^r f = Lf \quad \textit{for} \quad x \in \bar{D}. \end{aligned} \tag{3.2}$$

PROOF. *Step* 1. We construct the diffusion $X^3(t) = (\mathfrak{G}_j^3, D(\mathfrak{G}_j^3))$ in a neighborhood V_3 of ∂D_3, which satisfies (3.3) below and stops at $\partial V_3 \cap D$.

$$\begin{aligned} &\text{(i) } D(\mathfrak{G}_j^3) \supset D(A^3) \equiv \left\{ f \in C_b^2(V_3 \cap \bar{D});\ \left(\nabla\rho, \frac{\partial f}{\partial x}\right) = 0 \quad \text{for} \quad x \in \partial D_3 \right\} \\ &\text{(ii) } f \in D(A^3) \Rightarrow \mathfrak{G}_j^3 f = Lf \quad \text{in} \quad V_3 \cap D. \end{aligned} \tag{3.3}$$

For simplicity, let ∂D_3 be covered by two local charts, say (ψ', U') and (ψ'', U''), which satisfy (3.1). Consider a local stochastic differential equation for U':

$$dx'_i(t) = \sum_{j=1}^n \sigma'_{ij}(x'(t))\, dB'_j(t) + b'_i(x'(t))\, dt \qquad i = 1, \dots, n-1,$$

$$dx'_n(t) = \sum_{j=1}^n \sigma'_{nj}(x'(t))\, dB'_j(t)) + b'_n(x'(t)\, dt + d\theta(t)$$

(3.4)
$$\int_0^t \chi_{x'_n} = 0\ (x'(s))\, d\theta(s) = \theta(t), \qquad \int_0^t \chi_{x'_n} = 0\ (x'(s))\, ds = 0,$$

where L is expressed in (ψ', U') by

$$L = ½ \sum a'_{ij}(x) D_{ij} + \sum b'_i(x) D_i$$

with $a' = \sigma' \cdot \sigma'^*$. Watanabe [21] proved that there is a unique solution $x'(t:x)$ of (3.4). Let $\zeta_1 = \inf\{t \geqq 0;\ x'(t) \notin U'\}$. If $x'(\zeta_1) \in \partial(U' \cup U'')$, stop $x'(t)$ and put $\zeta_n = 0$ $n = 2, 3, \dots$. But if $x'(\zeta_1) \in (U'')^0$, consider a solution $x''(t:\psi''(\psi'^{-1}(x'(\zeta_1:x))))$ of a local stochastic differential equation for U'', where we take the Brownian motion $B''(t) = B'(t + \zeta_1) - B'(\zeta_1)$. Define $\zeta_2 = \inf\ \{t \geqq 0;\ x''(t) \notin U''$]. If $x''(\zeta_2) \in \partial(U' \cup U'')$, stop $x''(t)$ and put $\zeta_n = 0$ $n = 3, 4, \dots$. But if $x''(\zeta_2) \in (U'')^0$, consider (3.3) based on the Brownian motion $B'''(t) = B''(t + \zeta_2) - B''(\zeta_2)$, etc. Thus we define

$$\begin{aligned} X^3(t:x) &= \psi'^{-1}(x'(t:x)) && 0 \leqq t \leqq \zeta_1 \\ &= \psi''^{-1}(x''(t:z_1)) && \zeta_1 \leqq t \leqq \zeta_1 + \zeta_2 \quad z_1 = \psi''(X^3(\zeta_1:x)) \\ &\psi'^{-1}(x'(t:z_2)) && \zeta_1 + \zeta_2 \leqq t \leqq \zeta_1 + \zeta_2 + \zeta_3\ z_2 = \psi'(X^3(\zeta_1 + \zeta_2:x)), \end{aligned}$$

etc., $X^3(t:x)$ is a diffusion on $\bar{V}_3 \cap \bar{D}$ associated with (3.3) (see [11]).

Step 2. We construct the diffusion $X^2(t) = (\mathfrak{G}_j^2, D(\mathfrak{G}_j^2))$ in a neighborhood V_2 of ∂D_2, which satisfies (3.5) below and stops at $\partial V_2 \cap D$.

