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TAUBERIAN THEOREMS AND THE CENTRAL LIMIT THEOREM

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We prove Tauberian theorems for random walks with positive drift obeying the central limit theorem. The results include (i) conclusions involving certain averages, relevant to number-theoretic densities and extending results of Diaconis and Stein; (ii) pointwise conclusions, including the classical Borel-Tauber theorem and extending results of Schmaal, Stam and de Vries.

1. Results. Let X_n be independent and identically distributed random variables with law P, mean $\mu > 0$ and variance $\sigma^2 < \infty$, $S_n = \sum_1^n X_k$. Write P_n, Q_n, Φ for the laws of S_n, $(S_n - n\mu)/(\sigma\sqrt{n})$ and the standard normal $N(0, 1)$; thus $Q_n \Rightarrow \Phi$ (weak convergence) by the global central limit theorem.

We shall need the local central limit theorem (see e.g., Ibragimov and Linnik (1971), Theorems 4.4.1, 4.2.1, 4.2.2). We shall assume either

(LLT-I) some P_k has an absolutely continuous component, in which case $\| Q_n - \Phi \| \to 0$ as $n \to \infty$ (variation-norm), or

(LLT-II) P is lattice, supported by some arithmetic progression $a + hk$ with k integer and h maximal. We shall suppose for simplicity that $a = 0$ and $h = 1$, when, writing p_{nk} for $P_n(\{k\})$,

$$\sum_k \left| p_{nk} - \frac{1}{\sigma\sqrt{n}}\phi\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right)\right| \to 0 \qquad n \to \infty$$

We shall be concerned with $Ef(S_n)$ as $n \to \infty$ for suitable measurable functions f. We shall always suppose $f(x) = 0$ for $x < 0$ (as $S_n \to \infty$ and we are concerned with the behaviour of f at $+\infty$). In the lattice case (when S_n is supported by the integers) we suppose also that f is constant on each $[k, k + 1)$, $k = 0, 1, \ldots$.

THEOREM 1. *If P satisfies* (LLT-I) *or* (LLT-II) *and f is bounded and satisfies the conditions above, the following are equivalent*:

(1) $Ef(S_n) \to c$ $(n \to \infty)$ *for some P with mean $\mu > 0$ and variance $\sigma^2 < \infty$*

(2) $$Ef(S_n) \to c \qquad (n \to \infty) \quad \textit{for} \quad P = N(1, 1), \text{ i.e.,}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \exp\left\{-\frac{1}{2}(x - y)^2/x\right\} dy/\sqrt{x} \to c \qquad (x \to \infty)$$

(3) $$\frac{1}{\epsilon\sqrt{x}} \int_x^{x-\epsilon\sqrt{x}} f(y)\, dy \to c \qquad (x \to \infty) \quad \textit{for all} \quad \epsilon > 0.$$

By Theorem 1, if $Ef(S_n)$ converges with $\{S_n\}$ a random walk generated by *one* law P with positive mean and finite variance satisfying (LLT-I) or (LLT-II), $Ef(S_n)$ converges for *all* such P, and the limits are the same; such convergence is equivalent to that of the 'delayed averages' of f in (3).

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Taking P the Poisson law with parameter 1, (1) is

$$\textstyle\sum_{k=0}^{\infty} e^{-n}n^k f(k)/k! \to c \qquad n \to \infty$$

i.e., $f(n) \to c$ (B)—in the sense of Borel summability. Taking P the Bernoulli law with parameter p, (1) is

$$\textstyle\sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k} f(k) \to c \qquad n \to \infty;$$

i.e., $f(n) \to c$ ($\mathrm{E_p}$)—in the sense of Euler summability with parameter p. With P the geometric law with parameter $p = 1 - q$, (1) may be written

$$(1-q)^{n+1} \textstyle\sum_{k=0}^{\infty} \binom{n+k}{k} q^k f(k) \to c \qquad n \to \infty,$$

i.e., $f(n) \to c$ ($\mathrm{M\text{-}K_q}$)—in the sense of the Meyer-König method (usually written S_q: Meyer-König (1949)). With $P = N(1, 1)$, (1) becomes (2), which is $f(x) \to c$ (V′)—the continuous Valiron method (Hardy and Littlewood (1916)). We thus have (Meyer-König (1949); see also Jurkat (1956)).

COROLLARY 1. *For bounded f, the Borel* (B), *Euler* ($\mathrm{E_p}$), $0 < p < 1$, *Meyer-König* ($\mathrm{M\text{-}K_q}$), $0 < q < 1$ *and continuous Valiron* (V′) *methods are all equivalent, to each other and to* (3).

