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CONSTRUCTION OF A MARTINGALE WITH GIVEN ABSOLUTE VALUE

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Let Y be a nonnegative submartingale. A martingale M is constructed on an enlarged space, with the property that $Y = |M|$.

1. A well-known consequence of Jensen's inequality is that the absolute value of a martingale is a submartingale. Gilat, in [4], has proved the converse result that every nonnegative submartingale is equal in law to the absolute value of a martingale. More precisely, given a nonnegative submartingale $Y = (Y_t, t \geq 0)$ defined on a probability space $(\Omega, \mathscr{F}, P)$, there exists a martingale $M = (M_t, t \geq 0)$ on another space $(\Omega', \mathscr{F}', P')$ such that $|M|$ and Y have the same law.

This leaves open the question of whether we can construct a martingale M on $(\Omega, \mathscr{F}, P)$ such that $|Y| = M$. Indeed, this may not be possible if $(\Omega, \mathscr{F}, P)$ does not contain sources of randomisation additional to Y. In such a case it is necessary to enlarge $(\Omega, \mathscr{F}, P)$, by taking its product with another probability triple. Protter and Sharpe [8], and Maisonneuve [5], have shown how M may then be constructed in the case where Y and Y_- are strictly positive, and Barlow and Yor [1] have given a construction of M in the case where the unique increasing previsible process B such that $Y - B$ is a martingale satisfies $\int_0^t I(Y_{s-} > 0)\, dB_s = 0$.

In this paper we give a construction of M for a general nonnegative Y. The basic idea of the construction when Y_- is not zero is to define a set of points Γ at which M changes sign, and to do this in such a way that M is a martingale. A simple calculation suggests that the probability of a jump in the interval dt should be $(1/2Y_{t-})\, dB_t$ if $\Delta B_t = 0$, and $\frac{1}{2}\Delta B_t/(Y_{t-} + \Delta B_t)$ if $\Delta B_t \neq 0$. Thus if Y and Y_- are strictly positive then the points of Γ form a discrete set, and the sign change presents no problem. For a general Y, however, Γ may have accumulation points. The problem of defining M after a right accumulation point of Γ is solved by providing the points of Γ with a random sign, and defining the sign of M_t to be the sign of the last point in Γ before t. The left accumulation points of Γ do not present such a difficulty. A different approach must be adopted on the set $\{Y_- = 0\}$, and, as in [1], a random sign is assigned to each excursion of Y_- from 0.

We may summarise the results of this paper in the following theorem.

Theorem 2. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a complete filtered probability space satisfying the usual conditions, and Y be a nonnegative submartingale$/(\mathscr{F}_t)$. Suppose there exists a random variable ϕ on $(\Omega, \mathscr{F}, P)$ independent of $\mathscr{F}_\infty$ and with continuous distribution function. Then we may construct a filtration $(\mathscr{M}_t)$ and a process M_t such that*

(i) $\mathscr{F}_t \subset \mathscr{M}_t$ *for* $t \geq 0$;
(ii) *every martingale*$/(\mathscr{F}_t)$ *is a martingale*$/(\mathscr{M}_t)$;
(iii) M *is a martingale*$/(\mathscr{M}_t)$;
(iv) $|M| = Y$.

2. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space satisfying the usual conditions. If

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Γ is a subset of $\Omega \times [0, \infty)$ we set $\Gamma(\omega) = \{t \geq 0 : (\omega, t) \in \Gamma\}$, the cut of Γ at ω. Let $\#(\Gamma)(\omega)$ be the number of points in $\Gamma(\omega)$. Given nonnegative random variables S, T with $S < T$ we denote the stochastic interval $\{(\omega, t) : S(\omega) < t < T(\omega)\}$ by (S, T), and the graph of S, that is $\{(\omega, t) : S(\omega) = t\}$ by $[S]$. Let $X = (X_t, t \geq 0)$ by any process with left limits: we define the processes X^T and X^{T-} by $X_t^T = X_{t \wedge T}$, $X_t^{T-} = X_t I(t < T) + X_{T-} I(t \geq T)$. Throughout this paper by martingale (resp submartingale) we mean martingale (resp submartingale) with paths which are right-continuous and with left limits. If X is a martingale and T is a stopping time then X^T is a martingale, and if T is previsible X^{T-} is also a martingale.

If $(\mathscr{G}_t)$ is any filtration, and R is a nonnegative random variable, we define the σ-field $\mathscr{G}_R$ by

$$\mathscr{G}_R = \sigma(X_R, X \text{ a process optional}/(\mathscr{G}_t)).$$

3. Now let Y be a nonnegative submartingale/$(\mathscr{F}_t)$. Let B be the increasing previsible process such that $Y - B$ is a martingale, let $A_t = \int_0^t I(Y_{s-} > 0)\, dB_s$, and let $A = A^c + A^d$ be the decomposition of A into continuous and pure jump parts, as in [3], IV. T37.

We shall assume there exists a σ-field $\mathscr{H} \subset \mathscr{F}$ independent of $\mathscr{F}_\infty$ and carrying a random variable with a continuous distribution function; if no such σ-field exists then we may construct one by taking a suitable product enlargement of $(\Omega, \mathscr{F}, P)$. The space $(\Omega, \mathscr{H}, P)$ is then sufficiently rich to support any sequence of independent random variables.