(3.5)
$$\text{(i) } D(\mathfrak{G}_j^2) \subset D(A^2) \equiv \left\{ f \in C_b^2(\bar{V}_2 \cap \bar{D});\ \left(\nabla\rho, \frac{\partial f}{\partial x}\right) = 0 \quad \text{for} \quad x \in \partial D_2 \right\}.$$
$$\text{(ii) } f \in D(A^2) \Rightarrow \mathfrak{G}_j^2 f = Lf \quad \text{in} \quad V_2 \cap \bar{D}.$$

Let ∂D_2 be also covered by two local charts (ψ', U') and (ψ'', U''), which satisfy (3.1). The local stochastic differential equations on ∂D_2, for U', are:

(3.6)
$$dx'_i(t) = \sum_{j=1}^{n-1} \sigma'_{ij}(x'_1(t), \dots, x'_{n-1}(t), 0)\, dB'_j(t) + b'_i(x'_1(t), \dots, x'_{n-1}(t), 0)\, dt \qquad i = 1, \dots, n-1.$$

By the same procedure as in Step 1, we have the diffusion $\tilde{X}^2$ on ∂D_2 associated with the infinitesimal operator $L_{/\!/}$ defined in ∂D by

(3.7)
$$L_{/\!/} = L - \{\textstyle\sum_{i=1}^n \{b_i - ½ \sum_{j=1}^n D_j a_{ij}\} D_i](x) \sum_{k=1}^n (D_k\rho(x)) \dfrac{\partial}{\partial x_k}.$$

Let $Y(t)$ be the solution of (1.5), and set $\tau_2 = \inf\{t \geqq 0;\ Y(t) \in \partial(V_3 \cap D)\}$. We define $X^2(t)$ as follows:

$$\begin{aligned} X^2(t:x) &= Y(t:x) && 0 \leqq t \leqq \tau_2 \\ &= \tilde{X}^2(t - \tau_2 : Y(\tau_2:x) && \pi_2 \leqq t \quad \text{and} \quad Y(\tau_2:x) \in \partial D_2 \\ &= Y(\tau_2:x) && \tau_2 \leqq t \quad \text{and} \quad Y(\tau_2:x) \notin \partial D_2 \end{aligned}$$

By Theorem 1.1, it is clear that $X^2(t)$ satisfies (3.5).

Step 3. By repeating the same procedure as in Step 1, we define the diffusion $Z(t)$ in R^n such that we connect $Y(t)$ of (1.5) with $X^3(t)$ of Step 1.

Step 4.

$$
\begin{aligned}
(3.8) \qquad X^r(t{:}x) &= Z(t{:}x) && 0 \leqq t \leqq \tau_3 \\
&= \tilde{X}^2(t-\tau_3{:}\ Z(\tau_3{:}x)) && \tau_3 \leqq t,
\end{aligned}
$$

where we set $\tau_3 = \inf\{t \geqq 0;\ Y(t) \in \partial D_2\}$, and where $\tilde{X}^2(t)$ is based on the suitable Brownian motion. From Theorems 1.1, 1.4 and Step 2, it follows that $X^r(t)$ defined by (3.8) satisfies (3.2).

4. The degenerate Neumann problem. Let F be a $C_b(\bar{D})$-function and G be a $C_b(\partial D)$-function. Denote by $\theta^r(t)$ the local time of $X^r(t)$ at ∂D_3, and set $\tau_4 = \inf\{t \geqq 0;\ X^r(t) \in \partial D_2\}$.

DEFINITION 4.1. We say that a $C(\bar{D} - D_0)$-function u satisfies

$$\bar{L}u = F \text{ on } D \cup \partial D_2, \quad \text{and} \quad \overline{\left(\frac{\partial}{\partial v}\right)}u = G \text{ on } \partial D_3,$$

if $u(X^r(t \wedge \tau_4)) - \int_0^{t\wedge\tau_4} F(X^r(s))\,ds - \int_0^{t\wedge\tau_4} G(X^r(s))\,d\theta^r(s)$ is a P_x-martingale for any $x \in D \cup \partial D_3$ and any $t \geqq 0$.

DEFINITION 4.2. We say that a $C_b(\partial D)$-function u satisfies

$$\bar{L}_{/\!/}\, u = G \text{ on } \partial D_2,$$

if $u(\tilde{X}^2(t)) - \int_0^t G(\tilde{X}^2(s))\,ds$ is a P_x-martingale for any $x \in \partial D_2$ and any $t \geqq 0$.

DEFINITION 4.3. We say that a $C(\bar{D} - \partial D_0)$-function u is a stochastic solution of (N), if

$$\bar{L}u = F \quad \text{on} \quad D \cup \partial D_2, \quad \text{and} \quad \overline{\left(\frac{\partial}{\partial v}\right)}u = G \quad \text{on} \quad \partial D_3,$$

$$\text{and} \quad \overline{L_{/\!/}}\, u = F - b_+ G \quad \text{on} \quad \partial D_2,$$

where $b_+(x) = [\Sigma_{i=1}^n (b_i - \frac{1}{2}\Sigma_{j=1}^n D_j a_{ij}) D_i\rho](x)$.