This result may be restated in terms of number-theoretic densities. For A a subset of the positive integers Z_+, let f be constant on each $[n, n+1)$ with $f(n) = I_A(n)$. We obtain (Diaconis and Stein (1978)):

COROLLARY 2. *For* $A \subset Z_+$, *the following are equivalent*:

$$I_A(n) \to c \qquad (\mathrm{B}),$$

$$I_A(n) \to c \qquad (\mathrm{E_p}) \qquad \textit{for some (all)} \quad p \in (0, 1)$$

$$\frac{1}{\epsilon\sqrt{n}} \textstyle\sum_{n \le k < n+\epsilon\sqrt{n}} I_A(k) \to c \qquad \textit{for all} \quad \epsilon > 0.$$

The summability methods above are included in the *circle family* (cf., Meyer-König (1949)). We may thus say that $A \subset Z_+$ has *circle density* $c \in [0, 1]$ when any (all) of the statements of Corollary 2 hold. We shall see below that this is stronger than saying A has density c in the usual (Cesàro) sense.

The lattice case (LLT-II) is the most interesting; here our results may be restated in more classical language. Write $f(n) - f(n-1) = a_n$; so $f(x) = \sum_{0 \le n \le x} a_n$ and, in the usual notation for sequences, with $s_n := \sum_{k=0}^{n} a_k$ we have $f(x) = s_{[x]}$. We write $f(n) \to c$ (R) if

$$\frac{1}{x}\int_0^x \{\textstyle\sum_{e^{\sqrt{n}} \le t} a_n\}\, dt \to c \qquad x \to \infty$$

(the Riesz typical mean $R(e^{\sqrt{n}}, 1)$; see Chandrasekharan and Minakshisundaram (1952)).

THEOREM 2. *For f bounded, the following are equivalent*:

(4) $$f(n) \to c \qquad (\mathrm{B})$$

(5) $$f(n) \to c \qquad (\mathrm{R})$$

(6) *there exists* $\epsilon_n \to 0$ *such that* $\dfrac{1}{(n+1)} \sum_{k=0}^{n} f(k) = c + o(1/\sqrt{n}) + \dfrac{1}{(n+1)} \sum_{k=0}^{n} \epsilon_k$

(7) $$\sqrt{2/\pi}\int_{-\infty}^{\infty} f(y^2)\exp\{-2(x-y)^2\}\,dy \to c \qquad x\to\infty$$

(8) $$\frac{1}{\epsilon}\int_x^{x+\epsilon} f(y^2)\,dy \to c \qquad (x\to\infty) \quad \textit{for all } \epsilon > 0.$$

COROLLARY. *The following are equivalent to* (1)–(8) *under the conditions above*:

(9) $\frac{1}{\epsilon\sqrt{x}}\int_x^{x+\epsilon\sqrt{x}} f(y)\,dy \to c$ $(x\to\infty)$ *for two positive values of* ϵ *with irrational quotient*

(10) $\frac{1}{\epsilon}\int_x^{x+\epsilon} f(y^2)\,dy \to c$ $(x\to\infty)$ *for two positive values of* ϵ *with irrational quotient.*

The results above link (1) for random walks with *positive* mean to Borel (and related methods of) summability. By contrast, for random walks with *zero* mean it is Cesàro summability which is linked to (1); see Davydov and Ibragimov (1971), Davydov (1974).

That (2) and (4) are equivalent (under the weaker Tauberian condition $f(n) = o(\sqrt{n})$) is due to Hardy and Littlewood (1916) and Meyer-König (1949). That (4) implies (5) for bounded f is due to Karamata (1937), (1938). That (6) implies (4) is due essentially to Hardy (1949), Theorem 149 (see also Diaconis and Stein (1978), Theorem 3). Hardy did not have the last term on the right in (6). Including this, however, we can reverse the implication in Hardy's theorem to see that (6) is necessary as well as sufficient for (4), and thus obtain an explicit representation of bounded Borel-summable (or bounded 'circle-summable') sequences. Note that (6) always implies $f(n) = o(\sqrt{n})$ (multiply by $(n+1)$ and difference).