Let us define on $\mathscr{H}$ the following independent sequences of random variables:

(i) sequences ϕ_n, η_n, ψ_{nm}, $n \geq 1$, $m \geq 1$ of independent rv's taking values in $\{-1, 1\}$ and with $E\phi_n = E\eta_n = E\psi_{nm} = 0$;

(ii) a sequence ρ_n, $n \geq 1$ of independent rv's distributed uniformly on $[0, 1]$;

(iii) sequences λ_{nm}, $n \geq 1$, $m \geq 1$ of independent rv's with negative exponential distribution with mean 1.

Set, for $n \geq 1$,

$$J_t^n = \int_0^t (1/Y_{s-}) I_{[1/n, 1/(n-1))}(Y_{s-})\, dA_s^c \qquad (I_{[1,\infty)} \text{ if } n = 1).$$

Thus J^n is continuous and $J_t^n \leq nA_t^c$. Now for $n \geq 1$ and $j \geq 1$ let

$$T^{nj} = \inf\{t \geq 0 : J_t^n = \textstyle\sum_{k=1}^j \lambda_{nk}\}.$$

Let (S_j) be a sequence of disjoint previsible times exhausting the jumps of A^d, and set

$$R^j = \begin{cases} S_j & \text{if } \rho_j \leq \Delta A_{S_j}/(Y_{S_j-} + \Delta A_{S_j}) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $Y_{R_j-} > 0$, as A increases only on $\{Y_- > 0\}$. We associate with T^{nj} the sign ψ_{nj}, and with R^j the sign η_j. Let $T^{0j} = R^j$, $\psi_{0j} = \eta_j$, $\Gamma^n = \cup_{j=1}^\infty [T^{nj}]$, and $\Gamma = \cup_{n=0}^\infty \Gamma^n$.

LEMMA 1.

(i) $[T^{nj}] \subset \{1/n \leq Y_{t-} \leq 1/n - 1\}$ *for* $n \geq 1$;

(ii) *for each* $t \geqslant 0$, $n \geqslant 1$, $\#(\Gamma^n \cap [0, t])$ *is finite* a.s.;

(iii) a.s. *for all* t, $\#(\Gamma^0 \cap [0, t] \cap \{Y_- \geq 1/n\})$ *is finite*;

(iv) *a set* Ω_0 *exists with* $P(\Omega_0) = 1$, *and with the property that if* $[U, V]$ *is any stochastic interval such that* $V < \infty$ *and* $[U, V] \subset \{Y \geqslant 1/n\}$, *then* $\#(\Gamma \cap [U, V])$ *is finite on* Ω_0;

(v) *let* U *be a right* (resp *left*) *accumulation point of* Γ. *Then* $Y_U = 0$ (resp $Y_{U-} = 0$) *on* Ω_0.

(vi) *Let* $n \geqslant 1$, $j \geqslant 1$, *and let* U *be any nonnegative* rv *measurable with respect to* $\sigma(\mathscr{F}_\infty, T^{mk}, m \neq n, k \geq 1)$. *Then* $P(T^{nj} = U) = 0$;

(vii) Γ *has no double point* a.s.

PROOF. (i) is immediate from the definition of T^{nj}. We have $\#(\Gamma^n \cap [0, t]) = \sup\{r: \sum_{k=1}^{r} \lambda_{nk} \le J_t^n\}$, from which (ii) follows. For each j

$$P(\{R_j = S_j\} \cap \{S_j \le t\} \cap \{Y_{S_j-} \ge 1/n\} \mid \mathscr{F}_\infty) \le n\Delta A_{S_j} I(Y_{S_j-} \ge 1/n) I(S_j \le t),$$

as ρ_j is independent of $\mathscr{F}_\infty$. Thus by Borel-Cantelli, as $EA_t < \infty$, we have

$$P(R_j = S_j,\, S_j \ge t,\, Y_{S_j-} \ge 1/n \text{ for infinitely many } j) = 0,$$

proving (iii).

(iv) For each t and n let F_{nt} (resp G_{nt}) be the set on which (ii) (resp (iii)) fails. Set $\Omega_0^c = \cup_{n\ge1} \cup_{m\ge1} (F_{nm} \cup G_{nm})$, and note that (ii) and (iii) hold identically on Ω_0: the result is now immediate.

(v) Let $\omega \in \Omega_0$, and suppose that $Y_U(\omega) > 0$. Then $Y_U(\omega) > 1/n$ for some n, and by the right-continuity of Y there exists $V(\omega)$ with $V(\omega) > U(\omega)$ such that $Y_s(\omega) > 1/n$ for $U(\omega) \le s \le V(\omega)$. By (iv) there are only finitely many points of $\Gamma(\omega)$ in $[U(\omega), V(\omega)]$, so that $U(\omega)$ is not a right accumulation point of $\Gamma(\omega)$. Similarly, if $Y_{U-} > 0$, U cannot be a left accumulation point of Γ.

(vi) $P(T^{nj} = S) = P(\sum_{k=1}^{j} \lambda_{nj} = Z_S^n)$, and the last term is 0 as $\sum_{k=1}^{j} \lambda_{nk}$ has a continuous distribution and is independent of Z_S^n.

(vii) It is sufficient to show that $P(T^{mk} = T^{nj}) = 0$ when $(m, k) \ne (n, j)$. This is immediate if $n = m$, and follows from (vi) if $n \ne m$.