REMARK 4.1. If u is a stochastic solution of (N) and $u \in C^2(\bar{D} - \partial D_0)$, then u is a classical solution of (N).

PROOF. Applying the generalized Itô formula to u, we have

$$
\begin{aligned}
\text{Martingale} &= u(X^r(t \wedge \tau_4)) - \int_0^{t\wedge\tau_4} F(X^r(s))\,ds - \int_0^{t\wedge\tau_4} G(X^r(s))\,d\theta^r(s) \\
&= u(x) + \int_0^{t\wedge\tau_4} (Lu - F)(X^r(s))\,ds + \int_0^{t\wedge\tau_4} \left(\frac{\partial u}{\partial v} - G\right)(X^r(s))\,d\theta^r(s) \\
&\quad + \text{martingales}.
\end{aligned}
$$

Thus $Lu - F = 0$ on D, and $(\partial u/\partial v) - G = 0$ on ∂D_3. Since $(Lu - F)$ is continuous on $\bar{D} - \partial D_0$

$$(4.1) \qquad Lu = F \quad \text{on} \quad \bar{D} - \partial D_0.$$

In ∂D_2, we have

$$\text{Martingale} = u(\tilde{X}^2(t)) - \int_0^t (F - b_+ G)(\tilde{X}^2(s))\, ds$$

$$= u(x) + \int_0^t [L_{/\!/} u + b_+ G - F](\tilde{X}^2(s))\, ds + \text{martingales}.$$

Thus $L_{/\!/} u = F - b_+ G$ on ∂D_2. By (4.1), we have on ∂D_2

$$b_+ \frac{\partial u}{\partial v} = b_+[\textstyle\sum_{i=1}^n D_i \rho D_i u] = Lu - L_{/\!/} u = F + b_+ G - F = b_+ G.$$

Since $\partial D_2 = \Sigma_2$, it follows that $b_+ < 0$, and we conclude that $(\partial u/\partial v) = G$ on ∂D_2.

We introduce the Döeblin condition (see [1]).

The Döeblin condition. A Markov chain $x(k)$ in a space E is given. Let A be a Borel set in E. There is a constant measure $\psi(\cdot)$ such that $\psi(A) > 0$ and $P_x[x(k) \in B] \geqq \delta\psi(B)$ for any Borel set $B \subset A$, any $x \in E$ with a $k \geqq 0$ and $\delta > 0$.

REMARK 4.2. If a diffusion $x(t)$ in M satisfies the Döeblin condition then there is an invariant mesure $\mu(\cdot)$ on M such that for any bounded measurable function f, the inequality

$$|E_x f(x(t)) - \int_M f(x)\, \mu(dx)| \leqq c\, e^{-c't} \|f\|$$

holds with positive constants c and c' (see, for example, [1] and [3]).

THEOREM 4.1. *Let* (A.1), (C.1), *and* (C.2) *hold, and* $F(x) \in C_b(\bar{D})$, $G(x) \in C_b(\partial D_3 \cup \partial D_2)$. *Assume that* $\partial D_2 \neq \emptyset$, *and that*

(4.2) $\sup_{x \in D} E_x\{\tau_4\} < \infty$ *for each* $x \in D - \partial D_0$,

(4.3) $L_{/\!/}$ *is nondegenerate on* ∂D_2, *where* $L_{/\!/}$ *is restricted on* ∂D_2.

Then (i) *and* (ii) *are equivalent.*

(i). *There is a stochastic solution* u *of* (N), *and* u *differs from any other stochastic solution by a constant.*

(ii). $\int_{\partial D_2} [F - b_+ G](x) \mu\, (dx) = 0$, *where* $\mu(\cdot)$ *is any invariant measure of* $\tilde{X}^2(t)$.

REMARK 4.3. Since ∂D_2 is compact, it follows from (4.3) that $\tilde{X}^2(t)$ satisfies the Döeblin condition.

REMARK 4.4. It is not easy to verify (4.2) in general. The sufficient conditions for (4.2) are given in the Appendix (Lemmas 6.1 and 6.2). The other sufficient conditions are discussed in [3, 4, 5, 19, and etc.].