We remark briefly on the role of the two 'error terms' on the right of (6). The second may tend to zero arbitrarily slowly, but does so 'smoothly' in the sense that, when multiplied by $n+1$ and differenced to give ϵ_n, it still tends to zero. The first tends to zero at the specified rate $o(1/\sqrt{n})$, but in general gives $o(\sqrt{n})$ when subjected to the operations above. The representation (6) arises from the Karamata theory of regular variation and, like all representations of Karamata type, is essentially nonunique; see Seneta (1976), page 14, Bingham and Goldie (1980), II. Note that (6) may be interpreted as telling us (at least for bounded sequences) exactly how much stronger than Cesàro convergence the Borel convergence in (4) is.

We can obtain pointwise (rather than average) conclusions by imposing a Tauberian condition of slow-decrease type. Consider the one-sided Tauberian condition

(TC) $$\lim_{\epsilon\to 0+}\liminf_{x\to\infty}\inf_{y\in[x,x+\epsilon\sqrt{x}]} f(y) - f(x) \ge 0$$

and the corresponding two-sided condition

(TC′) $$|f(y)-f(x)| \to 0 \qquad x, y\to\infty, \quad |x-y| = o(\sqrt{x})$$

((TC) for f and $-f$).

THEOREM 3a. *With P as in Theorem* 1, (1) *implies*

$$f(x)\to c \qquad x\to\infty$$

if and only if (TC) *holds.*

If we strengthen the Tauberian condition from the one-sided (TC) to the two-sided (TC′), we can weaken the conditions on P and use the global rather than the local central limit theorem.

THEOREM 3b. *If P has positive mean and finite variance,* (1) *implies $f(x) \to c$ $(x \to \infty)$ if and only if* (TC′) *holds.*

Theorem 3b is Theorem 1 of Schmaal, Stam and de Vries (1976). On the other hand, the Poisson case of Theorem 3a is the classical *Borel-Tauber theorem* of Schmidt (1925); cf., Hardy (1949), Theorem 241:

COROLLARY. *If $f(n) \to c$* (B) *$(n \to \infty)$, then $f(n) \to c$ $(n \to \infty)$ if and only if the Tauberian condition*

$$\lim_{\epsilon\to 0+} \liminf_{n\to\infty} \min_{n\le m<n+\epsilon\sqrt{n}} f(m) - f(n) \geq 0$$

holds.

It is interesting to compare Theorem 2 with the following result of Chow (1973).

THEOREM. *For X_n* i.i.d. *random variables, the following are equivalent:*

$$EX_n = c, \qquad \operatorname{Var} X_n < \infty,$$

$$X_n \to c \qquad (\mathrm{B}) \quad \text{a.s.} \quad (n \to \infty),$$

$$X_n \to c \qquad (\mathrm{E_p}) \quad \text{a.s.} \quad \textit{for some (all)} \quad p \in (0, 1),$$

$$\frac{1}{\epsilon\sqrt{n}} \sum_{n\le k<n+\epsilon\sqrt{n}} X_k \to c \qquad \text{a.s.} \quad (n \to \infty) \quad \textit{for some (all)} \quad \epsilon > 0.$$

Note that, by a Borel-Cantelli argument, $\operatorname{Var} X_n < \infty$ implies $X_n = o(\sqrt{n})$ a.s. When $X_n = o(\sqrt{n})$, these statements are equivalent to analogues of (6)–(8); see Section 2 below. I am indebted for these remarks to Dr. J. Mijnheer.

2. Proofs. We first quote the following result (Beurling's generalisation of Wiener's Tauberian theorem; see Moh (1972), Peterson (1972)):

THEOREM B. *Let $\theta(x)$ be positive for large x, $o(x)$ at infinity, with*

$$\text{(B)} \qquad \theta(x + t\theta(x))/\theta(x) \to 1 \qquad (x \to \infty) \quad \textit{for all } t \in \mathbb{R}.$$

If $K \in L_1(\mathbb{R})$ with $\hat{K}(t) := \int_{-\infty}^{\infty} e^{ixt}K(t)\,dt \neq 0$ $(t \in \mathbb{R})$, and $\psi \in L_\infty(\mathbb{R})$,

$$(*) \qquad \int_{-\infty}^{\infty} \psi(y)K\left(\frac{x-y}{\theta(x)}\right) dy/\theta(x) \to A \int_{-\infty}^{\infty} K(y)\,dy \qquad x \to \infty$$

implies

$$(**) \qquad \int_{-\infty}^{\infty} \psi(y)H\left(\frac{x-y}{\theta(x)}\right) dy/\theta(x) \to A \int_{-\infty}^{\infty} H(y)\,dy \qquad x \to \infty$$

for all $H \in L_1(\mathbb{R})$.