Let ϵ_{nm} denote the mth excursion of Y_- from 0 the duration of which lies in the interval $[1/n, 1/n - 1)$, for $n \ge 1$, $m \ge 1$. To avoid too many subscripts we shall renumber the ϵ_{nm} so that they are indexed by a single integer n. Let α_n, β_n denote the left and right endpoints of ϵ_n: note that β_n is a stopping time, which might not be the case had we chosen another way of numbering these excursions. We associate with the excursion ϵ_n the sign ϕ_n.

Let

$$C_t = \textstyle\sum_n I_{[\alpha_n, \beta_n]}(t).$$

For $n \geqslant 0, j \geqslant 1$, let

$$S^{nj} = \inf\{t > T^{nj}: (\omega, t) \in \Gamma \cap \{Y_- = 0\}\}.$$

By Lemma 1(v), $S^{nj} > T^{nj}$ a.s. on $\{Y_{T^{nj}} > 0\}$. Set $\Lambda = \cup_{n\ge0} \cup_{j\ge1} [T^{nj}, S^{nj})$, and define a second sign-change process as follows:

$$G_t = \textstyle\sum_{n\ge0; j\ge1} I_{[T^{nj}, S^{nj})}(t)\, \psi_{nj} + I_{\Lambda^c}(t).$$

By the definition of S^{nj} the intervals $\{T^{nj}, S^{nj})$ are disjoint, and therefore $|G_t| = 1$.

Let $M_t = C_t G_t Y_t$. We have $|M_t| = Y_t$, and will prove that M is a martingale. We may note that if $Y_- > 0$ then C is constant, and that if A is 0 then $G = 1$. Thus the role of C is to make M a martingale on $\{Y_- = 0\}$, and that of G is to make M a martingale on $\{Y_- > 0\}$.

LEMMA 2. *M is right-continuous* a.s.

PROOF. On $\{Y_t = 0\}$ we have $|\limsup_{s\downarrow t} M_s| \le \limsup_{s\downarrow t} Y_s = 0$ so that M_{t+} exists and equals M_t. Let $\omega \in \Omega_0$, and suppose that $Y_t(\omega) > 0$. By right-continuity $Y_{s-}(\omega) > 0$ for $s \in (t, t + \epsilon(\omega))$ for some $\epsilon(\omega) > 0$, and hence $t \in [\alpha_n, \beta_n)$ for some n. Thus $C_t(\omega) = C_{t+}(\omega)$. As for G, if $(\omega, t) \in \Lambda$ then $(\omega, t) \in [T^{nj}, S^{nj})$ for some n, j, and consequently $G_{t+}(\omega) = G_t(\omega)$. If $(\omega, t) \notin \Lambda$ then as, by Lemma 1(v), t is not a right accumulation point of $\Gamma(\omega)$, $[t, t + \delta(\omega)] \cap \Gamma(\omega) = \varnothing$ for sufficiently small $\delta(\omega) > 0$, and thus $G_{t+}(\omega) = G_t(\omega)$. Hence M is right-continuous on Ω_0.

Since M is right-continuous, once we show that M is a martingale it will follow that M has left limits.

Now let $\mathscr{C}_t$ and $\mathscr{G}_t$ be the natural (right-continuous) filtrations of the processes C and G. Set $\mathscr{M}_t^0 = \mathscr{C}_t \vee \mathscr{G}_t \vee \mathscr{F}_t$, and $\mathscr{M}_t = \cap_{r>t} \mathscr{M}_r^0$: thus M is $(\mathscr{M}_t)$-adapted. It follows from the definition of $\mathscr{M}_{t-}$ (see [3], III, D27]), that $\mathscr{M}_{t-} = \mathscr{C}_{t-} \vee \mathscr{G}_{t-} \vee \mathscr{F}_{t-}$.

Let $\mathscr{H}_1 = \sigma(\lambda_{nm}, \rho_n, n \geqslant 1, m \geqslant 1)$, $\mathscr{H}_2 = \sigma(\phi_n, n \geqslant 1)$, and $\mathscr{H}_3 = \sigma(\eta_n, \psi_{nm}, n \geqslant 1, m \geqslant 1)$. Then $\mathscr{C}_t \subset \mathscr{F}_\infty \vee \mathscr{H}_2$, and $\mathscr{G}_t \subset \mathscr{F}_t \vee \mathscr{H}_1 \vee \mathscr{H}_3$. Note also that $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_3$, and $\mathscr{F}_\infty$ are independent.

We shall make use of some elementary results on conditional independence. Let $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ be sub-σ-fields of $\mathscr{F}$, with $\mathscr{E}_2 \subset \mathscr{E}_1$.

(i) $\mathscr{E}_1$ and $\mathscr{E}_3$ are conditionally independent of given $\mathscr{E}_2$ (we abbreviate this to $\mathscr{E}_1$ and $\mathscr{E}_3$ are c.i./$\mathscr{E}_2$) if and only if $E(e_3 \mid \mathscr{E}_1) = E(e_3 \mid \mathscr{E}_2)$ for all $e_3 \in b\mathscr{E}_3$.

(ii) If $\mathscr{E}_1$ and $\mathscr{E}_3$ are c.i./$\mathscr{E}_2$, and $\mathscr{E}_4$ is independent of $\mathscr{E}_1 \vee \mathscr{E}_3$ then $\mathscr{E}_1 \vee \mathscr{E}_4$ and $\mathscr{E}_3$ are c.i./$\mathscr{E}_2$.