PROOF OF THEOREM 4.1. *Step* 1. We prove that (i) $\Rightarrow$ (ii). By (4.3), Remarks 4.2 and 4.3, there is an invariant measure $\mu(\cdot)$ for $\tilde{X}^2(t)$. Let u be a stochastic solution of (N), and for any $x \in \partial D_2$

$$u(\tilde{X}^2(t{:}x)) - u(x) - \int_0^t [F - b_+ G](\tilde{X}^2(s))\, ds = \text{a } P_x\text{-martingale}.$$

Take the expectations of the terms above:

$$\frac{1}{t} E_x[u(\tilde{X}^2(t)) - u(x)] = \frac{1}{t} E_x\left[\int_0^t [F - b_+ G](\tilde{X}^2(s))\, ds\right].$$

By integration with respect to $\mu(\cdot)$, we obtain

$$0 = \frac{1}{t}\int_{\partial D_2} \mu(dx)\, E_x[u(\tilde{X}^2(t)) - u(x)]$$

$$= \frac{1}{t}\int_{\partial D_2} \mu(dx) \int_0^t E_x[F - b_+ G](\tilde{X}^2(s))\, ds = \int_{\partial D_2} \mu(dx)[F - b_+ G](x).$$

Step 2. We prove that (ii) $\Rightarrow$ "the existence of the stochastic solution". Set $u_{\partial D_2} = E_x \int_0^\infty [b_+ G - F](\tilde{X}^2(s))\, ds$ for $x \in \partial D_2$. From (ii), Remarks 4.2, and 4.3, it follows that the right-hand side of the expression above is well defined and bounded. We show that $u_{\partial D_2}$ is continuous. Let T and c be suitable constants.

$$|u_{\partial D_2}(x) - u_{\partial D_2}(y)| = \left|\int_0^\infty E[(b_+ G - F)(\tilde{X}^2(s{:}x)) - (b_+ G - F)(\tilde{X}^2(s{:}y))]\, ds\right|$$

$$\leqq 2\epsilon + \int_0^T |E[(b_+ G - F)(\tilde{X}^2(s{:}x)) - (b_+ G - F)(\tilde{X}^2(s{:}y))]|\, ds$$

$$\leqq 2\epsilon + |x - y|\, ce^{cT}.$$

Thus $u_{\partial D_2} \in C_b(\partial D_2)$. It is easy to prove that $\overline{L_{/\!/}}\, u_{\partial D_2} = F - b_+ G$ on ∂D_2. We define $u(x)$ on $\bar{D} - \partial D_0$ as follows:

$$u(x) = -E_x\left[\int_0^\infty F(X^r(t))\, dt\right] - E_x\left[\int_0^\infty G(X^r(t))\, d\theta^r(t)\right]$$

$$-E_x\left[\int_0^\infty (F - b_+ G)(X^r(t))\, \chi_{\partial D_2}(X^r(t))\, dt\right]$$

$$= -E_x\left[\int_0^{\tau_4} F(X^r(t))\, dt - \int_0^{\tau_4} G(X^r(t))\, d\theta^r(t)\right] \tag{4.4}$$

$$- E_x\left[\int_{\tau_4}^\infty (F - b_+ G)(\tilde{X}^2(t))\, dt\right]$$

$$= E_x\left[-\int_0^{\tau_4} F(X^r(t))\, dt - \int_0^{\tau_4} G(X^r(t))\, d\theta^r(t) + u_{\partial D_2}(X^r(\tau_4))\right].$$

Because of (4.2), $u(x)$ is well defined and bounded. Noting (4.2), we see that the first and the second terms of the last equality are continuous in x. A slight modification of the results of [4] shows that the third term of (4.4) is continuous with respect to x. Now, it is clear that $\bar{L}u = F$ on $D \cup \partial D_2$ and $\overline{(\partial/\partial v)}u = G$ on ∂D_3.

Step 3. We prove the uniqueness of the stochastic solution. Let u and u' be two stochastic solutions of (N). Set $v = u - u'$; then $v(\tilde{X}^2(t))$ is a P_x-martingale for any $x \in \partial D_2$. Since (4.3) holds, $\tilde{X}^2(t)$ is recurrent in ∂D_2. Let $U(x^0)$ be a ϵ-neighborhood of $x^0 \in \partial D_2$, and set $\zeta = \inf\{t \geqq 0; (\tilde{X}^2(t) \in U(x^0)\}$. Note that $E_x\zeta < \infty$ for any $x \in \partial D_2$, and that $v(\tilde{X}^2(t))$ is a closed martingale. By the continuity of v and the optimal sampling theorem, it follows that $v(x) = E_x v(\tilde{X}^2(\zeta)) = v(x^0)$ for any $x \in \partial D_2$. Thus $v(x) =$ constant on ∂D_2. By the hypothesis, $v(X^r(t \wedge \tau_4))$ is a P_x-martingale for any $x \in D \cup \partial D_2 \cup \partial D_3$. We conclude that

$$v(x) = E_x v(X^r(t \wedge \tau_4)) = \lim_{t\to\infty} E_x v(X^r(t \wedge \tau_4)) = E_x v(X^r(\tau_4))$$

$$= \text{constant} \quad \text{for any } x \in D \cup \partial D_2 \cup \partial D_3.$$