Extension: if $(*)$ holds for a class of kernels K whose Fourier transforms $\hat{K}$ have no common zero on R, then again $(**)$ holds for all $H \in L_1(\mathbb{R})$.

The case $\theta(x) \equiv 1$ is Wiener's Tauberian theorem (Wiener (1932), Widder (1941)). For the proof of Beurling's theorem see Peterson (1972); the extension may be proved as in Wiener (1932).

PROOF OF THEOREM 1. Since f is bounded and $Ef(S_n) = \int f(n\mu + x\sigma\sqrt{n})\, dQ_n(x)$, statements (1) with P and with $N(\mu, \sigma)$ are equivalent in the case (LLT-I). In the lattice case (LLT-II),

$$Ef(S_n) = \sum_k f(k)p_{nk}.$$

As f is bounded, (LLT-II) shows that

$$Ef(S_n) - \sum_k \frac{f(k)}{\sigma\sqrt{n}}\phi\left(\frac{k-n\mu}{\sigma\sqrt{n}}\right) \to 0 \qquad n\to\infty.$$

Writing $(k-n\mu)/(\sigma\sqrt{n}) = x_k$, $\Delta x_k = x_{k+1} - x_k = 1/\sqrt{n}$, this is

$$Ef(S_n) - \sum_k f(n\mu + x_k\,\sigma\sqrt{n})\phi(x_k)\Delta x_k \to 0 \qquad n\to\infty.$$

Now

$$\sum_k \left| \phi(x_k)\Delta x_k - \int_{x_k}^{x_{k+1}} \phi(x)\,dx \right| \to 0 \qquad n\to\infty,$$

as may be seen by using the monotonicity of ϕ on each half-line separately. As f is bounded, this gives

$$\sum_k f(n\mu + x_k\sigma\sqrt{n})\phi(x_k)\Delta x_k - \int f(n\mu + x\sigma\sqrt{n})\phi(x)\,dx \to 0 \qquad n\to\infty.$$

The integral here is $Ef(S_n)$ with P replaced by $N(\mu, \sigma)$. Combining, (1) with P and with $N(\mu, \sigma)$ are equivalent, as before.

Next, (1) with $P = N(\mu, \sigma)$ is

$$(1') \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\exp\left\{-\frac{1}{2}(y-n\mu)^2/(n\sigma^2)\right\} dy/\sqrt{n} \to c \qquad n\to\infty.$$

It is easy to see that we may let $n\to\infty$ here through continuous rather than discrete values (cf., e.g., Hardy and Littlewood (1916), 613). So we may take $\mu = 1$ by replacing n by n/μ. We may then also take $\sigma = 1$ by Beurling's Tauberian theorem with $\theta(x) = \sqrt{x}$, and $K(x)$, $H(x)$ as successively $\exp\{-\tfrac{1}{2}x^2\mu/\sigma^2\}$, $\exp\{-\tfrac{1}{2}x^2\}$ and the same functions reversed; thus (1′) and (2) are equivalent.

Using Beurling's theorem with $\theta(x) = \sqrt{x}$, $K(x) = \exp\{-\tfrac{1}{2}x^2\}$, $H(x) = I_{[-\epsilon,0]}(x)$, (2) implies (3) (Moh (1972), Theorem II).

If (3) holds for $\epsilon = \epsilon_1, \epsilon_2 > 0$ with ϵ_1/ϵ_2 irrational, write $K_j(x) = I_{[-\epsilon_j,0]}(x)$, $j = 1, 2$. Then $\hat{K}_j(t) = (1-\exp\{-i\epsilon_j t\})/(it) = 0$ for $t = 2n\pi/\epsilon_j$ (n a nonzero integer). Since $\hat{K}_1$ and $\hat{K}_2$ have no common zero for ϵ_1/ϵ_2 irrational, the extension to Beurling's theorem with $H(x) = \exp\{-\tfrac{1}{2}x^2\}$ yields (2), which completes the proof of Theorem 1.