(iii) Let $\mathscr{E}_2^n$ be a sequence of σ-fields, with $\mathscr{E}_2^n \subset \mathscr{E}_1$ and such that $\mathscr{E}_1$ and $\mathscr{E}_3$ are c.i./$\mathscr{E}_2^n$ for each n. Then $\mathscr{E}_1$ and $\mathscr{E}_3$ are c.i./$\cap_{n=1}^{\infty} \mathscr{E}_2^n$.

((i) is a corollary of [6], II, T51, and (ii) and (iii) follow easily from (i).)

Lemma 3. *$\mathscr{M}_t$ and $\mathscr{F}_\infty$ are* c.i./$\mathscr{F}_t$.

Remark. This implies that every martingale/$(\mathscr{F}_t)$ is a martingale/$(\mathscr{M}_t)$-see [2]. In particular Y is a submartingale/$(\mathscr{M}_t)$.

Proof. Suppose we have that $\mathscr{F}_\infty$ and $\mathscr{C}_t$ are c.i./$\mathscr{F}_t$. An application of the monotone class lemma now shows that $\mathscr{F}_\infty$ and $\mathscr{C}_t \vee \mathscr{F}_t$ are c.i./$\mathscr{F}_t$; hence, by (ii) above, $\mathscr{F}_\infty$ and $\mathscr{C}_t \vee \mathscr{F}_t \vee \mathscr{H}_1 \vee \mathscr{H}_3$ are c.i./$\mathscr{F}_t$. But $\mathscr{M}_t^0 \subset \mathscr{C}_t \vee \mathscr{F}_t \vee \mathscr{H}_1 \vee \mathscr{H}_3$, so that $\mathscr{M}_t^0$ and $\mathscr{F}_\infty$ are c.i./$\mathscr{F}_t$. Now $\mathscr{M}_t \subset \mathscr{M}_r^0$ for $r > t$, and therefore $\mathscr{M}_t$ and $\mathscr{F}_\infty$ are c.i./$\mathscr{F}_r$ for any $r > t$; the result follows by (iii).

A monotone class argument now shows that it is sufficient to check that for $f \in b\mathscr{C}_t$ of the form $f = \prod_{i=1}^n I(C_{t_i} = a_i)$, where $0 \le t_1 \le \cdots \le t_n \le t$ and $a_i = -1$, 0 or 1, we have $E(f \mid \mathscr{F}_\infty) \in \mathscr{F}_t$. We will treat only the case $n = 1$, as the proof given below may easily be extended to general n.

Now $\{|C_t| = 1\} = \cap_{m\geqslant 1} \cup_{n \geqslant m} \{Y_{s-} > 0, \text{ for } s \in (t, t + 1/n)\}$, which is $\mathscr{F}_{t+} = \mathscr{F}_t$ measurable. Thus $|C_t|$ is $(\mathscr{F}_t)$-adapted, so that if $a_1 = 0$ then $f \in \mathscr{F}_{t_1}$. Suppose that $|a_1| = 1$. Then

$$\begin{aligned} E(f \mid \mathscr{F}_\infty) &= E(\textstyle\sum_n I_{[\alpha_n, \beta_n)}(t_1) I(\phi_n = a_1) \mid \mathscr{F}_\infty) \\ &= \textstyle\sum_n I_{[\alpha_n, \beta_n)}(t_1) \cdot (1/2) \\ &= \tfrac{1}{2} I(|C_{t_1}| = 1) \in \mathscr{F}_t. \end{aligned}$$

Lemma 4.

(i) $E(\phi_n I(\alpha_n \ge t) \mid \mathscr{M}_{t-} \vee \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_3) = 0$;

(ii) $E(\psi_{nj} I(T^{nj} \ge t) \mid \mathscr{M}_{t-} \vee \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_2) = 0$.

Proof. (i) We have that $\mathscr{M}_{t-} \vee \mathscr{F}_\infty \subset \mathscr{H}_1 \vee \mathscr{H}_3 \vee \mathscr{C}_{t-} \vee \mathscr{F}_\infty$; since $\phi_n I(\alpha_n \ge t) \in \mathscr{C}_\infty \vee \mathscr{F}_\infty$ and $\mathscr{H}_1 \vee \mathscr{H}_3$ is independent of $\mathscr{C}_\infty \vee \mathscr{F}_\infty$ it is sufficient to verify that $E(\phi_n I(\alpha_n \ge t) \mid \mathscr{C}_{t-} \vee \mathscr{F}_\infty) = 0$. As in the previous lemma it is enough that $E\phi_n I(\alpha_n \ge t) \cdot fg = 0$, where $f = \prod_1^n I(C_{t_i} = a_i)$, $0 \le t_1 \le \cdots \le t_n < t$, $a_i = -1, 0, +1$, and $g \in b\mathscr{F}_\infty$. Suppose $m = 1$. Then since $E(\phi_n \mid \mathscr{F}_\infty) = 0$, by the independence of $\mathscr{H}_2$ and $\mathscr{F}_\infty$, the result is immediate if $a_1 = 0$. So let $|a_1| = 1$: then

$$Efg\phi_n I(\alpha_n \ge t) = \textstyle\sum_r E(\phi_n \phi_r g I(\alpha_n \ge t) I(\alpha_r \le t < \beta_r)).$$

The term in the sum with $r = n$ is zero, since $t_1 < t$, and as $E(\phi_n \phi_r \mid \mathscr{F}_\infty) = 0$ for $r \neq n$ the remaining terms are also zero. The proof for $m > 1$ is essentially the same.