THEOREM 4.2. *Set* $\tau_5 = \inf\{t \geqq 0; X^r(t) \in \partial D\}$. *Let* $F(x) \in C_b(D)$ *and* $G(x) \in C_b(\partial D_3)$. *Let* (A.1), (C.1), *and* (C.2) *hold. Assume that* $\partial D_2 = \varnothing$, *and that*

$$E_x \tau_5 < \infty \quad \textit{for each} \quad x \in \bar{D} - \partial D_0 \tag{4.5}$$

(4.6) *L is nondegenerate in a neighborhood of* ∂D_3.

Then, (i) *of Theorem* 4.1 *is equivalent to the following* (ii′):

(ii′). $\int_{\partial D_3} G(x) u_{\partial D_3}(dx) = 0$ *and* $\int_D F(x)\,\mu(dx) = 0$, *where* $u_{\partial D_3}$ *and* μ *are any invariant measures of* $X^r(t)$.

PROOF. Note that $\partial D_3 \neq \varnothing$, because (4.5) holds and $\partial D_2 = \varnothing$. Since (4.5) and (4.6) hold, we can prove that $X^r(t)$ satisfies the Döeblin condition on $D \cup \partial D_3$ in a way similar to that in Freidlin [3] and Ikeda [6]. Set $u(x) = -E_x \int_0^\infty F(X^r(t))\,dt - E_x \int_0^\infty G(X^r(t))\,d\theta^r(t)$, and a slight modification of the proof of Theorem 4.1 completes the proof.

5. On the stochastic solution of (N). In this section, we discuss the analytic meaning of the stochastic solution of (N). We follow step by step the approach of Stroock-Varadhan, which is used in the Fichera problem [19]. We assume that

(A.5). L is nondegenerate in a neighborhood of ∂D_3, and a_{ij}, $b_i \in C^3(R^n)$ for $i, j = 1, \dots, n$.

DEFINITION 5.1. Let the Banach space E be the pair (u, f) such that $u, f \in C_b(\bar{D})$. H is the subset of E such that u is smooth and $Lu = f$ on $D \cup \partial D_2$, $\partial u/\partial \nu = g$ on ∂D_3. The closure of H in E is denoted by $\bar{H}$.

THEOREM 5.1. *Let the hypothesis and* (ii) *of Theorem* 4.1 *hold. Then* $u_{\partial D_2} \in C_b^2(\partial D_2)$.

PROOF. By Theorem 4.1, $u_{\partial D_2} \in C_b(\partial D_2)$. Let U be an open domain in ∂D_2, and consider the following Dirichlet problem on ∂D_2:

$$(5.1)\qquad \begin{aligned} L_{/\!/} v &= F - b_+ G &&\text{on } U \\ v &= u_{\partial D_2} &&\text{on } \partial U. \end{aligned}$$

Since $L_{/\!/}$, restricted in ∂D_2, is nondegenerate and $(F - b_+ G) \in C_b(\partial D_2)$, there is unique $C_b^2(U)$-solution v. The stochastic representation of v is as follows:

$$\begin{aligned} v(x) &= E_x u_{\partial D_2}(\tilde{X}^2(\xi_1)) - E_x \int_0^{\xi_1} (F - b_+ G)(\tilde{X}^2(t))\,dt \\ &= E_x\left[\int_0^\infty (b_+ G - F)(\tilde{X}^2(t))\,ds\right] = u(x), \end{aligned}$$

where $\xi_1 = \inf\{t \geqq 0; \tilde{X}^2(t) \in \partial U\}$. Thus $u(x) \in C_b^2(\partial D_2)$.

THEOREM 5.2. *Let the hypothesis and* (ii) *of Theorem* 4.1 *hold. Assume that* (A.5) *holds and* $G \in C_b^1(\partial D)$. *Then*

$$(5.2)\qquad \bar{L}u = F \quad \text{on} \quad D \cup \partial D_2 \quad \text{and} \quad \overline{\left(\frac{\partial}{\partial \nu}\right)} u = G \quad \text{on} \quad \partial D_3,$$

if and only if $(u, F) \in \bar{H}$.