Proof of Theorem 2. Put $\epsilon = 2\log\lambda$ ($\lambda > 1$). As $f(x)$ is bounded ($o(\sqrt{x})$ would suffice), (3) is equivalent to

$$\frac{1}{2\sqrt{x}\log\lambda}\int_x^{x+2\sqrt{x}\log\lambda+\log^2\lambda} f(y)\,dy \to c \qquad x\to\infty \text{ for all } \lambda > 1.$$

Put $\Lambda(x) = \exp\{\sqrt{x}\}$; then $\Lambda^{-1}(x) = \log^2 x$, and for all $\lambda > 1$

$$\Lambda^{-1}(\lambda\Lambda(x)) = x + 2\sqrt{x}\log\lambda + \log^2\lambda.$$

So replacing x by $\Lambda^{-1}(x)$ and writing $\Lambda(y) = u$, this is

$$(3') \qquad \frac{1}{\log x}\int_x^{\lambda x} f(\log^2 u)\log u\,du/u \to c\log\lambda \qquad x\to\infty \quad \text{for all } \lambda > 1.$$

But this is of the form

$$[F(\lambda x) - F(x)]/L(x) \to c\log\lambda$$

with $F(x) = \int_1^x f(\log^2 u) \log u \, du/u$ and $L(x) = \log x$ slowly varying; for the theory of such asymptotic relations see de Haan (1970), Seneta (1976), II, Bingham and Goldie (1980), II, Section 4. From these results, each of the following is equivalent to (3′):

$$(11) \qquad F(x) - \frac{1}{x}\int_0^x F(u)\,du = \frac{1}{x}\int_0^x f(\log^2 u) \log u\, du \to c \log x \qquad x \to \infty,$$

$$(12) \qquad F(x) = \int_1^x f(\log^2 u) \log u \, du/u = \int_1^x [c + \delta(u)] \log u \, du/u + o(\log x),$$

where $\delta(x) \to 0$ as $x \to \infty$.

Writing $\log^2 u = v$ and replacing x by $\exp\{\sqrt{x}\}$, (11) is

$$\frac{1}{2\sqrt{x}e^{\sqrt{x}}}\int_0^x f(v) \exp\{\sqrt{v}\}\, dv \to c \qquad x \to \infty.$$

Write $\Lambda_1(x) = \sqrt{x}e^{\sqrt{x}}$; as $d\Lambda_1(x)/dx = \frac{1}{2}(1 + x^{-1/2})e^{\sqrt{x}}$ and $f(x) = 0(1)$ ($o(\sqrt{x})$ would suffice),

$$\tfrac{1}{2} f(x) e^{\sqrt{x}}\, dx = (f(x) + o(1))\, d\Lambda_1(x),$$

so this is

$$\frac{1}{\Lambda_1(x)}\int_0^x f(u)\, d\Lambda_1(u) \to c \qquad x \to \infty.$$

As $f(u) = \sum_{n \le u} a_n$, this is (replacing x by $\Lambda_1^{-1}(x)$ and writing $\Lambda_1(u) = v$)

$$\frac{1}{x}\int_0^x \{\textstyle\sum_{\Lambda_1(n) \le u} a_n\}\, du \to c \qquad x \to \infty,$$

or $s_n \to c$ (R, Λ_1, 1) in the notation of Riesz typical means. As $\Lambda_1(x) = \sqrt{x}e^{\sqrt{x}}$ is bounded above and below by positive powers of $\Lambda(x) = e^{\sqrt{x}}$, this is

$$(5') \qquad \frac{1}{x}\int_0^x \{\textstyle\sum_{\Lambda(n) \le u} a_n\}\, du \to c \qquad x \to \infty$$

or $s_n \to c$ (R, Λ, 1) (Chandrasekharan and Minakshisundaram (1952), 35), which is (5); thus (3) and (5) are equivalent.

We may rewrite (12) (equivalent to (1)–(3) and (5)) as

$$\frac{1}{x}\int_0^x f(u)\, du = c + o(1/\sqrt{x}) + \frac{1}{x}\int_0^x \delta(e^{\sqrt{u}})\, du \qquad x \to \infty$$

where $\delta(x) \to 0$. Take $x = n + 1$, $n = 0, 1, \ldots$, and recall $f(x) = s_n$ for $n \le x < n + 1$: then writing $\epsilon_n = \int_n^{n+1} \delta(e^{\sqrt{u}})\, du$, $\epsilon_n \to 0$ and

$$\frac{1}{(n+1)}\sum_{k=0}^n s_k = c + o(1/\sqrt{n}) + \frac{1}{(n+1)}\sum_{k=0}^n \epsilon_k,$$

which is (6).

Now assume (6). Then Theorem 149 of Hardy (1949) yields

$$s_n - \epsilon_n \to c\ (\mathrm{E_p}) \qquad \text{for all} \quad p \in (0, 1),$$

which by regularity of $\mathrm{E_p}$ is $s_n \to c$ ($\mathrm{E_p}$). Thus by Theorem 1 and its Corollaries, (1)–(6) are equivalent.