(ii) Since $\mathscr{M}_{t-} \vee \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_2 \subset \mathscr{G}_{t-} \vee \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_2$, $\psi_{nj} I(T^{nj} \ge t) \in \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_3$, and $\mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_3$ and $\mathscr{H}_2$ are independent, it is sufficient to prove that $E(\psi_{nj} I(T^{nj} \ge t) \mid \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{G}_{t-}) = 0$. Thus by the monotone class lemma it is enough that $Efg\psi_{nj} I(T^{nj}$

$\geq t) = 0$ for $f \in b(\mathscr{F}_\infty \vee \mathscr{H}_1)$, and g of the form $g = \prod_{i=1}^{m} I(G_{t_i} = a_i)$, where $0 \leq t_1 \leq \cdots \leq t_m < t$, and $a_i = \pm 1$. As before we shall verify this only for $m = 1$: the proof for $m > 1$ is very similar. We have

$$Efg\psi_{nj}I(T^{nj} \geq t) = Ef\psi_{nj}I(T^{nj} \geq t)(I_{\Lambda^c}(t_1)I(a_1 = 1) + \textstyle\sum_{r\geq 0; k\geq 1} I_{[T^{rk}, S^{rk})}(t_1)I(\psi_{rk} = a_1)).$$

All the terms in this last expression except those containing ψ's are $\mathscr{F}_\infty \vee \mathscr{H}_1$ measurable. The result therefore follows, since $E(\psi_{nj} \mid \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \sigma(\psi_{rk})) = 0$ for $(r, k) \neq (n, j)$, and $I(T^{nj} \geq t)I(T^{nj} \leq t_1 < S^{nj}) = 0$.

The following time-substitution result will be required later.

Lemma 5. *Let $(\mathscr{E}_t)$ and $(\mathscr{F}_t)$ be two filtrations, such that $\mathscr{E}_\infty$ and $\mathscr{F}_\infty$ are independent. Let Z be a martingale/$(\mathscr{E}_t)$, and K be a continuous, nondecreasing $(\mathscr{F}_t)$-adapted process, satisfying $0 \leq K_t \leq \lambda t$ for some $\lambda > 0$. Then $(\mathscr{F}_t \vee \mathscr{E}_{K_t}, t \geq 0)$ is a nondecreasing family of σ-fields, and $(Z_{K_t}, t \geq 0)$ is a martingale/$(\mathscr{F}_t \vee \mathscr{E}_{K_t})$.*

Proof. Z_{K_t} is right-continuous, as Z and K are.

The definition of $\mathscr{E}_{K_t}$ is given in Section 2: note that $(\mathscr{E}_{K_t}, t \geq 0)$ is not necessarily nondecreasing. Let X be any $(\mathscr{E}_t)$-optional process: then since X^s is also $(\mathscr{E}_t)$-optional, $X_{s\wedge K_t} \in \mathscr{E}_{K_t}$, and thus $X_{s\wedge K_t} \in \mathscr{F}_t \vee \mathscr{E}_{K_t}$ for any $s \geq 0$. Consequently, as $K_s \in \mathscr{F}_t$ when $s \leq t$, $X_{K_s} = X_{K_s \wedge K_t} \in \mathscr{F}_t \vee \mathscr{E}_{K_t}$, and therefore $\mathscr{E}_{K_s} \subset \mathscr{F}_t \vee \mathscr{E}_{K_t}$ when $s \leq t$, which proves that $(\mathscr{F}_t \vee \mathscr{E}_{K_t}, t \geq 0)$ is nondecreasing.

Now let $\mathscr{E}'_t = \mathscr{E}_t \vee \mathscr{F}_\infty$: then $\mathscr{E}_\infty$ and $\mathscr{E}'_t$ are c.i./$\mathscr{E}_t$, and thus Z is a martingale/$(\mathscr{E}'_t)$—see [2]. K_t is a bounded stopping time/$(\mathscr{E}'_t)$, so that $E(Z_{K_t} \mid \mathscr{E}'_{K_s}) = Z_{K_s}$. Therefore, as $Z_{K_s} \in \mathscr{F}_s \vee \mathscr{E}_{K_s} \subset \mathscr{E}'_{K_s}$, Z_{K_t} is a martingale/$(\mathscr{F}_t \vee \mathscr{E}_{K_t})$.

Theorem 1. *M is a martingale/$(\mathscr{M}_t)$.*

Proof. It is sufficient to prove that for $s < r$

$$E(M_r \mid \mathscr{M}_{s-}) = E(M_s \mid \mathscr{M}_{s-}). \tag{1}$$

For suppose that (1) is true, if (s_n) decreases to s with $s_1 < r$ then, since $\mathscr{M}_s = \cap_n \mathscr{M}_{s_n-}$, $E(M_r \mid \mathscr{M}_{s_n-})$ converges a.s. and in L^1 to $E(M_r \mid \mathscr{M}_s)$. On the other hand

$$E \mid E(M_{s_n} \mid \mathscr{M}_{s_n-}) - M_s \mid \leq E \mid M_{s_n} - M_s \mid.$$

This last expression tends to 0 as $n \to \infty$, as M is right-continuous, $\mid M_{s_n} - M_s \mid \leq E(Y_r \mid \mathscr{F}_{s_n}) + Y_s$, and $(E(Y_r \mid \mathscr{F}_{s_n}), n \geq 1)$ is uniformly integrable. Thus $E(M_r \mid \mathscr{M}_s) = M_s$, and M is a martingale.