REMARK 5.1. Theorem 5.2 means that there is a sequence $\{u_n\}$ on H, such that $u_n \to u$, $Lu_n \to F$, and $\partial u/\partial \nu \to G$ on ∂D.

In order to prove Theorem 5.2, we present several lemmas.

LEMMA 5.1. *If* $(u, F) \in \bar{H}$, *then* (5.2) *holds.*

PROOF. By the hypothesis, there is a sequence (u^i, F^i) such that $Lu^i = F^i$ on $D \cup \partial D_1 \cup \partial D_2$ and $\partial u^i/\partial \nu = G$ on ∂D_3. By Remark 4.1, (5.2) holds for each i, and note that the limit of martingales is also a martingale.

LEMMA 5.2. *If* (5.2) *holds, then for any* $\lambda \geqq 0$

$$e^{-\lambda(t\wedge\tau_4)}u(X^r(t\wedge\tau_4)) - \int_0^{t\wedge\tau_4} e^{-\lambda s}(F-\lambda u)(X^r(s))\,ds \tag{5.3}$$
$$-\int_0^{t\wedge\tau_4} e^{-\lambda s}G(X^r(s))\,d\theta^r(s)$$

is a P_x*-martingale.*

PROOF.

$$d[u(X^r(t\wedge\tau_4))] = \chi_{\tau_4>t}F(X^r(t))\,dt + \chi_{\tau_4>t}G(X^r(t))d\theta^r(t) + dM(t),$$

where $M(t)$ is a P_x-martingale. Using the generalized Itô formula, we have

$$d[e^{-\lambda(t\wedge\tau_4)}u(X^r(t\wedge\tau_4))] = [\chi_{\tau_4>t}e^{-\lambda(t\wedge\tau_4)}(F-\lambda u)(X^r(t))]\,dt$$
$$+ \chi_{\tau_4>t}e^{-\lambda(t\wedge\tau_4)}G(X^r(t))\,d\theta^r(t) + e^{-\lambda(t\wedge\tau_4)}dM(t).$$

Since the last term of the above is a P_x-martingale, the lemma is proved.

DEFINITION 5.2. If the form (5.3) of a $C_b(D)$-function u is a P_x-martingale for any $x \in D \cup \partial D_2 \cup \partial D_3$, then we say that

$$\overline{(L-\lambda)}u = F - \lambda u \quad \text{on } D\cup\partial D_2 \quad\text{and}\quad \overline{\left(\frac{\partial}{\partial\nu}\right)}u = G \quad \text{on } \partial D_3.$$

DEFINITION 5.3. H_λ is the subset of E with smooth u such that $(L-\lambda)u = f$ on $D \cup \partial D_2$ and $\partial u/\partial\nu = G$ on ∂D_3. $\bar{H}_\lambda$ is the closure of H_λ.

LEMMA 5.3. *If* $\overline{(L-\lambda)}u = f$ *on* $D \cup \partial D_2$ *and* $\overline{(\partial/\partial\nu)}u = g$ *on* ∂D_3, *then*

$$u(x) = -E_x\left[\int_0^{\tau_4} e^{-\lambda t}f(X^r(t))\,dt + \int_0^{\tau_4} e^{-\lambda t}g(X^r(t))\,d\theta^r(t)\right]$$
$$+ E_x[e^{-\lambda\tau_4}u(X^r(\tau_4))] \quad \textit{for } x \in D\cup\partial D_2\cup\partial D_3.$$

PROOF. This follows directly from Definition 5.2 for $t \to \infty$,

LEMMA 5.4. *If u is given by*

$$u(x) = -E_x\left[\int_0^{\tau_4} e^{-\lambda t}f(X^r(t))\,dt + \int_0^{\tau_4} e^{-\lambda t}g(X^r(t))\,d\theta^r(t)\right]$$
$$+ E_x[e^{-\lambda\tau_4}h(X^r(\tau_4))],$$

then $(u, f) \in \bar{H}_\lambda$.

PROOF. Without loss of generality, we can and do assume that f and g are smooth.

Step 1. Let U_2 be a neighborhood of ∂D_2, *and* $\Gamma_2 \equiv \partial U_2 \cap \bar{D}^c$ be smooth. Set $D' = U_2 \cup D$. Let L be nondegenerate on $\bar{D}' - \bar{D}$. By a method similar to that in Section 3, we construct the diffusion $X^5(t)$ on $\bar{D}'$ such that $X^5(t) \simeq X^r(t)$ for $t \leqq \tau_4$ and $X^5(t)$ is associated with L outside of D and stops at Γ_2. Set $\xi_2 = \inf\{t \geqq 0; X^5(t) \in \Gamma_2\}$. Define

$$v(x) = -E_x\left[\int_0^{\xi_2} e^{-\lambda t}f(X^5(t))\,dt + \int_0^{\xi_2} e^{-\lambda t}g(X^5(t))\,d\theta^r(t)\right].$$

Note that $X^5(t) \in \bar{D}' - D$ for $t \geqq \tau_4$ a.s. In a way similar to that in the proof of Theorem 4.1, we see that $v(x)$ is a $C_b(\bar{D}')$-function.