Next, write (5), (5′) with $\phi(x) = f(\Lambda^{-1}(x)) = f(\log^2 x)$ as

$$\frac{1}{x}\int_0^x \phi(u)\,du \to c \qquad x \to \infty.$$

By the theory of Frullani integrals (Aljančić and Karamata (1956), Ostrowski (1976), Bingham and Goldie (1980), II, Section 6) this is equivalent to

$$\text{(13)} \qquad \frac{1}{\log \lambda}\int_x^{\lambda x} \phi(t)\,dt/t \to c \qquad x \to \infty \quad \text{for all} \quad \lambda > 1.$$

Writing $t = e^u$ and replacing $\log \lambda (\lambda > 1)$ by $\lambda > 0$, this is

$$\frac{1}{\lambda}\int_x^{x+\lambda} f(\Lambda^{-1}(e^u))\,du = \frac{1}{\lambda}\int_x^{x+\lambda} f(u^2)\,du \to c \qquad x \to \infty \quad \text{for all} \quad \lambda > 0,$$

which is (8); so (8) is equivalent to (1)–(6). That (7) is equivalent to (2) for bounded f follows by (12.15.9) of Hardy (1949). Alternatively, the equivalence of (7) and (8) follows by Wiener's Tauberian theorem ($\theta(x) \equiv 1$ in Theorem B): take $K(x) = \exp\{-2x^2\}$, $H(x) = I_{[-\epsilon,0]}(x)$ to pass from (7) to (8); (8) includes (10); take $K_j(x) = I_{[-\epsilon_j,o]}(x)$, $j = 1, 2$, $H(x) = \exp\{-2x^2\}$ to pass from (10) to (7)).

Putting $y^2 = u$ and replacing x by $\sqrt{x}$ (8) is

$$\frac{1}{2\epsilon}\int_x^{x+2\epsilon\sqrt{x}+\epsilon^2} f(u)\,du/\sqrt{u} \to c.$$

As f is bounded, this is equivalent to

$$\frac{1}{2\epsilon}\int_x^{x+2\epsilon\sqrt{x}} f(u)\,du/\sqrt{u} \to c.$$

But $1/\sqrt{u} = [1 + o(1)]/\sqrt{x}$, uniformly in $u \in [x, x + 2\epsilon\sqrt{x}]$, so this is

$$\frac{1}{2\epsilon\sqrt{x}}\int_x^{x+2\epsilon\sqrt{x}} f(u)[1 + o(1)]\,du \to c,$$

which as f is bounded is (3). Taking $\epsilon = \epsilon_1, \epsilon_2$ the equivalence of (9) and (10) follows. Alternatively, the equivalence of (8) and (10) follows from the theory of Frullani integrals (see (13) and Bingham and Goldie (1980), II, Section 6). In fact we only need f bounded below here, as this ensures the Tauberian condition

$$\text{(14)} \qquad \liminf_{\lambda\downarrow 1} \liminf_{x\to\infty} \int_x^{\lambda x} \phi(t)\,dt/t \geq 0.$$

This completes the proof of Theorem 2 and its Corollary. We remark that f bounded below suffices also for the implication from (7) to (8), by the one-sided version of Wiener's Tauberian theorem (Widder (1941), 216).

Proof of Theorem 3a.

(I) First, from $Ef(S_n)$ and (TC), we deduce boundedness of f. In the lattice case (LLT-II), this follows by Vijayaraghavan's theorem (Hardy (1949), Theorem 238 and page 313). We need only check that

$$\sum_{k\leq n\mu-\nu\sqrt{n}} p_{nk} \to 0, \qquad \sum_{k\geq n\mu+\nu\sqrt{n}} p_{nk} \to 0 \qquad n, \nu \to \infty,$$

both immediate from the (global) central limit theorem, and

$$\frac{1}{\sqrt{n}}\sum_{k\geq n\mu+\nu\sqrt{n}} (k - n\mu)p_{nk} \to 0 \qquad n, \nu \to \infty.$$