So let s, r be fixed with $s < r$. Let T_1 be the debut of $\{Y_- = 0\} \cap (s, \infty)$, and T_2 be the debut of $(\{Y_- = 0\} \cap (s, \infty)) \backslash (T_1, \infty)$. This set is previsible, and is also the graph of T_2: thus T_2 is previsible, and $T_2 > s$. Let T_3 be the debut of $\Gamma \cap (s, \infty)$, and $T = T_1 \wedge T_3$.

We have $M_r = M_r I(r \geq T) + M_r I(r < T)$. First let us verify that $E(M_r I(r \geq T) \mid \mathscr{M}_{s-}) = 0$. For this it is enough that

$$E(C_r G_r I(Y_r \neq 0) I(r \geq T) \mid \mathscr{M}_{s-} \vee \mathscr{F}_\infty) = 0. \tag{2}$$

Substituting for C_r and G_r, and rearranging the terms, we have

$$C_r G_r I(Y_r \neq 0) I(r \geq T) = \textstyle\sum_m \phi_m f_m + \sum_{m,n,j} \phi_m \psi_{nj} g_{mnj}, \tag{3}$$

where $f_m = I(Y_r \neq 0)I(r \geq T)I_{[\alpha_m, \beta_m)}(r)(I_{\Lambda^c}(r) + I(T = T_1) \sum_{n,j} \psi_{nj} I_{[T^{nj}, S^{nj})}(r))$, and

$$g_{mnj} = I(Y_r \neq 0)I(r \geq T)I(T < T_1)I_{[\alpha_m, \beta_m)}(r)I_{[T^{nj}, S^{nj})}(r).$$

Lemma 6.
(i) $g_{mnj} I(T^{nj} < s) = 0$ a.s.;
(ii) $f_m I(\alpha_m < s) = 0$ a.s.

Proof. (i) If $T < T_1$ and $r \ge T$ then $T_3 \le r$, and if $T^{nj} < s$ then $S_{nj} \le T_3$. Consequently $I(r \ge T)I(T < T_1)I(T^{nj} < s < r < S^{nj}) = 0$, from which it is immediate that $g_{mnj}I(T_{nj} < s) = 0$.

(ii) On $\{\alpha_m < s < \beta_m\}$ we have $T_1 = \beta_m$; thus $\{T_1 \le r\} \cap \{\alpha_m < s < r < \beta_m\} = \varnothing$. Therefore $I(T = T_1)I(T \le r)I_{[\alpha_m, \beta_m)}(r)I(\alpha_m < s) = 0$, and it remains to show that $P(F) = 0$, where $F = \{Y_r \ne 0\} \cap \{r \ge T\} \cap \{\alpha_m < s < r < \beta_m\} \cap \{r \in \Lambda^c\}$.

Suppose $\omega \in F \cap \Omega_0$, where Ω_0 is the set of probability one introduced in Lemma 1. Now $r \notin \Gamma(\omega)$, and $Y_r(\omega) \ne 0$; therefore, by Lemma 1(v), r is not a right accumulation point of $\Gamma(\omega)$, and so there exists $\epsilon(\omega) > 0$ such that $[r, r + \epsilon(\omega)) \cap \Gamma(\omega) = \varnothing$. Let $\tau(\omega) = \sup\{u < r : u \in \Gamma(\omega)\}$. As $T_1(\omega) = \beta_m(\omega)$, and $T(\omega) \le r$, it follows that $T_3(\omega) \le r$, and hence that $(s, r + \epsilon(\omega)) \cap \Gamma(\omega) \ne \varnothing$. Thus $\tau(\omega) \ge s$. However, $Y_-(\omega)$ is never 0 on $(\alpha_m(\omega), \beta_m(\omega))$, which implies firstly that $\tau(\omega) < r$, and secondly that $\tau(\omega)$ is not a left accumulation point of $\Gamma(\omega)$. Since $\tau(\omega)$ cannot be a right accumulation point of $\Gamma(\omega)$ either, $\tau(\omega) = T^{nj}(\omega)$ for some n, j.

By the definition of τ, $(\tau(\omega), r + \epsilon(\omega)) \cap \Gamma(\omega) = \varnothing$, and thus $S^{nj}(\omega) \ge r + \epsilon(\omega)$. But then $r \in [T_{nj}(\omega), S^{nj}(\omega)) \subset \Gamma(\omega)$, which is a contradiction. Therefore $F \cap \Omega_0 = \varnothing$, and so $P(F) = 0$, completing the proof of the lemma.

Resuming the proof of Theorem 2, by Lemma 6 we may rewrite (3) in the form

$$C_rG_rI(Y_r \ne 0)I(r \ge T) = \textstyle\sum_m \phi_m f_m I(\alpha_m \ge s) + \sum_{m,n,j} \phi_m \psi_{nj} g_{mnj} I(T^{nj} \ge s). \tag{4}$$

We have $f_m \in \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_3$, and $g_{nmj} \in \mathscr{F}_\infty \vee \mathscr{H}_1$. By Lemma 4, $E(\phi_m f_m I(\alpha_m \ge s) \mid \mathscr{M}_{s-} \vee \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_3) = 0$, and $E(\phi_m g_{mnj} \psi_{nj} I(T^{nj} \ge s) \mid \mathscr{M}_{s-} \vee \mathscr{F}_\infty \vee \mathscr{H}_1 \vee \mathscr{H}_2) = 0$, which establishes (2).