Step 2. Let U_3 be an open neighborhood of ∂D_3, and $\Gamma_3 = \partial U_3 \cap D$ be smooth. Consider the following problem:

$$(5.4)\qquad \begin{aligned} &(L-\lambda)w = f \quad \text{on } U_3 \cap D \\ &\frac{\partial w}{\partial \nu} = g \quad \text{on } \partial D_3, \quad \text{and} \quad w = v \quad \text{on } \Gamma_3. \end{aligned}$$

Since L is nondegenerate on $U_3 \cap D$, (5.4) has the unique $C_b^2(U_3 \cap D)$-function w. Let $X^3(t)$ be the diffusion constructed in Step 1 of the proof of Theorem 3.1, and $\xi_3 = \inf\{t \geqq 0; X^3(t) \in \Gamma_3\}$. The stochastic representation of w is given by

$$\begin{aligned} w(x) &= -E_x\left[\int_0^{\xi_3} e^{-\lambda t} f(X^3(t))\,dt + \int_0^{\xi_3} e^{-\lambda t} g(X^3(t))\,d\theta^r(t)\right] \\ &\quad + E_x[e^{-\lambda \xi_3}v(X^3(\xi_3))] = -E_x\left[\int_0^{\xi_2} e^{-\lambda t}f(X^5(t))\,dt\right. \\ &\quad \left.+ \int_0^{\xi_2} e^{-\lambda t} g(X^5(t))\,d\theta^r(t)\right] = v(x) \quad \text{for } x \in U_3 \cap D. \end{aligned}$$

Thus $v(x) \in C_b^2(U_3 \cap D)$.

Step 3. Let U_3' be an open neighborhood of ∂D_3 such that $U_3' \subset U_3$ and $\Gamma_3' \equiv \partial U_3' \cap D$ be smooth. Consider the Dirichlet problem:

$$(5.5)\qquad \begin{aligned} &(L-\lambda)W = f \quad \text{on } (U_3')^c \cap D' \\ &W = v \quad \text{on } \Gamma_3', \quad \text{and} \quad W = 0 \quad \text{on } \Gamma_2. \end{aligned}$$

By a modification of the argument in Freidlin [4], it follows that there is the unique C_b^2-solution W of (5.5). Let $\xi_4 = \inf\{t \geqq 0; X^5(t) \in \Gamma_3'\}$, and

$$\begin{aligned} W(x) &= -E_x\left[\int_0^{\xi_2 \wedge \xi_4} e^{-\lambda t}f(X^5(t))\,dt\right] + E_x[e^{-\lambda \xi_4}v(X^5(\xi_4)); \xi_2 > \xi_4] \\ &= v(x) \quad \text{for } x \in (U_3')^c \cap D'. \end{aligned}$$

We conclude that $v(x) \in C_b^2((U_3')^c \cap D')$.

Step 4. By Steps 2 and 3, it is proved that $(L-\lambda)v = f$ on $U_3 \cap D$, and on $(U_3')^c \cap D'$. Since $[U_3 \cap D] \cap [(U_3')^c \cap D'] \neq \phi$, it follows that

$$\begin{aligned} &(L-\lambda)v = f \quad \text{on } D' \\ &\frac{\partial v}{\partial \nu} = g \quad \text{on } \partial D_3. \end{aligned}$$

Let V be a smooth function such that $V = 0$ on D and $V > 0$ on $\bar{D}' - \bar{D}$. Set $f = f' - \mathrm{N}u_{\partial D_2}V$ with a positive integer N. By a slight modification of Steps 1 − 3, we can and do set $\lambda = \lambda_0 + NV$.

$$\begin{aligned} v^N(x) &= -E_x\int_0^{\xi_2}\left\{[F - Nu_{D_2}V](X^5(t))\exp\left[-\lambda_0 t - N\int_0^t V(X^5(s))\,ds\right]\right\}dt \\ &\quad + E_x\int_0^{\xi_2}\left\{G(X^5(t))\exp\left[-\lambda_0 t - N\int_0^t V(X^5(s))\,ds\right]\right\}d\theta^r(t). \end{aligned}$$

For each N, it follows that

$$(L - \lambda_0)v^N = f' - Nu_{D_2}V = f' \quad \text{on } D \cup D_2$$
$$\frac{\partial}{\partial \nu} v^N = G \quad \text{on } \partial D_3.$$

It is not difficult to show that $\| v^N - u \| \to 0$ on $D \cup \partial D_2$ as $N \to \infty$.