But the left-hand side is

$$E[n^{-1/2}(S_n - n\mu)I\{(S_n - n\mu)/\sqrt{n} \geq \nu\}] \leq \left\{E\left[\frac{1}{n}(S_n - n\mu)^2 I\{(S_n - n\mu)^2/n \geq \nu\}\right]\right\}^{1/2} \to 0 \quad \nu \to \infty$$

as the variance is finite. In the absolutely continuous case (LLT-I), we use the continuous version of Vijayaraghavan's theorem (Karamata (1937), Theorem IV). We need to check that (with p_n the density of P_n)

$$\int_0^\infty p_n(x)\,|\sqrt{x} - \sqrt{y_n}|\,dx = O(1) \qquad (n \to \infty;\ z_n := (y_n - n\mu)/\sqrt{n} \text{ bounded}).$$

First, $y_n = n\mu + O(\sqrt{n}) > 0$ for large n as $\mu > 0$. But (cf., Schmaal, Stam and de Vries (1976), 80)

$$\begin{aligned} |\sqrt{S_n^+} - \sqrt{y_n}| = |S_n^+ - y_n|/(\sqrt{S_n^+} + \sqrt{y_n}) &\leq |S_n^+ - y_n|/\sqrt{y_n} \\ &\leq |S_n - y_n|/\sqrt{y_n} \end{aligned}$$

(as $y_n > 0$), so

$$\begin{aligned} E|\sqrt{S_n^+} - \sqrt{y_n}| &\leq y_n^{-1/2}\{E(|S_n - y_n|^2)\}^{1/2} \\ &= O(n^{-1/2}\{E((S_n - n\mu - \sigma z_n\sqrt{n})^2\}^{1/2}) \\ &= O(n^{-1/2}\{E((S_n - n\mu)^2 + \sigma^2 n z_n^2\}^{1/2}) \\ &= O(1) \end{aligned}$$

as required.

(II) Using the boundedness of f, we obtain (3) as in the proof of Theorem 1.

(III) From (3) and (TC), $f(x) \to c$ follows by a classical argument.
For, for all $\epsilon > 0$ there exist $\Delta > 0$, $X < \infty$ with

$$f(y) - f(x) \geq -\epsilon \qquad \text{for all} \quad x + \delta\sqrt{x} \geq y \geq x \geq X, \quad 0 < \delta \leq \Delta.$$

But then $y + \delta\sqrt{y} \geq x + \delta\sqrt{x} \geq y \geq X$, so also $f(x + \delta\sqrt{x}) - f(y) \geq -\epsilon$. So

$$\frac{1}{\delta\sqrt{x}}\int_x^{x+\delta\sqrt{x}} [f(y) - f(x)]\,dy \geq -\epsilon,$$

$$\frac{1}{\delta\sqrt{x}}\int_x^{x+\delta\sqrt{x}} [f(x + \delta\sqrt{x}) - f(y)]\,dy \geq -\epsilon.$$

So

$$f(x) \leq \epsilon + \frac{1}{\delta\sqrt{x}}\int_x^{x+\delta\sqrt{x}} f(y)\,dy \leq c + 2\epsilon$$

for large x, by (3), and

$$f(x + \delta\sqrt{x}) \geq -\epsilon + \frac{1}{\delta\sqrt{x}}\int_x^{x+\delta\sqrt{x}} f(y)\,dy \geq c - 2\epsilon$$

for large x. So lim sup $f(\cdot) \leq c + 2\epsilon$, lim inf $f(\cdot) \geq c - 2\epsilon$, and thus $f(\cdot) \to c$, as required.

PROOF OF THEOREM 3b.

(I′) The proof of boundedness of f is of course contained in (I) above, but with a two-sided Tauberian condition much simpler arguments than Vijayaraghavan's suffice; cf., Karamata (1938), 56, Schmaal, Stam and de Vries (1976), Lemma 5.

(II′) We have f bounded and

$$Ef(S_n) = \int f(n\mu + x\sigma\sqrt{n})\, dQ_n(x) \to c,$$

where $Q_n \Rightarrow \Phi$. Let $f_n(x) := f(n\mu + x\sigma\sqrt{n})$; then the f_n are uniformly bounded, and for all $\delta_k > 0$, $n_k \to \infty$ (TC′) shows that $|y_k - z_k| < \delta_k$ implies

$$f_{n_k}(y_k) - f_{n_k}(z_k) = f(n_k\mu + y_k\sigma\sqrt{n_k}) - f(n_k\mu + z_k\sigma\sqrt{n_k}) \to 0 \qquad n \to \infty.$$

Hence (Topsøe (1967), Theorem 1; cf., Billingsley and Topsøe (1967), Ranga-Rao (1962))

$$\int f(n\mu + x\sigma\sqrt{n})\, d\Phi(x) \to c \qquad n \to \infty,$$

which is (1) with $P = N(\mu, \sigma)$. Then (3) follows as in Theorem 1.