To complete the proof of the theorem we must show that $E(M_r I(r < T) \mid \mathscr{M}_s) = M_s$. Let $U = Y^{T_1 \wedge (T_2-)}$: thus, $U_t = \int_0^t I_{[0,T_1]}(u) I_{[0,T_2)}(u)\, dY_u$, U is a submartingale/$(\mathscr{M}_t)$, and $U - B^{T_1 \wedge (T_2-)}$ is a martingale/$(\mathscr{M}_t)$. Now $Y_{T_1}I(T_1 < T_2) = Y_{T_2-}I(T_1 = T_2) = 0$, and hence $U_t = Y_t I(t < T_1)$ and thus $Y_t I(t < T) = U_t I(t < T_3)$ for $s \le t$. By the definition of T we have $C_rG_rI(r < T) = C_sG_sI(r < T)$, so that $E(M_rI(r < T) \mid \mathscr{M}_s) = C_sG_sE(U_rI(r < T_3) \mid \mathscr{M}_s)$. We have that $U_sI(s = T_3) = 0$ by Lemma 1(v), since $T_3 = s$ only if s is a right accumulation point of Γ, and therefore that $U_sI(s < T_3) = Y_s$.

It is therefore sufficient to prove that $U_tI(t < T_3)$ is a martingale/$(\mathscr{M}_t)$ for $t \ge s$, or, equivalently, that $V_t = I(t \ge s)(U_tI(t < T_3) - U_s)$ is a martingale/$(\mathscr{M}_t)$ for $t \ge 0$.

A simple application of Itô's formula for semimartingales ([7], IV, 21, 23) shows that if Z is a nonnegative submartingale with decomposition $Z = Z_0 + K + N$ ($K_0 = N_0 = 0$, K previsible increasing, N a martingale), and D is a nonnegative, decreasing optional process with $D_s = 1$ then $I_{[s,\infty)}(Z_tD_t - Z_s)$ is a martingale if and only if

$$\int_s^t Z_u\, dD_u + \int_s^t D_{u-}\, dK_u \quad \text{is a martingale.} \tag{5}$$

So set $D_t = I(t < T_3)$, and $Z = U$. Evaluating (5) and using the fact that $B_t^{T_1 \wedge (T_2-)} - B_s = A_t^{T_1} - A_s$, we see that V is a martingale/$(\mathscr{M}_t)$ if and only if

$$-I_{[T_3,\infty)}(t)\, U_{T_3} + I_{[s,\infty)}(t)(A_t^{T_1 \wedge T_3} - A_s) \quad \text{is a martingale/}(\mathscr{M}_t). \tag{6}$$

Let T_4' be the debut of $\cup_{n=1}^\infty \Gamma^n \cap (s, \infty)$, and T_5' be the debut of $\Gamma^0 \cap (s, \infty)$. Let T_4 and T_5 be the restrictions of T_4', T_5' to $\{Y_{T_4'} > 0\}$, $\{Y_{T_5'} > 0\}$ respectively. Then, since by Lemma 1(v) T_4 and T_5 cannot be right accumulation points of Γ, $[T_4] \subset \cup_{n=1}^\infty \Gamma^n$ and $[T_5] \subset \cup_{j=1}^\infty [S^j]$. Thus T_5 is accessible, and $P(T_4 = T_5) = 0$. Therefore $U_{T_3} = U_{T_4}I(T_3 = T_4) + U_{T_5}I(T_3 = T_5)$. Further, by Lemma 1(vi) $\Delta Y_{T^{nj}} = 0$ a.s. for $n \ge 1, j \ge 1$, so that $\Delta U_{T_4} = 0$ a.s. Hence

$$U_{T_3} I_{[T_3,\infty)}(t) = U_{T_4-} I_{[T_4,\infty)}(t \wedge T_3) + U_{T_5} I_{[T_5,\infty)}(t \wedge T_3),$$

and (6) is immediate from Lemma 7 and Proposition 8.

Lemma 7. *$W_t = U_{T_5} I_{[T_5,\infty)}(t \wedge T_3) - \int_s^{t \wedge T_3} I_{(s,T_1]}(u)\, dA_u^d$ is a martingale/$(\mathscr{M}_t)$.*

PROOF. As W jumps only on $\cup_{j=1}^{\infty} [S_j]$ it is sufficient to verify that $E(\Delta W_{S_j} | \mathscr{M}_{S_j-}) = 0$ for each j. Now $\Delta W_{S_j} = I(s < S_j)I(S_j \geq T)[Y_{S_j} I(S_j = R^j) - \Delta A_{S_j}]$. As ρ_j is independent of $\mathscr{M}_{S_j-} \vee \mathscr{F}_\infty$ we have $E(I(S_j = R^j) | \mathscr{M}_{S_j-} \vee \mathscr{F}_\infty) = \Delta A_{S_j}/(\Delta A_{S_j} + Y_{S_j-})$. Finally, $E(Y_{S_j} | \mathscr{M}_{S_j-}) = Y_{S_j-} + \Delta A_{S_j}$, and so as $I(s < S_j)I(S_j \geq T) \in \mathscr{M}_{S_j-}$ a simple computation completes the proof.