LEMMA 5.5. *$(u, f) \in \bar{H}$ if and only if $(u, f - \lambda u) \in \bar{H}_\lambda$.*

PROOF. Clear.

Now, Lemmas 5.1 – 5.5 complete the proof of Theorem 5.2.

6. Appendix. We present sufficient conditions for (4.2).

LEMMA 6.1. *Let* (A.1), (C.1), *and* (C.2) *hold. Assume that $\partial D_0 = \phi$ and that* (A.6). *L is nondegenerate inside D.*
Then

$$\sup_{x \in \bar{D}} E_x \tau_4 < \infty. \tag{6.1}$$

PROOF. *Step* 1. Let U_i' $i = 1, 2, 3$ be ϵ-neighborhoods, which are mutually disjoint. Set $U_i = U_i' \cap D$ and $r(x) = \text{distance}\,[x, \partial D]$. We define a $C_b^2(\bar{D})$-function $W_1(x)$ with adjustable constants K, k, q:

$$\begin{aligned} W_1(x) &= K - kr(x) && x \in U_1 \\ &= kr(x) && x \in U_2 \\ &= K - kr^2(x) && x \in U_3 \\ &= K - kr^q(x) && x \in D_\epsilon^0 = \{x \in D; r(x) \geqq 2\epsilon\}. \end{aligned} \tag{6.2}$$

(To construct $W_1(x)$, we set $W_1'(x)$ as (6.2), and $W_1(x)$ is given by a mollifier of $W_1'(x)$.) We have $LW_1(x) \leqq -1$ for $x \in D$ and $W_1(x) \geqq 0$, with suitable K, k, and q.

Step 2. Applying the Itô formula to $W_1(x)$, we obtain

$$E_x W_1(X^r(\tau_4 \wedge N)) - W_1(x)$$
$$= E_x \int_0^{\tau_4 \wedge N} LW_1(X^r(s))\, ds + E_x \int_0^{\tau_4 \wedge N} \left(\frac{\partial}{\partial \nu} W_1(X^r(s))\, d\theta^r(s) \right) \leqq - E_x \tau_4 \wedge N.$$

Since $W_1(x)$ is nonnegative and bounded in $\bar{D}$, we find that

$$\sup_{x \in \bar{D}} E_x \tau_4 \wedge N \leqq \sup_{x \in \bar{D}} W_1(x) < \infty.$$

The proof is completed by letting $N \uparrow \infty$.

Let U_0' be a neighborhood of the boundary ∂D_0, and set $U_0 = U_0' \cap D$.

DEFINITION 6.1. We say that the $\sum_0$-bondary ∂D_0 is reflective, if there is a $C^2(U_0)$-function $V(x)$ such that

$$LV(x) \leqq -1 \quad \text{in } U_0, \quad \text{and} \quad V(x) \geqq 0 \quad \text{in } U_0. \tag{A.3}$$

LEMMA 6.2. *Let* (A.1), (C.1), *and* (C.2) *hold. Assume that* (A.6) *holds and that ∂D_0 is reflective. Then* (4.2) *holds.*

PROOF. *Step* 1. Let U_0' be a neighborhood of ∂D_0, which is disjoint to $\cup_{i=1}^3 U_i'$. Set $U_0 = U_0' \cap D$. By using the same procedure as in Lemma 6.1, we obtain the $C^2(\bar{D} - \partial D_0)$-

function $W_2(x)$ such that

$$W_2(x) = V(x) \quad \text{in } U_0$$
$$= W_1(x) \quad \text{otherwise.}$$

By the hypothesis, we have

$$W_2(x) \geqq 0 \quad \text{in } \bar{D} - \partial U_0, \quad \text{and} \quad LW_2(x) \leqq -1 \quad \text{in } \bar{D} - \partial D_0.$$

Step 2. Applying the Itô formula to W_2, we have for $x \in \bar{D} - \partial D_0$

$$E_x[\tau_4 \wedge N] \leqq W_2(x) - E_x W_2(X^r(\tau_4 \wedge N)) \leqq W_2(x).$$

As $N \uparrow \infty$, it follows that

$$E_x \tau_4 \leqq W_2(x) < \infty \quad \text{for each } x \in \bar{D} - \partial D_0.$$

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