(III′) By (TC′), for all $\eta > 0$, $f(y)$ is within η of $f(x)$ in $[x, x + \epsilon\sqrt{x}]$ for large enough x and small enough ϵ, whence (3) gives $f(\cdot) \to c$, completing the proof.

3. Remarks.

1. Apart from B, E_p, M-K_q, the circle methods include the discrete Valiron methods V_α, $\alpha > 0$:

$$\sqrt{\alpha/\pi n}\, \textstyle\sum_{k=0}^{\infty} \exp\{-\alpha(n-k)^2/n\} f(k) \to c \qquad n \to \infty$$

and the Taylor methods T_α, $0 < \alpha < 1$:

$$(1-\alpha)^{n+1} \textstyle\sum_{k=0}^{\infty} \binom{n+k}{k} \alpha^k f(n+k) \to c \qquad n \to \infty.$$

The methods B, $V_{1/2}$ and its continuous analogue $V'_{1/2}$ (denoted simply V′ in Section 1) are equivalent for $f(n) = o(\sqrt{n})$ (Hardy and Littlewood (1916), Theorem 3.4; Meyer-König (1949), Section 5). With P the law of $1 + X$, X geometric with parameter $p = 1 - q$, (1) is $f(n + 1) \to c$ (T_q). For the equivalence of all the circle methods for bounded sequences, see Meyer-König (1949) Section 6. For the analogue of Hardy's result (Hardy (1949), Theorem 149) for circle methods, see Parameswaran (1959).

2. Also related to the central limit theorem is the family of $F(a, q)$-methods introduced by Meir (1963) (cf., Meyer-König (1949), Section 5.2), who studied their Tauberian constants. Swaminathan (1974) proves for them the analogue of Hardy's result, while Sitaraman and Swaminathan (1977) obtain the analogue of the Borel-Tauber theorem.

3. Theorem 2 shows that Borel summability B implies Cesàro summability C_1 for bounded sequences $f(n)$. For unbounded sequences this need not hold (e.g., if $f(n)$ grows exponentially. Consider $\sum z^n$ outside the unit circle; Hardy (1949), IX). However, $f(n) = O(\sqrt{n})$ suffices (Parameswaran (1975)), and so does $f(n) - f(n-1)$ bounded below ($a_n = O_L(1)$); see below.

4. Laws of the iterated logarithm for Borel and Euler summability have been obtained by Lai (1974a), (1974b). These provide analogues to the laws of the iterated logarithm for Cesàro and Abel summability by Gaposhkin (1965), and provide interesting complements to the work of Chow (1973) cited in Section 1.

5. Our results link (1) with (3), a stronger mode of convergence ('circle-convergence') than Cesàro convergence. When $\mu = 0$, the results of Davydov and Ibragimov (1971) and Davydov (1974) link (1) with Cesàro convergence of f and convergence in law of $n^{-1} \sum_1^n f(S_k)$. When $\mu > 0$ one can link Cesàro convergence of f with convergence of $n^{-1} \sum_1^n Ef(S_k)$, and of convergence of $n^{-1} \sum_1^n f(S_k)$, almost surely or in probability, under suitable conditions; see Bingham and Goldie (1981).

6. In Theorem 2 the boundedness of f plays the role of a Tauberian condition. The question arises as to whether, or to what extent, this Tauberian condition can be weakened.

One possibility is to use instead the weaker Tauberian condition $f(n) = o(\sqrt{n}\,)$. The proof given above shows that under it (3), (5), (6), (8) are equivalent, (2), (4) are equivalent, and (6) $\Rightarrow$ (4). The difficulty is to obtain (2) $\Rightarrow$ (3) (the step where the Wiener Tauberian theory is used above).

One-sided Tauberian conditions are also relevant. Assume, for instance, $a_n := f(n) - f(n-1) = O_L(1)$. Then (2) implies $f(n) \to c\ (C_1)$ by a result of Rajagopal (1960) (Theorem 2, $k = ½$). Then $f(n) = o(\sqrt{n})$ by a result of Minakshisundaram and Rajagopal (1948) (Theorem 2, 0-form, $r = 1$, $\theta(x) = \sqrt{x}$, $\phi(x) = x$) and we are back in the situation just considered. The corresponding two-sided results with $a_n = O(1)$ are classical (Hardy and Littlewood (1916), Theorems 3.11, 3.12).

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