PROPOSITION 8. *$X_t = U_{T_4-} I_{[T_4,\infty)}(t \wedge T_3) - \int_s^{t \wedge T_3} I_{(s,T_1]}(u)\, dA_u^c$ is a martingale/$(\mathscr{M}_t)$.*

PROOF. Let $V_t^n = \sum_{j=1}^{\infty} I_{[T^{nj},\infty)}(t)$ for $n \geq 1$. We shall show first that $V^n - J^n$ is a martingale/$(\mathscr{M}_t)$. Let n be fixed, and let $\tilde{\mathscr{F}}_t = \mathscr{F}_t \vee \sigma(V_s^n, s \leq t)$. Now certainly $\tilde{\mathscr{F}}_t$ and $\tilde{\mathscr{F}}_\infty$ are c.i./$\tilde{\mathscr{F}}_t$; therefore by property (ii) of conditional independence given above, $\tilde{\mathscr{F}}_t \vee \mathscr{H}_2 \vee \mathscr{H}_3 \vee \sigma(\lambda_{rj}, r \neq n, j \geq 1)$ and $\tilde{\mathscr{F}}_\infty$ are c.i./$\tilde{\mathscr{F}}_t$. Thus, as $\mathscr{M}_t^0 \subset \tilde{\mathscr{F}}_t \vee \mathscr{H}_1 \vee \mathscr{H}_3 \vee \sigma(\lambda_{rj}, r \neq n, j \geq 1)$, $\mathscr{M}_t^0$ and $\tilde{\mathscr{F}}_\infty$ are c.i./$\tilde{\mathscr{F}}_t$, and so, by property (iii), $\mathscr{M}_t$ and $\tilde{\mathscr{F}}_\infty$ are c.i./$\tilde{\mathscr{F}}_{t+}$. It is therefore sufficient to show that $V^n - J^n$ is a martingale/$(\tilde{\mathscr{F}}_{t+})$.

Let $N_t = \max\{j : \sum_{i=1}^{j} \lambda_{ni} \leq t\}$. By the definition of the λ_{ni}, N_t is a Poisson process, and so $N_t - t$ is a martingale relative to its natural filtration. Applying Lemma 5, with $Z_t = N_t - t$, $\mathscr{E}_t = \sigma(Z_s, s \leq t)$, $\mathscr{F}_t = \mathscr{F}_t$, and $K = J^n$, we deduce that $N_{J_t^n} - J_t^n$ is a martingale/$(\mathscr{F}_t \vee \mathscr{E}_{J_t^n})$. However $N_{J_t^n} = V_t^n$, and thus as $(\mathscr{F}_t \vee \mathscr{E}_{J_t^n}, t \geq 0)$ is a nondecreasing family of σ-fields, $\sigma(V_s^n, s \leq t) \subset \mathscr{F}_t \vee \mathscr{E}_{J_t^n}$. Therefore $\tilde{\mathscr{F}}_t \subset \mathscr{F}_t \vee \mathscr{E}_{J_t^n}$, which implies that $V^n - J^n$ is a martingale/$(\tilde{\mathscr{F}}_t)$. Since $V^n - J^n$ is right-continuous, $V^n - J^n$ is also a martingale/$(\tilde{\mathscr{F}}_{t+})$.

Let $X_t^n = \int_s^t I_{(s,T_3]}(u) U_{u-}\, d(V^n - J^n)_u$ for each $n \geq 1$. X^n is a local martingale/$(\mathscr{M}_t)$, and as $E(\int_0^t (U_{u-} I_{(s,T_3]}(u))^2\, d[V^n - J^n, V^n - J^n]_u)^{1/2} \leq E(U_{T_4-} I(t \geq T_4)) \leq EY_t < \infty$, X^n is a martingale/$(\mathscr{M}_t)$. Let $H_t^n = \int_s^t I_{(s,T_3]}(u) U_{u-}\, dV_u^n$: evaluating this integral we have that $H_t^n = I_{[T_4,\infty)}(t \wedge T_3) I(T_4 \in \Gamma^n) U_{T_4-}$. Set $\tilde{H}_t^n = \int_s^t I_{(s,T_3]}(u) U_{u-}\, dJ_u^n$. Then, as $U_{u-} = I(u \leq T_1) Y_{u-}$, $\tilde{H}_t^n = \int_s^{t \wedge T_3} I_{(s,T_1]}(u) I(1/n \leq Y_{u-} < 1/(n-1))\, dA_u^c$. The martingale $V^n - J^n$ is of finite variation, and $I_{(s,T_3]} U_-$ is previsible, so that $X^n = H^n - \tilde{H}^n$.

Now $\sum_{n=1}^{\infty} I(T_4 \in \Gamma^n) = I(T_4 < \infty)$ a.s., and therefore $\sum_1^{\infty} H_t^n = U_{T_4-} I_{[T_4,\infty)}(t \wedge T_3)$ a.s. Also, $\{Y_- > 0\} \cap (s, T_1) = (s, T_1)$, and as A^c is continuous, $\sum_1^{\infty} \tilde{H}_t^n = \int_s^{t \wedge T_3} I_{(s,T_1](u)}\, dA_u^c$, and consequently $X_t = \sum_1^{\infty} H_t^n - \sum_1^{\infty} \tilde{H}_t^n$. For each n, $|X_t - \sum_1^n (H_t^n - \tilde{H}_t^n)|$ is dominated by $X_t + Y_t + A_t^c$, which is integrable. Therefore $\sum_1^n (H_t^n - \tilde{H}_t^n) \to X_t$ in L^1 as well as a.s., and X is a martingale/$(\mathscr{M}_t)$.

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